Formal quasi-modular forms

and formal multiple Eisenstein series & formal multiple zeta values

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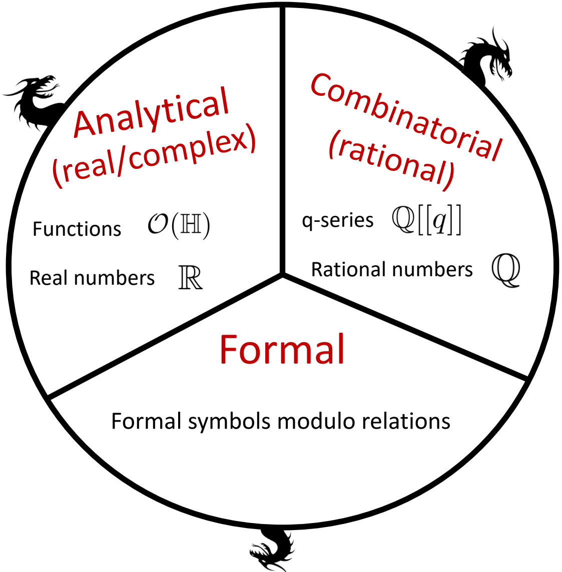
Ghidorah principle

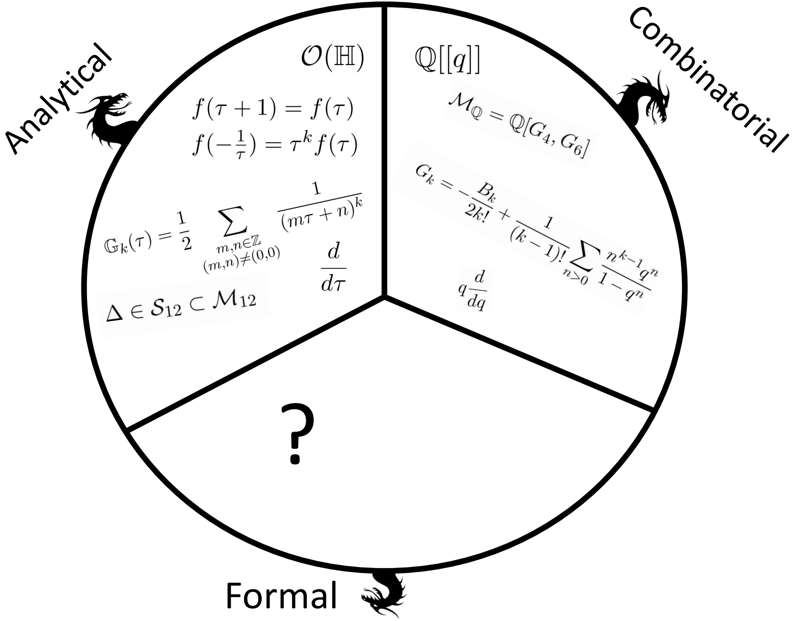
One object with three different point of views

