A simultaneous q-analogue of finite and symmetrized multiple zeta values

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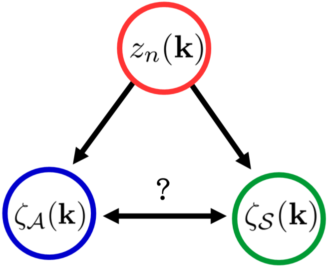

joint work with Y. Takeyama and K. Tasaka

CIRM, Luminy, 26th June 2017

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A simultaneous **q-analogue** of **finite** and **symmetrized** multiple zeta values

A simultaneous **q-analogue** of **finite** and **symmetrized** multiple zeta values

$$\zeta_{\mathcal{A}}(\mathbf{k})$$

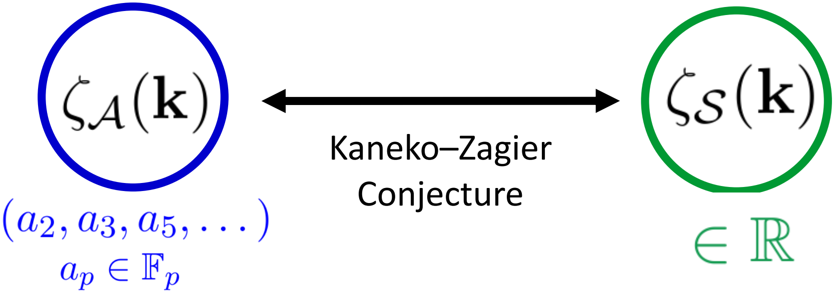
$$(a_2, a_3, a_5, \ldots)$$

$$a_p \in \mathbb{F}_p$$

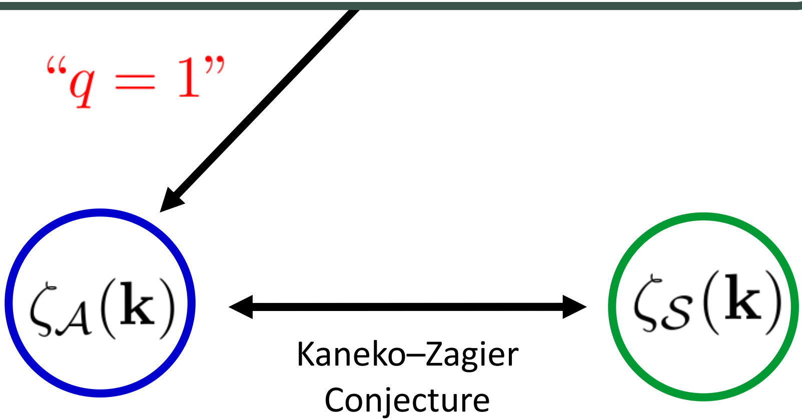
$$\zeta_{\mathcal{S}}(\mathbf{k})$$

$$\in \mathbb{R}$$

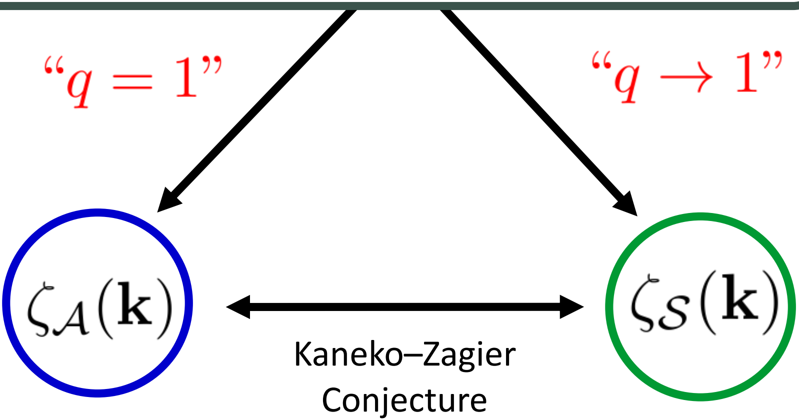
A simultaneous **q-analogue** of **finite** and **symmetrized** multiple zeta values