キングギドラ

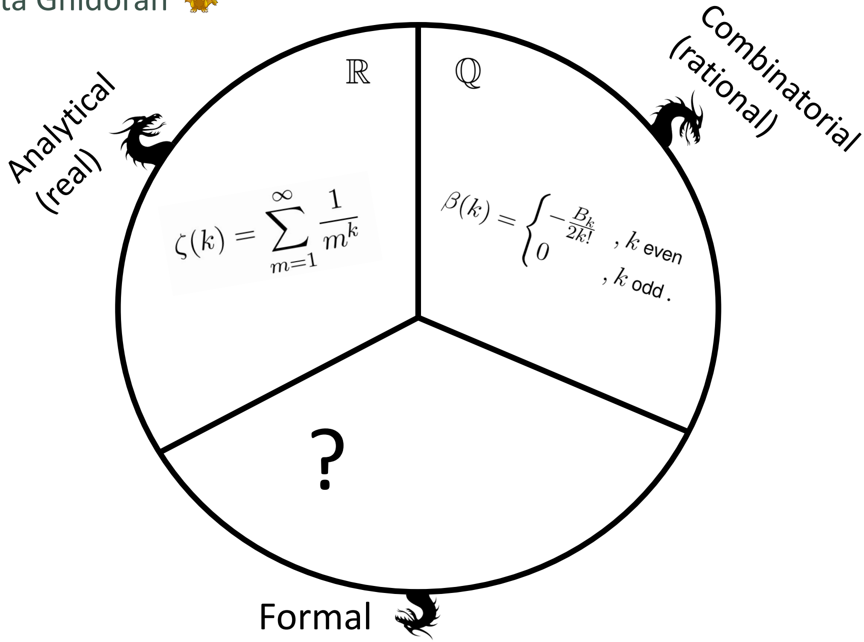


A real picture of ghidorah

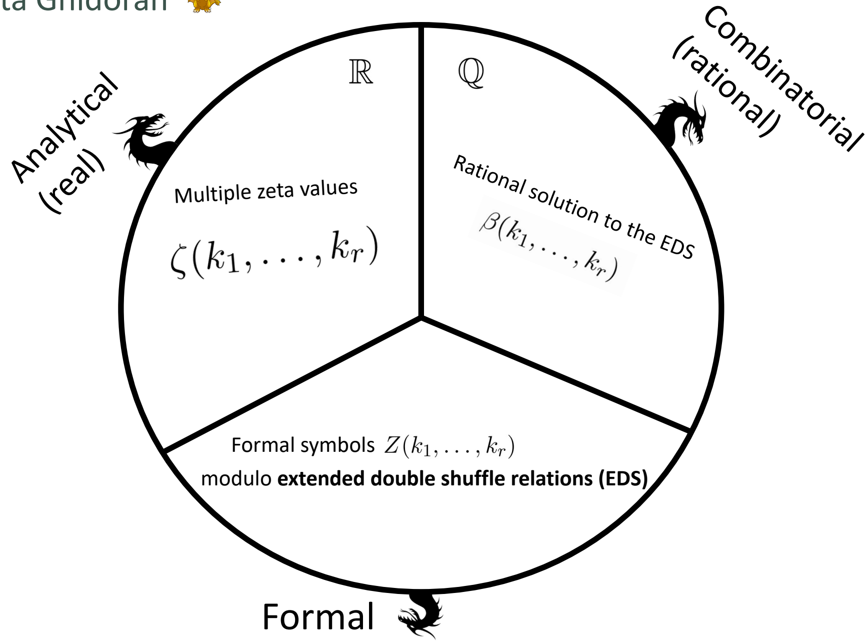
Modular Ghidorah



Riemann zeta Ghidorah

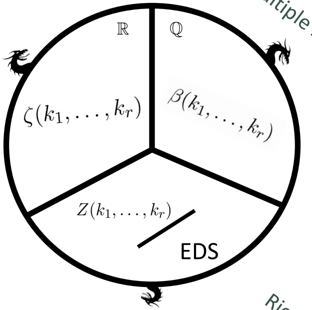


Multiple zeta Ghidorah

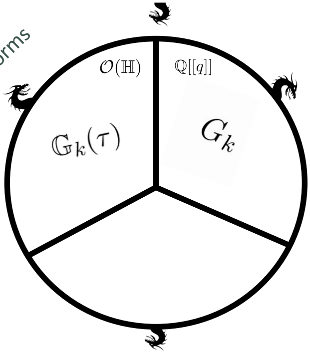


Overview

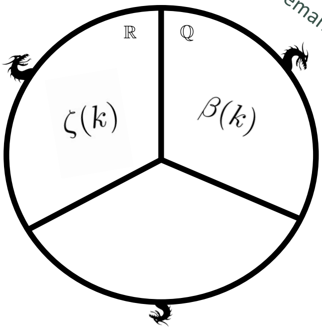
Multiple zeta values



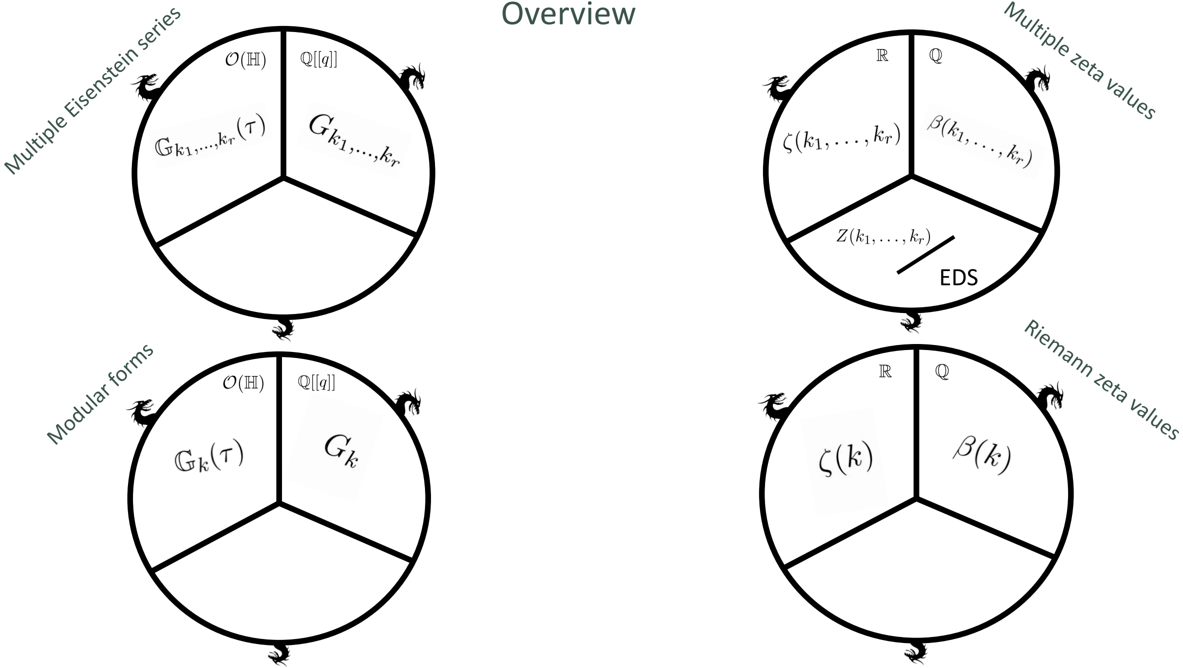
Modular forms



Riemann zeta values



Overview


Multiple Eisenstein series
$\mathcal{O}(\mathbb{H})$
$\mathbb{Q}[[q]]$
$\mathbb{G}_{k_1,\ldots,k_r}(\tau)$
$G_{k_1,\ldots,k_r}$
Multiple zeta values
$\mathbb{R}$
$\mathbb{Q}$
$\zeta(k_1,\ldots,k_r)$
$\beta(k_1,\ldots,k_r)$
$Z(k_1,\ldots,k_r)$
EDS
Modular forms
$\mathcal{O}(\mathbb{H})$
$\mathbb{Q}[[q]]$
$\mathbb{G}_k(\tau)$
G_k
Riemann zeta values
$\mathbb{R}$
$\mathbb{Q}$
$\zeta(k)$
$\beta(k)$

Plan of this talk

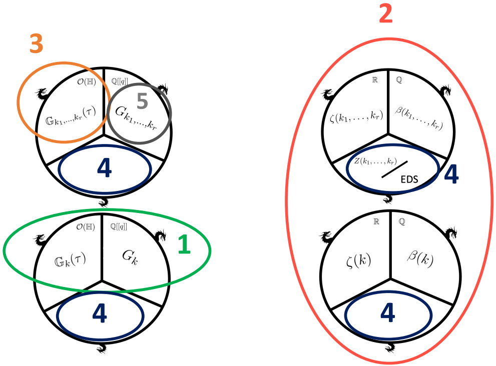
1 Modular forms

2 Multiple zeta values & Double shuffle relations

3 Multiple Eisenstein series

4 Formal multiple Eisenstein series & formal quasi modular forms & formal multiple zeta values

5 Realizations & Combinatorial multiple Eisenstein series



Complex upper half plane: $\mathbb{H} = \{x + iy \in \mathbb{C} \mid x, y \in \mathbb{R}\,, y > 0\}$.

Definition

A holomorphic function $f \in \mathcal{O}(\mathbb{H})$ is called a **modular form of weight** $k \in \mathbb{Z}$ if it satisfies

- $f(\tau + 1) = f(\tau)\,,$
- $f(-\frac{1}{\tau}) = \tau^k f(\tau)\,,$

for all $\tau \in \mathbb{H}$ and if it has a Fourier expansion of the form

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n\,. \qquad (a_n \in \mathbb{C}, q = e^{2\pi i\tau})$$

- $\mathcal{M}_k$: space of all modular forms of weight k.
- The space of **cusp forms** of weight k is defined by

$$\mathcal{S}_k = \{f \in \mathcal{M}_k \mid f = \sum_{\mathbf{n=1}}^{\infty} a_n q^n\} = \ker(\text{projection to const. term})\,.$$

For even $k \geq 4$ the **Eisenstein series** are defined by

$$\mathbb{G}_k(\tau) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m\tau + n)^k} .$$

These have a Fourier expansion of the form

$$\mathbb{G}_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n ,$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ is the divisor sum.

Proposition

For every even $k \geq 4$ we have $\mathbb{G}_k \in \mathcal{M}_k$, $\mathcal{M}_k = \mathbb{C}\mathbb{G}_k \oplus \mathcal{S}_k$ and

$$\mathcal{M} = \bigoplus_{k=0}^{\infty} \mathcal{M}_k = \mathbb{C}[\mathbb{G}_4, \mathbb{G}_6] .$$

Are derivatives of modular forms again modular forms?... No

Define the **Eisenstein series of weight two** by

$$\mathbb{G}_2(\tau) = \zeta(2) + (-2\pi i)^2 \sum_{n=1}^{\infty} \sigma_1(n) q^n \,,$$

and the space of **quasi-modular forms** by (see Kimura-sans talk)

$$\widetilde{\mathcal{M}} = \mathbb{C}[\mathbb{G}_2, \mathbb{G}_4, \mathbb{G}_6] \,.$$

The first non-trivial cusp form is the **discriminant function** Δ

$$\begin{aligned}\Delta(\tau) &= q\prod_{n=1}^{\infty}(1-q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \dots\,, \\ &= 2400\cdot 6!\cdot G_4(\tau)^3 - 420\cdot 7!\cdot G_6(\tau)^2\,,\end{aligned}$$

where

$$G_k(\tau) = (2\pi i)^{-k}\mathbb{G}_k(\tau) = -\frac{B_k}{2k!} + \frac{1}{(k-1)!}\sum_{n>0}\sigma_{k-1}(n)q^n\,.$$

Theorem

For $k \geq 0$ the map $\mathcal{M}_k \to \mathcal{S}_{k+12}$ given by $f \mapsto \Delta\cdot f$ is an isomorphism of $\mathbb{C}$-vector spaces.

Definition

For $k_1 \geq 2, k_2, \ldots, k_r \geq 1$ define the **multiple zeta value** (MZV)

$$\zeta(k_1, \ldots, k_r) = \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R} \,.$$

By r we denote its **depth** and $k_1 + \cdots + k_r$ will be called its **weight**.

- $\mathcal{Z}$: $\mathbb{Q}$-algebra of MZVs
- $\mathcal{Z}_k$: $\mathbb{Q}$-vector space of MZVs of weight k.

MZVs can also be written as **iterated integrals**, e.g.

$$\zeta(2,3) = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} \int_0^{t_2} \frac{dt_3}{t_3} \int_0^{t_3} \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_5}{1-t_5} \,.$$

There are two different ways to express the product of MZV in terms of MZV.

Harmonic product (coming from the definition as iterated sums)

Example in depth two ($k_1, k_2 \geq 2$)

$$\begin{aligned}\zeta(k_1)\cdot\zeta(k_2) &= \sum_{m>0}\frac{1}{m^{k_1}}\sum_{n>0}\frac{1}{n^{k_2}}\\ &= \sum_{m>n>0}\frac{1}{m^{k_1}n^{k_2}} + \sum_{n>m>0}\frac{1}{m^{k_1}n^{k_2}} + \sum_{m=n>0}\frac{1}{m^{k_1+k_2}}\\ &= \zeta(k_1,k_2)+\zeta(k_2,k_1)+\zeta(k_1+k_2)\,.\end{aligned}$$

Shuffle product (coming from the expression as iterated integrals)

Example in depth two ($k_1, k_2 \geq 2$)

$$\zeta(k_1)\cdot\zeta(k_2) = \sum_{j=2}^{k_1+k_2-1}\left(\binom{j-1}{k_1-1}+\binom{j-1}{k_2-1}\right)\zeta(j,k_1+k_2-j)\,.$$

These two product expressions give various $\mathbb{Q}$-linear relations between MZV.

Example

$$\begin{aligned}\zeta(2)\cdot\zeta(3) &\overset{\text{harmonic}}{=} \zeta(2,3)+\zeta(3,2)+\zeta(5)\\ &\overset{\text{shuffle}}{=} \zeta(2,3)+3\zeta(3,2)+6\zeta(4,1)\,.\end{aligned}$$

$$\Longrightarrow 2\zeta(3,2)+6\zeta(4,1) \overset{\text{double shuffle}}{=} \zeta(5)\,.$$

But there are more relations between MZV. e.g.:

$$\sum_{m>n>0}\frac{1}{m^2n} = \zeta(2,1) = \zeta(3) = \sum_{m>0}\frac{1}{m^3}.$$

These follow from regularizing the double shuffle relations

$\rightsquigarrow$**extended double shuffle relations**.

Conjecture

All relations among MZVs are consequences of the extended double shuffle relations.

Conjecture

The space $\mathcal{Z}$ is graded by weight, i.e.

$$\mathcal{Z} = \bigoplus_{k \geq 0} \mathcal{Z}_k \,.$$

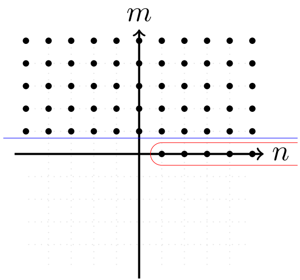
- There are various different families of relations which conjecturally give all relations among MZV.
- There are several "modular phenomena", e.g. Broadhurst-Kreimer conjecture (see bonus slides)

Let $\tau \in \mathbb{H}$. We define an order $\succ$ on the lattice $\mathbb{Z}\tau + \mathbb{Z}$ by setting

$$\lambda_1 \succ \lambda_2 :\Leftrightarrow \lambda_1 - \lambda_2 \in P$$

for $\lambda_1, \lambda_2 \in \mathbb{Z}\tau + \mathbb{Z}$ and the following set of positive lattice points

$$P := \{m\tau + n \in \mathbb{Z}\tau + \mathbb{Z} \mid m > 0 \vee (m = 0 \wedge n > 0)\} = U \cup R\,.$$



In other words: $\lambda_1 \succ \lambda_2$ **iff** λ_1 **is above or on the right of** λ_2.

Definition

For integers $k_1 \geq 3, k_2, \ldots, k_r \geq 2$, we define the **multiple Eisenstein series** by

$$\mathbb{G}_{k_1,\ldots,k_r}(\tau) = \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \ldots \lambda_r^{k_r}}.$$

It is easy to see that these are holomorphic functions in the upper half plane and that they fulfill the **harmonic product**, i.e. it is for example

$$\mathbb{G}_4(\tau) \cdot \mathbb{G}_3(\tau) = \mathbb{G}_{4,3}(\tau) + \mathbb{G}_{3,4}(\tau) + \mathbb{G}_7(\tau).$$

Definition

For $k_1, \dots k_r \geq 1$ we define the q-series $g(k_1, \dots, k_r) \in \mathbb{Q}[[q]]$ by

$$g(k_1, \dots, k_r) = \sum_{\substack{m_1 > \cdots > m_r > 0 \\ n_1, \dots, n_r > 0}} \frac{n_1^{k_1 - 1}}{(k_1 - 1)!} \cdots \frac{n_r^{k_r - 1}}{(k_r - 1)!} q^{m_1 n_1 + \cdots + m_r n_r} .$$

Theorem (Gangl-Kaneko-Zagier 2006 ($r = 2$), B. 2012 ($r \geq 2$))

The multiple Eisenstein series $\mathbb{G}_{k_1, \dots, k_r}(\tau)$ have a Fourier expansion of the form

$$\mathbb{G}_{k_1, \dots, k_r}(\tau) = \zeta(k_1, \dots, k_r) + \sum_{n > 0} a_n q^n \qquad \left(q = e^{2\pi i \tau}\right)$$

and they can be written explicitly as a $\mathcal{Z}[2\pi i]$-linear combination of q-analogues of multiple zeta values g. In particular, $a_n \in \mathcal{Z}[2\pi i]$.

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Examples

$$\mathbb{G}_k(\tau) = \zeta(k) + (-2\pi i)^k g(k)\,,$$

$$\mathbb{G}_{3,2}(q) = \zeta(3,2) + 3\zeta(3)(-2\pi i)^2 g(2) + 2\zeta(2)(-2\pi i)^3 g(3) + (-2\pi i)^5 g(3,2)\,.$$

③ Multiple Eisenstein series - Extended definitions

There are different ways to extend the definition of $\mathbb{G}_{k_1,\dots,k_r}$ to $k_1, \dots, k_r \geq 1$

- Formal double zeta space realization $\mathbb{G}_{r,s}$ (Gangl-Kaneko-Zagier, 2006)

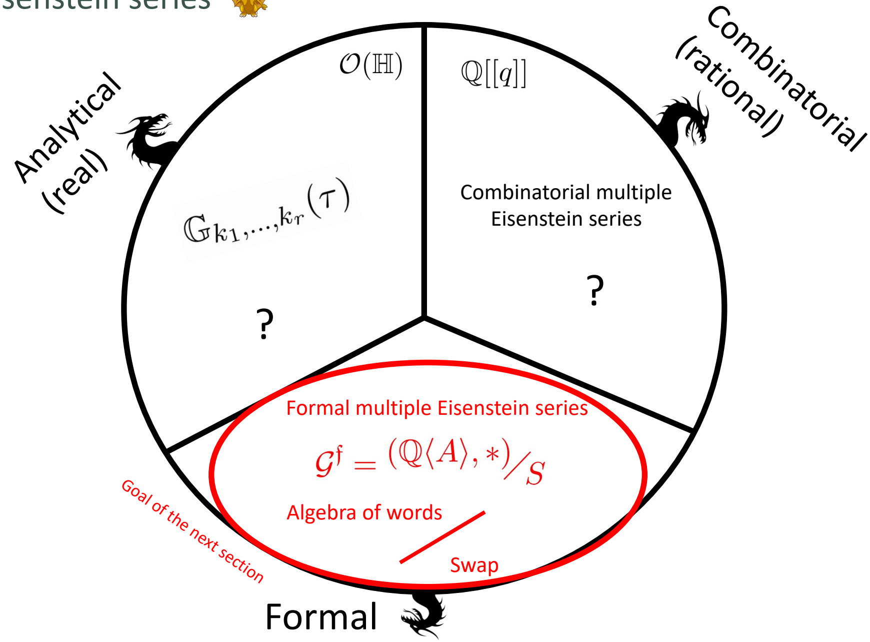
$$\mathbb{G}_{k_1} \cdot \mathbb{G}_{k_2} + (\delta_{k_1,2} + \delta_{k_2,2}) \frac{\mathbb{G}'_{k_1+k_2-2}}{2(k_1+k_2-2)} = \mathbb{G}_{k_1,k_2} + \mathbb{G}_{k_2,k_1} + \mathbb{G}_{k_1+k_2}$$
$$= \sum_{j=2}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) \mathbb{G}_{j,k_1+k_2-j}\,, \qquad (k_1+k_2 \geq 3)\,.$$

- Finite double shuffle version $\mathbb{G}_{r,s}$ (Kaneko, 2007).
- Shuffle regularized multiple Eisenstein series $\mathbb{G}^{ш}_{k_1,\dots,k_r}$ (B.-Tasaka, 2017).
- Harmonic regularized multiple Eisenstein series $\mathbb{G}^{*}_{k_1,\dots,k_r}$ (B., 2019).

Observation

- No version of these objects satisfy the double shuffle relations for all indices/weights.
- The derivative is always somewhere as an extra term.

Multiple Eisenstein series



Define the alphabet A by

$$A = \left\{ \begin{bmatrix} k \\ d \end{bmatrix} \mid k \geq 1,\, d \geq 0 \right\} .$$

On $\mathbb{Q}A$ we define the product $\diamond$ for $k_1, k_2 \geq 1$ and $d_1, d_2 \geq 0$ by

$$\begin{bmatrix} k_1 \\ d_1 \end{bmatrix} \diamond \begin{bmatrix} k_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ d_1 + d_2 \end{bmatrix} .$$

This gives a commutative non-unital $\mathbb{Q}$-algebra $(\mathbb{Q}A, \diamond)$.

Definition

Define the **quasi-shuffle product** $*$ on $\mathbb{Q}\langle A\rangle$ as the $\mathbb{Q}$-bilinear product, which satisfies $1 * w = w * 1 = w$ for any word $w \in \mathbb{Q}\langle A\rangle$ and

$$aw * bv = a(w * bv) + b(aw * v) + (a \diamond b)(w * v)$$

for any letters $a, b \in A$ and words $w, v \in \mathbb{Q}\langle A\rangle$.