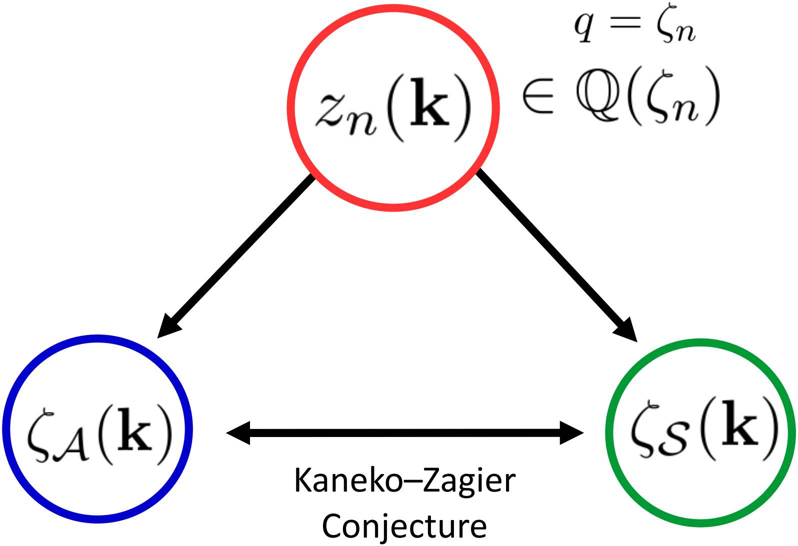
A simultaneous q-analogue of finite and symmetrized multiple zeta values



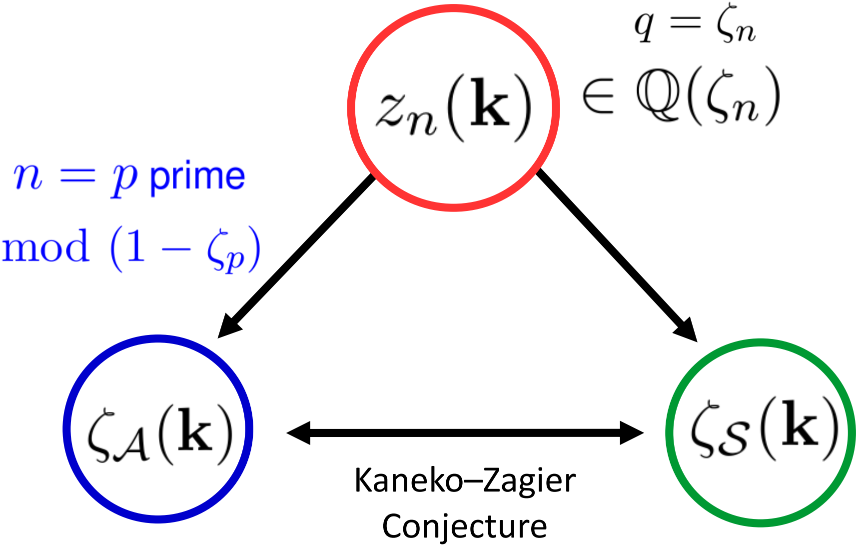
A simultaneous q-analogue of finite and symmetrized multiple zeta values



Overview

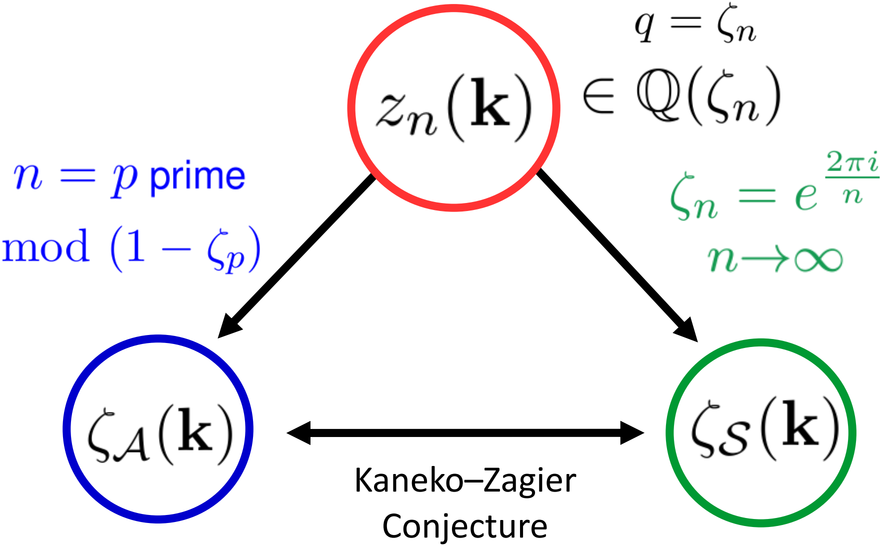

$q = \zeta_n$
$z_n(\mathbf{k}) \in \mathbb{Q}(\zeta_n)$
$\zeta_{\mathcal{A}}(\mathbf{k})$
$\zeta_{\mathcal{S}}(\mathbf{k})$
Kaneko–Zagier
Conjecture