Proposition

$(\mathbb{Q}\langle A\rangle, *)$ is a commutative $\mathbb{Q}$-algebra.

- For $k_1, \dots, k_r \geq 1, d_1, \dots, d_r \geq 0$ we use the following notation to write words in $\mathbb{Q}\langle A \rangle$:

$$\begin{bmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{bmatrix} := \begin{bmatrix} k_1 \\ d_1 \end{bmatrix} \dots \begin{bmatrix} k_r \\ d_r \end{bmatrix}.$$

- **weight:** $k_1 + \cdots + k_r$
- **depths:** r

In smallest depths the quasi-shuffle product is given by

$$\begin{bmatrix} k_1 \\ d_1 \end{bmatrix} * \begin{bmatrix} k_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} k_1, k_2 \\ d_1, d_2 \end{bmatrix} + \begin{bmatrix} k_2, k_1 \\ d_2, d_1 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 \\ d_1 + d_2 \end{bmatrix},$$

$$\begin{bmatrix} k_1 \\ d_1 \end{bmatrix} * \begin{bmatrix} k_2, k_3 \\ d_2, d_3 \end{bmatrix} = \begin{bmatrix} k_1, k_2, k_3 \\ d_1, d_2, d_3 \end{bmatrix} + \begin{bmatrix} k_2, k_1, k_3 \\ d_2, d_1, d_3 \end{bmatrix} + \begin{bmatrix} k_2, k_3, k_1 \\ d_2, d_3, d_1 \end{bmatrix} + \begin{bmatrix} k_1 + k_2, k_3 \\ d_1 + d_2, d_3 \end{bmatrix} + \begin{bmatrix} k_1, k_2 + k_3 \\ d_1, d_2 + d_3 \end{bmatrix}.$$

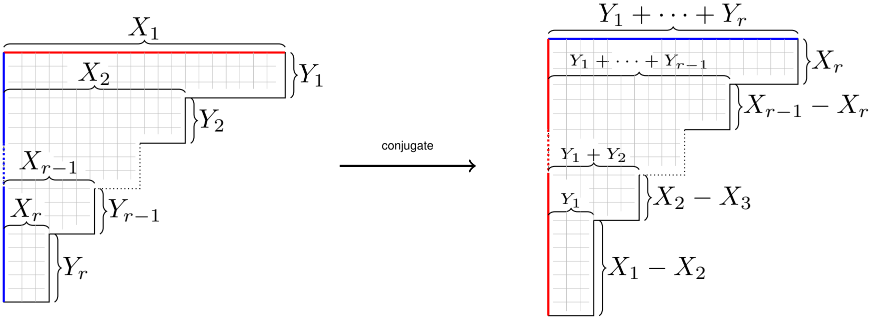
We define in depth $r \geq 1$ by the following formal power series in $\mathbb{Q}\langle A\rangle[[X_1, Y_1, \dots, X_r, Y_r]]$

$$\mathfrak{A}\binom{X_1, \dots, X_r}{Y_1, \dots, Y_r} := \sum_{\substack{k_1,\dots,k_r \geq 1 \\ d_1,\dots,d_r \geq 0}} \begin{bmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{bmatrix} X_1^{k_1-1} \dots X_r^{k_r-1} \frac{Y_1^{d_1}}{d_1!} \dots \frac{Y_r^{d_r}}{d_r!}.$$

With this the quasi-shuffle product in smallest depths reads

$$\mathfrak{A}\binom{X_1}{Y_1} * \mathfrak{A}\binom{X_2}{Y_2} = \mathfrak{A}\binom{X_1, X_2}{Y_1, Y_2} + \mathfrak{A}\binom{X_2, X_1}{Y_2, Y_1} + \frac{\mathfrak{A}\binom{X_1}{Y_1+Y_2} - \mathfrak{A}\binom{X_2}{Y_1+Y_2}}{X_1 - X_2}.$$

The conjugation of a Young diagram with $X_1Y_1 + \cdots + X_rY_r$ boxes and r stairs:



$$\begin{pmatrix} X_1, \ldots, X_r \\ Y_1, \ldots, Y_r \end{pmatrix} \longmapsto \begin{pmatrix} Y_1 + \cdots + Y_r, \ldots, Y_1 + Y_2, Y_1 \\ X_r, X_{r-1} - X_r, \ldots, X_1 - X_2 \end{pmatrix}$$

Definition

We define the **swap** as the linear map $\sigma : \mathbb{Q}\langle A\rangle \to \mathbb{Q}\langle A\rangle$ by $\sigma(1) = 1$ and for $r \geq 1$ on the generators of $\mathbb{Q}\langle A\rangle$ by

$$\sigma\left(\mathfrak{A}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r}\right) := \mathfrak{A}\binom{Y_1+\dots+Y_r,\dots,Y_1+Y_2,Y_1}{X_r,X_{r-1}-X_r,\dots,X_1-X_2},$$

where σ is applied coefficient-wise on the left, i.e. $\sigma\left(\left[\begin{smallmatrix}k_1,\dots,k_r\\d_1,\dots,d_r\end{smallmatrix}\right]\right)$ is defined as the coefficient of $X_1^{k_1-1}\dots X_r^{k_r-1}\frac{Y_1^{d_1}}{d_1!}\dots\frac{Y_r^{d_r}}{d_r!}$ on the right-hand side.

$$\sigma\left(\begin{bmatrix}k\\d\end{bmatrix}\right) = \frac{d!}{(k-1)!}\begin{bmatrix}d+1\\k-1\end{bmatrix}, \qquad (k \geq 1, d \geq 0).$$

Define S as the ideal in $(\mathbb{Q}\langle A\rangle, *)$ generated by all $\sigma(w) - w$ for $w \in \mathbb{Q}\langle A\rangle$, i.e.

$$S = \langle \sigma(w) - w \mid w \in \mathbb{Q}\langle A\rangle\rangle_{\mathbb{Q}} * \mathbb{Q}\langle A\rangle\,.$$

Definition

The algebra of **formal multiple Eisenstein series** is defined by

$$\mathcal{G}^{\mathfrak{f}} = {}^{\mathbb{Q}\langle A\rangle}\!\big/_{S}$$

and we denote the class of a word $\left[\begin{smallmatrix} k_1,\ldots,k_r \\ d_1,\ldots,d_r \end{smallmatrix}\right]$ by $\mathrm{G}\left(\begin{smallmatrix} k_1,\ldots,k_r \\ d_1,\ldots,d_r \end{smallmatrix}\right)$.

We obtain a commutative $\mathbb{Q}$-algebra $(\mathcal{G}^{\mathfrak{f}}, *)$, where each element is swap invariant. We write

$$\mathfrak{G}\binom{X_1, \ldots, X_r}{Y_1, \ldots, Y_r} := \sum_{\substack{k_1, \ldots, k_r \geq 1 \\ d_1, \ldots, d_r \geq 0}} \mathrm{G}\binom{k_1, \ldots, k_r}{d_1, \ldots, d_r} X_1^{k_1 - 1} \ldots X_r^{k_r - 1} \frac{Y_1^{d_1}}{d_1!} \ldots \frac{Y_r^{d_r}}{d_r!} .$$

Since the formal multiple Eisenstein series are swap invariant and their product is given by $*$ we have in particular

$$\mathfrak{G}\binom{X_1}{Y_1} = \mathfrak{G}\binom{Y_1}{X_1} ,$$

$$\mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2} = \mathfrak{G}\binom{Y_1 + Y_1, Y_1}{X_2, X_1 - X_2} ,$$

$$\mathfrak{G}\binom{X_1}{Y_1} * \mathfrak{G}\binom{X_2}{Y_2} = \mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2} + \mathfrak{G}\binom{X_2, X_1}{Y_2, Y_1} + \frac{\mathfrak{G}\binom{X_1}{Y_1 + Y_2} - \mathfrak{G}\binom{X_2}{Y_1 + Y_2}}{X_1 - X_2} .$$

All relations we will present in this talk are consequences of the three relations above.

Let $\partial : (\mathbb{Q}A, \diamond) \to (\mathbb{Q}A, \diamond)$ be the derivation defined for $k \geq 1, d \geq 0$ by

$$\partial\left(\begin{bmatrix} k \\ d \end{bmatrix}\right) = k \begin{bmatrix} k+1 \\ d+1 \end{bmatrix}.$$

This gives a derivation on $\mathbb{Q}\langle A \rangle$ (with respect to the concatenation product), defined by

$$\partial\left(\begin{bmatrix} k_1, \ldots, k_r \\ d_1, \ldots, d_r \end{bmatrix}\right) = \sum_{j=1}^{r} k_j \begin{bmatrix} k_1, \ldots, k_j + 1, \ldots, k_r \\ d_1, \ldots, d_j + 1, \ldots, d_r \end{bmatrix}.$$

Lemma

- ∂ is a derivation on $(\mathbb{Q}\langle A\rangle, *)$.
- The derivation ∂ commutes with the swap, i.e. $\partial\sigma = \sigma\partial$.

Theorem

∂ is a derivation on $(\mathcal{G}^{\mathfrak{f}}, *)$.

$$\partial\left(\mathrm{G}\binom{k_1,\ldots,k_r}{d_1,\ldots,d_r}\right) = \sum_{j=1}^{r} k_j\,\mathrm{G}\binom{k_1,\ldots,k_j+1,\ldots,k_r}{d_1,\ldots,d_j+1,\ldots,d_r}.$$

Conjecture

There exist a unique derivation $\mathfrak{d}$ on $(\mathbb{Q}\langle A\rangle, *)$ such that

- $\mathfrak{d}$ commutes with σ.
- The triple $(\partial, W, \mathfrak{d})$ satisfies the commutation relations of an $\mathfrak{sl}_2$-triple, i.e.

$$[W, \partial] = 2\partial, \quad [W, \mathfrak{d}] = -2\mathfrak{d}, \qquad [\mathfrak{d}, \partial] = W\,,$$

  where W is the weight operator, multiplying a word $\left[\begin{smallmatrix} k_1,\dots,k_r \\ d_1,\dots,d_r \end{smallmatrix}\right]$ by its weight $k_1 + \ldots + k_r + d_1 + \ldots + d_r$.

This would imply an $\mathfrak{sl}_2$-action on $\mathcal{G}^{\mathfrak{f}}$. In depth one this derivation seems to be given by

$$\mathfrak{d}\, \mathrm{G}\binom{k}{d} = d\, \mathrm{G}\binom{k-1}{d-1} - \frac{1}{2}\delta_{k+d,2}\,,$$

which correspond to the classical derivation for quasi-modular forms (the derivative with respect to G_2).

On $\mathbb{Q}\langle A\rangle$ we can define another product $ш$ by $w \,ш\, v = \sigma(\sigma(w) * \sigma(v))$ for $w, v \in \mathbb{Q}\langle A\rangle$. For any $f, g \in \mathcal{G}^{\mathfrak{f}}$ we have $f \,ш\, g - f * g = 0$.

Proposition

For $k_1, k_2 \geq 1, d_1, d_2 \geq 0$ we have

$$\begin{aligned}
\mathrm{G}\binom{k_1}{d_1}\mathrm{G}\binom{k_2}{d_2} &= \mathrm{G}\binom{k_1, k_2}{d_1, d_2} + \mathrm{G}\binom{k_2, k_1}{d_2, d_1} + \mathrm{G}\binom{k_1 + k_2}{d_1 + d_2} \\
&= \sum_{\substack{l_1 + l_2 = k_1 + k_2 \\ e_1 + e_2 = d_1 + d_2}} \left(\binom{l_1 - 1}{k_1 - 1}\binom{d_1}{e_1}(-1)^{d_1 - e_1} + \binom{l_1 - 1}{k_2 - 1}\binom{d_2}{e_1}(-1)^{d_2 - e_1} \right) \mathrm{G}\binom{l_1, l_2}{e_1, e_2} \\
&\quad + \frac{d_1! d_2!}{(d_1 + d_2 + 1)!}\binom{k_1 + k_2 - 2}{k_1 - 1}\mathrm{G}\binom{k_1 + k_2 - 1}{d_1 + d_2 + 1},
\end{aligned}$$

where we sum over all $l_1, l_2 \geq 1$ and $e_1, e_2 \geq 0$ in the second expression

The special case $d_1 = d_2 = 0$ is similar to the double shuffle relations of MZV.

Most of the relations we will obtain are among G, where the bottom entries are zero. For shorter notation we will denote these for $k_1, \dots, k_r \geq 1$ by

$$\mathrm{G}(k_1, \dots, k_r) := \mathrm{G}\binom{k_1, \dots, k_r}{0, \dots, 0}.$$

Instead of $*$ we will just write products of G (i.e. this will not denote the concatenation of words)

Example

$$\begin{aligned} \mathrm{G}(2)\,\mathrm{G}(3) &= \mathrm{G}(2,3) + \mathrm{G}(3,2) + \mathrm{G}(5) \\ &= \mathrm{G}(2,3) + 3\,\mathrm{G}(3,2) + 6\,\mathrm{G}(4,1) + 3\,\mathrm{G}\binom{4}{1}. \end{aligned}$$

Compare this to the previous example of multiple zeta values. Also notice: $3\,\mathrm{G}\binom{4}{1} = \partial\,\mathrm{G}(3)$.

Theorem (B.-van Ittersum 2021+)

For all $k_1, k_2 \geq 1$ with $k = k_1 + k_2 \geq 4$ even we have

$$\frac{1}{2}\left(\binom{k_1+k_2}{k_2} - (-1)^{k_1}\right)\mathrm{G}(k) = \sum_{\substack{j=2 \\ j \text{ even}}}^{k-2}\left(\binom{k-j-1}{k_1-1} + \binom{k-j-1}{k_2-1} - \delta_{j,k_1}\right)\mathrm{G}(j)\,\mathrm{G}(k-j)$$
$$+ \frac{1}{2}\left(\binom{k-3}{k_1-1} + \binom{k-3}{k_2-1} + \delta_{k_1,1} + \delta_{k_2,1}\right)\mathrm{G}\binom{k-1}{1}.$$

Proof sketch:

- Define an action of the group ring $\mathbb{Z}[\mathrm{Gl}_2(\mathbb{Z})]$ on the generating series in depth two.
- Above equality follows by describing the double shuffle relations in terms of this action together with some identities in $\mathbb{Z}[\mathrm{Gl}_2(\mathbb{Z})]$.

(See bonus slides for details)

Corollary

- For even $k \geq 4$ we have

$$\frac{k+1}{2}\,\mathrm{G}(k) = \mathrm{G}\binom{k-1}{1} + \sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 2 \text{ even}}} \mathrm{G}(k_1)\,\mathrm{G}(k_2)\,.$$

- For all even $k \geq 6$ we have

$$\frac{(k+1)(k-1)(k-6)}{12}\,\mathrm{G}(k) = \sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 4 \text{ even}}} (k_1-1)(k_2-1)\;\mathrm{G}(k_1)\,\mathrm{G}(k_2)\,.$$

Example

$$\mathrm{G}(8) = \frac{6}{7}\,\mathrm{G}(4)^2, \quad \mathrm{G}(10) = \frac{10}{11}\,\mathrm{G}(4)\,\mathrm{G}(6), \quad \mathrm{G}(12) = \frac{84}{143}\,\mathrm{G}(4)\,\mathrm{G}(8) + \frac{50}{143}\,\mathrm{G}(6)^2\,.$$

Notice that for $k \geq 3$ we have $\frac{1}{k-2}\,\mathrm{G}\binom{k-1}{1} = \partial\,\mathrm{G}(k-2) = \mathrm{G}'(k-2)$.

Corollary

- For $m \geq 1$ we have $\mathrm{G}(2m) \in \mathbb{Q}[\mathrm{G}(2), \mathrm{G}'(2), \mathrm{G}''(2)] = \mathbb{Q}[\mathrm{G}(2), \mathrm{G}(4), \mathrm{G}(6)]$ and

$$\mathrm{G}(2m) = -\frac{B_{2m}}{2(2m)!}(-24\,\mathrm{G}(2))^m + \text{terms with } \mathrm{G}'(2) \text{ and } \mathrm{G}''(2)\,.$$

- For $m \geq 2$ we have $\mathrm{G}(2m) \in \mathbb{Q}[\mathrm{G}(4), \mathrm{G}(6)]$.

Compare the first part with the formula by Euler for Riemann zeta values: $\zeta(2m) = -\frac{B_{2m}}{2(2m)!}(-24\zeta(2))^m$.