Overview



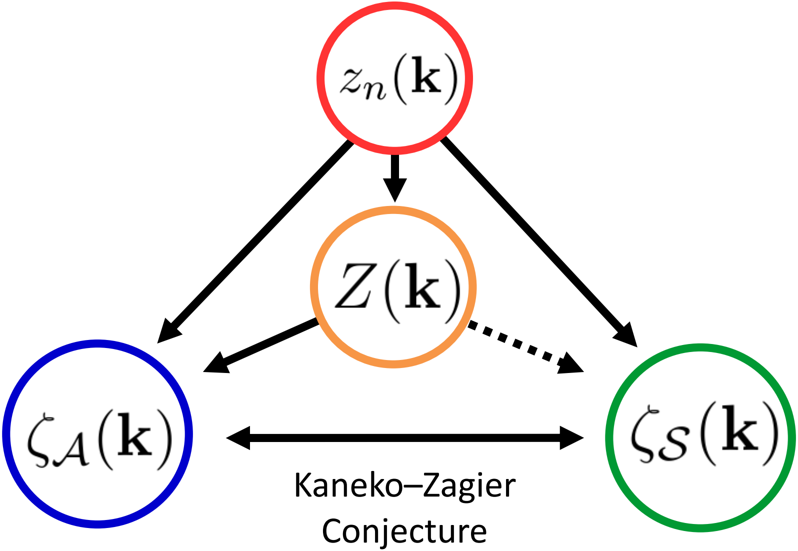
$q = \zeta_n$
$z_n(\mathbf{k})$
$\in \mathbb{Q}(\zeta_n)$
$n = p$ prime
$\mod (1 - \zeta_p)$
$\zeta_{\mathcal{A}}(\mathbf{k})$
$\zeta_{\mathcal{S}}(\mathbf{k})$
Kaneko–Zagier
Conjecture

Overview



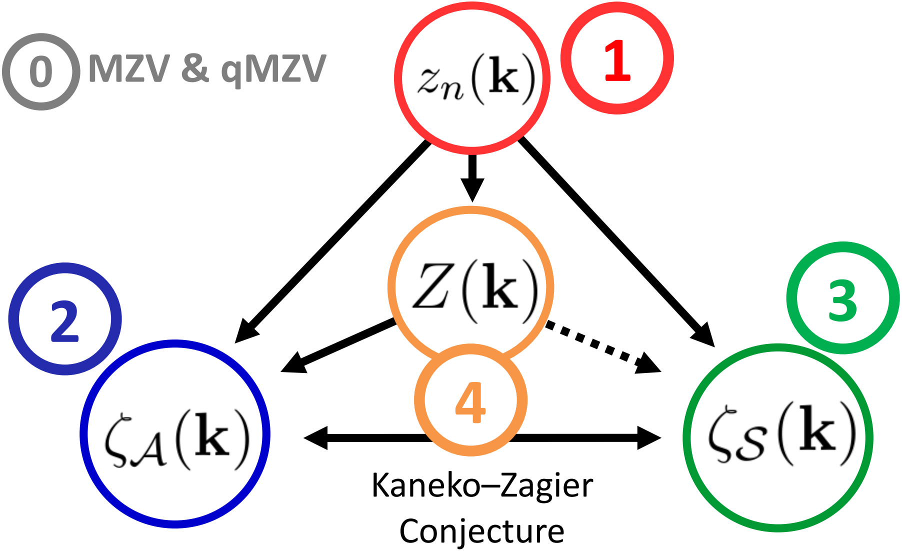
$q = \zeta_n$
$z_n(\mathbf{k}) \in \mathbb{Q}(\zeta_n)$
$n = p$ prime
$\bmod (1 - \zeta_p)$
$\zeta_n = e^{\frac{2\pi i}{n}}$
$n \to \infty$
$\zeta_{\mathcal{A}}(\mathbf{k})$
$\zeta_{\mathcal{S}}(\mathbf{k})$
Kaneko–Zagier Conjecture