Example

$$\mathrm{G}(4) = \frac{2}{5}\,\mathrm{G}(2)^2 + \frac{1}{5}\,\mathrm{G}'(2)\,,$$
$$\mathrm{G}(6) = \frac{8}{35}\,\mathrm{G}(2)^3 + \frac{6}{35}\,\mathrm{G}(2)\,\mathrm{G}'(2) + \frac{1}{70}\,\mathrm{G}''(2)\,.$$

$$\widehat{\mathcal{G}}^{\mathfrak{f}} = \mathbb{Q} + \langle \mathrm{G}(k_1,\dots,k_r) \mid r \geq 1, k_1,\dots,k_r \geq 1 \rangle_{\mathbb{Q}} \subset \mathcal{G}^{\mathfrak{f}}\,.$$

By the definition of the quasi-shuffle product, it is easy to see that $(\widehat{\mathcal{G}}^{\mathfrak{f}}, *)$ is a subalgebra of $(\mathcal{G}^{\mathfrak{f}}, *)$.
Applying ∂ to the generators of $\widehat{\mathcal{G}}^{\mathfrak{f}}$ gives

$$\partial\left(\mathrm{G}(k_1,\dots,k_r)\right) = \sum_{j=1}^{r} k_j \, \mathrm{G}\binom{k_1,\dots,k_j+1,\dots,k_r}{0,\dots,1,\dots,0}\,.$$

Proposition (B.-van Ittersum 2021+)

$\widehat{\mathcal{G}}^{\mathfrak{f}}$ is closed under ∂.

Conjecture

We have $\widehat{\mathcal{G}}^{\mathfrak{f}} = \mathcal{G}^{\mathfrak{f}}$.

Question

What are the "constant terms" of formal multiple Eisenstein series?

- To define formal cusp forms, we want to determine the projection onto the constant term of formal multiple Eisenstein series.
- This leads to the question of which relations are additionally satisfied for MZV compared to MES.
- This will give a definition of formal multiple zeta values.
- The following construction is motivated/inspired by a conjectural construction of combinatorial multiple Eisenstein series together with their behavior as $q \to 1$.

We define the following two subsets of the alphabet A

$$A_0 = \left\{ \begin{bmatrix} k \\ 0 \end{bmatrix} \mid k \geq 1 \right\}, \qquad A^1 = \left\{ \begin{bmatrix} 1 \\ d \end{bmatrix} \mid d \geq 0 \right\}.$$

With this we define the following ideal in $(\mathbb{Q}\langle A \rangle, *)$ generated by the set $A^* \backslash (A^1)^* (A_0)^*$

$$N = \left(A^* \backslash (A^1)^* (A_0)^* \right)_{\mathbb{Q}\langle A \rangle},$$

The elements in $A^* \backslash (A^1)^* (A_0)^*$ are exactly those elements which are <u>not</u> of the form

$$\begin{bmatrix} 1, \dots, 1, k_1, \dots, k_r \\ d_1, \dots, d_s, 0, \dots, 0 \end{bmatrix}.$$

In addition to the ideal N, we define the following ideal:

$$P = \left\langle \begin{bmatrix} 1, \dots, 1, k_1, \dots, k_r \\ d_1, \dots, d_s, 0, \dots, 0 \end{bmatrix} - \begin{bmatrix} 1, \dots, 1 \\ d_1, \dots, d_s \end{bmatrix} * \begin{bmatrix} k_1, \dots, k_r \\ 0, \dots, 0 \end{bmatrix} \middle| d_s \geq 1, k_1 \geq 2 \right\rangle_{\mathbb{Q}} * \mathbb{Q}\langle A \rangle.$$

Definition

The algebra of **formal multiple zeta values** is defined by

$$\mathcal{Z}^{\mathfrak{f}} = {\mathcal{G}^{\mathfrak{f}}}\big/{(N+P)}\,.$$

We denote the canonical projection by

$$\pi : \mathcal{G}^{\mathfrak{f}} \longrightarrow \mathcal{Z}^{\mathfrak{f}}\,.$$

This map can be seen as the formal version of the "projection onto the constant term".

Claim: The ideals N and P capture the additional relations satisfied by multiple zeta values, which are not satisfied by multiple Eisenstein series.

Proposition (B.-van Ittersum 2021+)

The map $\pi_{|\widehat{\mathcal{G}}^{\mathfrak{f}}} : \widehat{\mathcal{G}}^{\mathfrak{f}} \to \mathcal{Z}^{\mathfrak{f}}$ is surjective.

Definition

For $k_1, \ldots, k_r \geq 1$ we define the **formal multiple zeta value** $\zeta^{\mathfrak{f}}(k_1, \ldots, k_r)$ by

$$\zeta^{\mathfrak{f}}(k_1, \ldots, k_r) = \pi(\mathrm{G}(k_1, \ldots, k_r)) \,.$$

Proposition

We have $\partial \mathcal{G}^{\mathfrak{f}} \subset \ker(\pi)$.

Corollary

- *(Double shuffle relations in depth two)* For $k_1, k_2 \geq 1$ we have

$$\zeta^{\mathfrak{f}}(k_1)\zeta^{\mathfrak{f}}(k_2) = \zeta^{\mathfrak{f}}(k_1,k_2) + \zeta^{\mathfrak{f}}(k_2,k_1) + \zeta^{\mathfrak{f}}(k_1+k_2)$$
$$= \sum_{l_1+l_2=k_1+k_2} \left(\binom{l_1-1}{k_1-1} + \binom{l_1-1}{k_2-1} \right) \zeta^{\mathfrak{f}}(l_1,l_2) + \delta_{k_1+k_2,2}\zeta^{\mathfrak{f}}(2)\,.$$

  In particular we obtain the relation $\zeta^{\mathfrak{f}}(3) = \zeta^{\mathfrak{f}}(2,1)$ by taking $k_1 = 1, k_2 = 2$.

- *(Euler relation)* For $m \geq 1$ we have

$$\zeta^{\mathfrak{f}}(2m) = -\frac{B_{2m}}{2(2m)!}\left(-24\zeta^{\mathfrak{f}}(2)\right)^m\,.$$

Theorem (B.-Kühn-Matthes 2021+)

The formal multiple zeta values satisfy the extended double shuffle relations.

- Our formal multiple zeta values should be isomorphic (after switching to the shuffle regularization) to the classical definition of formal multiple zeta values (Racinet).
- There is a 1:1 correspondence between objects satisfying the extended double shuffle relations and the objects satisfying the relations in $\mathcal{Z}^{\mathfrak{f}}$.

In contrast to the analytic case, we start by defining formal quasi-modular forms before formal modular forms.

Definition

We define the algebra of **formal quasi-modular forms** $\widetilde{\mathcal{M}}^{\mathfrak{f}}$ as the smallest subalgebra of $\mathcal{G}^{\mathfrak{f}}$ which satisfies the following two conditions

- $\mathrm{G}(2) \in \widetilde{\mathcal{M}}^{\mathfrak{f}}$.
- $\widetilde{\mathcal{M}}^{\mathfrak{f}}$ is closed under ∂.

④ Formal (quasi) modular forms - Basic facts

Proposition (Seen for the classical case in Kawasetsu-sans talk)

- We have $\widetilde{\mathcal{M}}^{\mathfrak{f}} = \mathbb{Q}[\mathrm{G}(2), \mathrm{G}(4), \mathrm{G}(6)] = \mathbb{Q}[\mathrm{G}(2), \mathrm{G}'(2), \mathrm{G}''(2)]$.
- The Ramanujan differential equations are satisfied:

$$\begin{aligned} \mathrm{G}'(2) &= 5\,\mathrm{G}(4) - 2\,\mathrm{G}(2)^2\,, \\ \mathrm{G}'(4) &= 8\,\mathrm{G}(6) - 14\,\mathrm{G}(2)\,\mathrm{G}(4)\,, \\ \mathrm{G}'(6) &= \frac{120}{7}\,\mathrm{G}(4)^2 - 12\,\mathrm{G}(2)\,\mathrm{G}(6)\,. \end{aligned}$$

- The Chazy equation is satisfied

$$\mathrm{G}'''(2) + 24\,\mathrm{G}(2)\,\mathrm{G}''(2) - 36\,\mathrm{G}'(2)^2 = 0\,.$$

$$\frac{k+1}{2}\,\mathrm{G}(k) = \mathrm{G}\binom{k-1}{1} + \sum_{\substack{k_1+k_2=k \\ k_1,k_2 \geq 2 \text{ even}}} \mathrm{G}(k_1)\,\mathrm{G}(k_2)\,.$$

Definition

- The algebra of **formal modular forms** $\mathcal{M}^{\mathfrak{f}}$ is defined as the subalgebra of $\mathcal{G}^{\mathfrak{f}}$ generated by all $\mathrm{G}(k)$ with $k \geq 4$ even. (Alternative definition: $\mathcal{M}^{\mathfrak{f}} = \ker \mathfrak{d}_{|\widetilde{\mathcal{M}}^{\mathfrak{f}}}$)
- We define the algebra of **formal cusp forms** by $\mathcal{S}^{\mathfrak{f}} = \ker \pi_{|\mathcal{M}^{\mathfrak{f}}}$.

The first example of a non-zero formal cusp form appears in weight 12 and we write

$$\Delta^{\mathfrak{f}} = 2400 \cdot 6! \cdot \mathrm{G}(4)^3 - 420 \cdot 7! \cdot \mathrm{G}(6)^2 \,.$$

Proposition

- We have $\mathcal{M}^{\mathfrak{f}} = \mathbb{Q}[\mathrm{G}(4), \mathrm{G}(6)]$ and $\mathcal{M}^{\mathfrak{f}}_k = \mathbb{Q}\,\mathrm{G}(k) \oplus \mathcal{S}^{\mathfrak{f}}_k$.
- We have $\Delta^{\mathfrak{f}} \in \mathcal{S}^{\mathfrak{f}}_{12}$ and $\partial \Delta^{\mathfrak{f}} = -24\,\mathrm{G}(2)\Delta^{\mathfrak{f}}$.

$$\frac{1}{432}\Delta^{\mathfrak{f}} = 48\,\mathrm{G}(2)^2\,\mathrm{G}'(2)^2 + 32\,\mathrm{G}'(2)^3 - 32\,\mathrm{G}(2)^3\,\mathrm{G}''(2) - 24\,\mathrm{G}(2)\,\mathrm{G}'(2)\,\mathrm{G}''(2) - \mathrm{G}''(2)^2 \,.$$

Besides the mentioned basic facts we are also working on the following:

- Connection to the formal double zeta space of Gangl, Kaneko & Zagier. (see bonus slides)
- Rankin-Cohen brackets (see Kimura-sans talk) as a consequence of the $\mathfrak{sl}_2$-action on $\widetilde{\mathcal{M}}^{\mathfrak{f}}$.
- A formal version of "vanishing order at $i\infty$" by considering the kernels of

$$\pi_a : \mathcal{G}^{\mathfrak{f}} \longrightarrow \mathcal{G}^{\mathfrak{f}} \big/ (N+P)^a\,, \qquad (a \geq 1)$$

- Miller basis, Dimension formulas.

Not clear: How to formalize other important structures, such as Hecke operators ?

Definition

Let A be a (differential) $\mathbb{Q}$-algebra. A **realization of $\mathcal{G}^{\mathfrak{f}}$ in A** is an (differential) algebra homomorphism

$$\varphi : \mathcal{G}^{\mathfrak{f}} \longrightarrow A\,.$$

- $A = \mathbb{R}$: Multiple zeta values (derivation = zero map).
- $A = \mathbb{Q}$: Rational solution to extended double shuffle.
- $A = \mathbb{Q}[[q]]$: Combinatorial multiple Eisenstein series (derivation = $q\frac{d}{dq}$).
- $A = \mathcal{O}(\mathbb{H})$: ("Analytical") multiple Eisenstein series (derivation = $(2\pi i)\frac{d}{d\tau}$).

Theorem (B.-Kühn-Matthes 2021+)

For any field A of characteristic zero, there exist a realization of $\mathcal{G}^{\mathfrak{f}}$ in A, which factors through π.

- This follows from the fact that we know that for any field A of characteristic zero, there exists a solution to the extended double shuffle relations.
- For $A = \mathbb{R}$ these are given, for example, by (harmonic regularized) multiple zeta values.
- For $A = \mathbb{Q}$, there is no explicit construction known so far for depth ≥ 4.

Definition

For $k_1, \dots k_r \geq 1, d_1, \dots, d_r \geq 0$ define the q-series

$$g\binom{k_1, \dots, k_r}{d_1, \dots, d_r} = \sum_{\substack{m_1 > \dots > m_r > 0 \\ n_1, \dots, n_r > 0}} \frac{m_1^{d_1} n_1^{k_1 - 1}}{(k_1 - 1)!} \cdots \frac{m_r^{d_r} n_r^{k_r - 1}}{(k_r - 1)!} q^{m_1 n_1 + \dots + m_r n_r} .$$

Theorem (B.-van Ittersum 2021+)

The following gives a realization of $\mathcal{G}^{\mathfrak{f}}$ in $\mathbb{R}$

$$\varphi : \mathrm{G}\binom{k_1, \dots, k_r}{d_1, \dots, d_r} \longmapsto \lim_{q \to 1}^{*} (1-q)^{k_1 + \dots + k_r + d_1 + \dots + d_r} g\binom{k_1, \dots, k_r}{d_1, \dots, d_r} ,$$

where $\lim^{*}_{q \to 1}$ is a "(harmonic) regularized limit". This realization factors through π and we have

$$\varphi(\mathrm{G}(k_1, \dots, k_r)) = \zeta^{*}(k_1, \dots, k_r) .$$

$$\mathfrak{G}\binom{X_1}{Y_1} = \mathfrak{G}\binom{Y_1}{X_1}, \qquad \mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2} = \mathfrak{G}\binom{Y_1 + Y_1, Y_1}{X_2, X_1 - X_2},$$

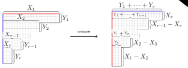
$$\mathfrak{G}\binom{X_1}{Y_1}\mathfrak{G}\binom{X_2}{Y_2} = \mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2} + \mathfrak{G}\binom{X_2, X_1}{Y_2, Y_1} + \frac{\mathfrak{G}\binom{X_1}{Y_1+Y_2} - \mathfrak{G}\binom{X_2}{Y_1+Y_2}}{X_1 - X_2}.$$

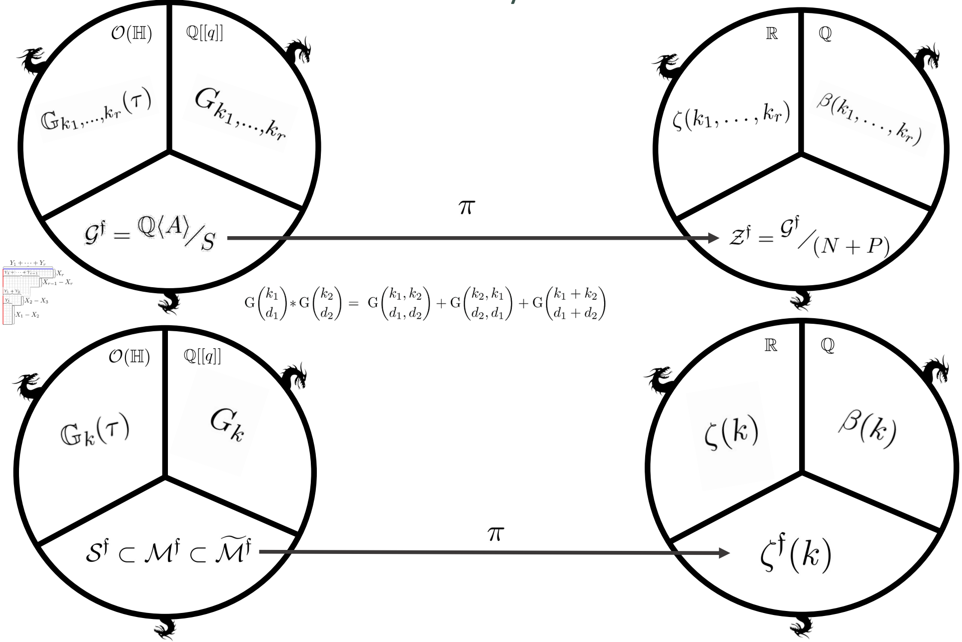
Theorem (B.-Kühn-Matthes 2021+, B.-Burmester 2021+)

There exist power series $\mathfrak{G}\binom{Y_1}{X_1}, \mathfrak{G}\binom{X_1,X_2}{Y_1,Y_2} \in \mathbb{Q}[[q]][[X_1, X_2, Y_1, Y_2]]$ which satisfy the above equations and where the coefficients of $\mathfrak{G}\binom{Y_1}{X_1}$ are given by (derivatives of) Eisenstein series. (See bonus slides)

- This gives combinatorial proofs of the classical identities for quasi-modular forms.
- There exists a construction for depth ≥ 3, which conjecturally gives a realization of $\mathcal{G}^{\mathfrak{f}}$. See the talkslides of Annika Burmesters talk "Combinatorial multiple Eisenstein series" at the JENTE Seminar (https://sites.google.com/view/jente-seminar/home).

Summary





$$\mathrm{G}\binom{k_1}{d_1} * \mathrm{G}\binom{k_2}{d_2} = \mathrm{G}\binom{k_1, k_2}{d_1, d_2} + \mathrm{G}\binom{k_2, k_1}{d_2, d_1} + \mathrm{G}\binom{k_1 + k_2}{d_1 + d_2}$$

$\operatorname{gr}_r^{\mathrm{D}} \mathcal{Z}_k$: MZV of weight k and depth r modulo lower depths MZV.

Conjecture (Broadhurst-Kreimer, 1997)

The generating series of the dimensions of the weight- and depth-graded parts of multiple zeta values is given by

$$\sum_{k,r\geq 0} \dim_{\mathbb{Q}} \left(\operatorname{gr}_r^{\mathrm{D}} \mathcal{Z}_k\right) X^k Y^r = \frac{1+\mathsf{E}(X)Y}{1-\mathsf{O}(X)Y+\mathsf{S}(X)Y^2-\mathsf{S}(X)Y^4},$$

where

$$\mathsf{E}(X) = \frac{X^2}{1-X^2}, \quad \mathsf{O}(X) = \frac{X^3}{1-X^2}, \quad \mathsf{S}(X) = \frac{X^{12}}{(1-X^4)(1-X^6)} = \sum_{k\geq 0} \dim \mathcal{S}_k X^k.$$

Observe that

$$\begin{aligned} &\frac{1+\mathsf{E}(X)Y}{1-\mathsf{O}(X)Y+\mathsf{S}(X)Y^2-\mathsf{S}(X)Y^4} \\ &= 1 + \left(\mathsf{E}(X)+\mathsf{O}(X)\right)Y + \Big(\big(\mathsf{E}(X)+\mathsf{O}(X)\big)\,\mathsf{O}(X) - \mathsf{S}(X)\Big)\,Y^2 + \cdots . \end{aligned}$$

In 2006 Gangl, Kaneko and Zagier introduced for $k \geq 1$ the **formal double zeta space** in weight k as

$$\mathcal{D}_k = \left\langle Z_k, Z_{k_1,k_2}, P_{k_1,k_2} \mid k_1 + k_2 = k, k_1, k_2 \geq 1 \right\rangle_{\mathbb{Q}} \Big/ \text{(1)}$$

where they divide out the following relations for $k_1, k_2 \geq 1$

$$\begin{aligned} P_{k_1,k_2} &= Z_{k_1,k_2} + Z_{k_2,k_1} + Z_{k_1+k_2} \\ &= \sum_{l_1+l_2=k_1+k_2} \left(\binom{l_1-1}{k_1-1} + \binom{l_1-1}{k_2-1} \right) Z_{l_1,l_2}\,. \end{aligned} \tag{1}$$

Proposition

For all $k \geq 1$ the following gives a $\mathbb{Q}$-linear map $\mathcal{D}_k \to \mathcal{G}^{\mathfrak{f}}$

$$\begin{aligned}
Z_k &\longmapsto \mathrm{G}(k) - \delta_{k,2}\,\mathrm{G}(2)\,, \\
Z_{k_1,k_2} &\longmapsto \mathrm{G}(k_1,k_2) + \frac{1}{2}\left(\delta_{k_2,1}\,\mathrm{G}\binom{k_1}{1} - \delta_{k_1,1}\,\mathrm{G}\binom{k_2}{1} + \delta_{k_1,2}\,\mathrm{G}\binom{k_2+1}{1}\right)\,, \\
P_{k_1,k_2} &\longmapsto \mathrm{G}(k_1)\,\mathrm{G}(k_2) + \frac{1}{2}\left(\delta_{k_1,2}\,\mathrm{G}\binom{k_2+1}{1} + \delta_{k_2,2}\,\mathrm{G}\binom{k_1+1}{1}\right)\,.
\end{aligned}$$

The double shuffle relations for formal multiple Eisenstein series in lowest depth are

$$\begin{aligned} P\binom{X_1, X_2}{Y_1, Y_2} &= \mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2} + \mathfrak{G}\binom{X_2, X_1}{Y_2, Y_1} + \frac{\mathfrak{G}\binom{X_1}{Y_1+Y_2} - \mathfrak{G}\binom{X_2}{Y_1+Y_2}}{X_1 - X_2} \\ &= \mathfrak{G}\binom{X_1+X_2, X_2}{Y_1, Y_2-Y_1} + \mathfrak{G}\binom{X_1+X_2, X_1}{Y_2, Y_1-Y_2} + \frac{\mathfrak{G}\binom{X_1+X_2}{Y_1} - \mathfrak{G}\binom{X_1+X_2}{Y_2}}{Y_1 - Y_2} \end{aligned} \tag{2}$$

with $P\binom{X_1,X_2}{Y_1,Y_2} = \mathfrak{G}\binom{X_1}{Y_1}\mathfrak{G}\binom{X_2}{Y_2}$. Define the action of the group ring $\mathbb{Z}[\mathrm{Gl}_2(\mathbb{Z})]$ on the formal Laurent series $\mathcal{L} = \mathbb{Q}\langle A\rangle((X_1, X_2, Y_1, Y_2))$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Gl}_2(\mathbb{Z})$ and $R \in \mathcal{L}$ by

$$R_{|\gamma}\binom{X_1, X_2}{Y_1, Y_2} = R\binom{aX_1 + bX_2, cX_1 + dX_2}{\det(\gamma)(dY_1 - cY_2), \det(\gamma)(-bY_1 + aY_2)}$$

and then extend it linearly to all elements in $\mathbb{Z}[\mathrm{Gl}_2(\mathbb{Z})]$.

⑥ Bonus - Action of $\mathrm{Gl}_2(\mathbb{Z})$ - 2

Now define the following elements in $\mathrm{Gl}_2(\mathbb{Z})$

$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \delta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

The equation (2) then becomes $P = \mathfrak{G} \mid (1+\epsilon) + R^* = \mathfrak{G} \mid T(1+\epsilon) + R^{\sqcup\!\sqcup}$ with

$$R^*\begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix} = \frac{\mathfrak{G}\binom{X_1}{Y_1+Y_2} - \mathfrak{G}\binom{X_2}{Y_1+Y_2}}{X_1 - X_2}, \quad R^{\sqcup\!\sqcup}\begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix} = \frac{\mathfrak{G}\binom{X_1+X_2}{Y_1} - \mathfrak{G}\binom{X_1+X_2}{Y_2}}{Y_1 - Y_2}.$$

Lemma

For $A = \epsilon U \epsilon$ we have

$$\mathfrak{G} \mid (1-\sigma) = P \mid (1-\delta)(1 + A - SA^2) - (R^* - R^{\sqcup\!\sqcup} \mid (T^{-1}\epsilon)) \mid (1 + A + A^2).$$

Considering the coefficients in above Lemma gives the Theorem on products of G.

⑥ Bonus - Combinatorial MES explicit

Theorem (B.-Kühn-Matthes 2020+, B.-Burmester 2020+)

The following series are swap invariant and their coefficients satisfy the quasi-shuffle product

$$\mathfrak{G}\binom{X_1}{Y_1} = \beta\binom{X_1}{Y_1} + \mathfrak{g}\binom{X_1}{Y_1},$$

$$\begin{aligned}\mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2} = \beta\binom{X_1, X_2}{Y_1, Y_2} &- \beta\binom{X_1 - X_2}{Y_2}\mathfrak{g}\binom{X_1}{Y_1 + Y_2} - \frac{1}{2}\mathfrak{g}\binom{X_1}{Y_1 + Y_2} \\ &+ \beta\binom{X_2}{Y_2}\mathfrak{g}\binom{X_1}{Y_1} + \beta\binom{X_1 - X_2}{Y_1}\mathfrak{g}\binom{X_2}{Y_1 + Y_2} + \mathfrak{g}\binom{X_1, X_2}{Y_1, Y_2}.\end{aligned}$$

Here β is a rational realization of $\mathcal{Z}^{\mathfrak{f}}$, such that the depth one objects are exactly the constant terms of the Eisenstein series G_k and

$$\mathfrak{g}\binom{X_1, \ldots, X_r}{Y_1, \ldots, Y_r} = \sum_{\substack{m_1 > \cdots > m_r > 0 \\ n_1, \ldots, n_r > 0}} \prod_{j=1}^{r} e^{X_j n_j + Y_j m_j} q^{m_j n_j}.$$