Overview



$z_n(\mathbf{k})$
$Z(\mathbf{k})$
$\zeta_{\mathcal{A}}(\mathbf{k})$
$\zeta_{\mathcal{S}}(\mathbf{k})$
Kaneko–Zagier
Conjecture

Plan of the talk



Definition

For $k_1 \geq 2\,, k_2, \dots, k_r \geq 1, \mathbf{k} = (k_1, \dots, k_r)$ define the **multiple zeta value**

$$\zeta(\mathbf{k}) = \zeta(k_1, \dots, k_r) = \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}$$

and the **multiple zeta-star value**

$$\zeta^\star(\mathbf{k}) = \zeta^\star(k_1, \dots, k_r) = \sum_{m_1 \geq \cdots \geq m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}\,.$$

By r we denote its **depth** and $\mathrm{wt}(\mathbf{k}) = k_1 + \cdots + k_r$ will be called its **weight**.

- The product of two MZ(S)V can be expressed as a linear combination of MZ(S)V with the same weight (**harmonic product**). e.g:

$$\zeta(k_1) \cdot \zeta(k_2) = \zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1 + k_2)\,,$$
$$\zeta^\star(k_1) \cdot \zeta^\star(k_2) = \zeta^\star(k_1, k_2) + \zeta^\star(k_2, k_1) - \zeta^\star(k_1 + k_2)\,.$$

Definition

For $k_1 \geq 2\,, k_2, \dots, k_r \geq 1$ and $\mathbf{k} = (k_1, \dots, k_r)$ define the q-**multiple zeta value** by

$$\zeta(k_1, \dots, k_r; q) = \sum_{m_1 > \cdots > m_r > 0} \frac{q^{(k_1-1)m_1} \dots q^{(k_r-1)m_r}}{[m_1]_q^{k_1} \cdots [m_r]_q^{k_r}} \in \mathbb{Q}[[q]]\,,$$

where $[m]_q = \frac{1-q^m}{1-q} = 1 + q + \cdots + q^{m-1}$.

- They satisfy: $\lim\limits_{q\to 1} \zeta(k_1, \dots, k_r; q) = \zeta(k_1, \dots, k_r)$.
- The harmonic product version for these objects reads

$$\begin{aligned}\zeta(k_1; q) \cdot \zeta(k_2; q) =& \zeta(k_1, k_2; q) + \zeta_q(k_2, k_1; q) + \zeta_q(k_1 + k_2; q) \\ & + (1-q)\zeta_q(k_1 + k_2 - 1; q)\,,\end{aligned}$$

- The product structures of ζ, $\zeta^\star$ and $\zeta(\dots; q)$ are all examples of Quasi shuffle products (See M. Hoffmans talk on Friday).

Definition

- For $n \geq 1$ and an index set $\mathbf{k} = (k_1, \dots, k_r)$ with $k_1, \dots, k_r \geq 1$ we define

$$z_n(\mathbf{k}; q) = z_n(k_1, \dots, k_r; q) = \sum_{n > m_1 > \cdots > m_r > 0} \frac{q^{(k_1-1)m_1} \dots q^{(k_r-1)m_r}}{[m_1]_q^{k_1} \cdots [m_r]_q^{k_r}}$$

  and similarly $z_n^\star(\mathbf{k}; q)$ by summing over $n > m_1 \geq \cdots \geq m_r > 0$.

- Let ζ_n a primitive n-th root of unity and write for $q = \zeta_n$

$$z_n(\mathbf{k}) := z_n(\mathbf{k}; \zeta_n) \in \mathbb{Q}(\zeta_n)\,.$$

Remark

Clearly the value $z_n(\mathbf{k}; \zeta_n) \in \mathbb{C}$ depends on the choice of ζ_n. But all the results we will present in the following are true for all ζ_n and we therefore just write $z_n(\mathbf{k})$.

In depth one we have

$$\sum_{k=1}^{\infty} z_n(k) \left(\frac{x}{1-\zeta_n} \right)^k = \frac{nx}{1-(1+x)^n} + 1$$

and therefore in particular $z_n(k) \in (1-\zeta_n)^k \mathbb{Q}$.

$$z_n(1) = \frac{n-1}{2}(1-\zeta_n)\,, \quad z_n(2) = -\frac{n^2-1}{12}(1-\zeta_n)^2\,,$$
$$z_n(3) = \frac{n^2-1}{24}(1-\zeta_n)^3\,, \quad z_n(4) = \frac{(n^2-1)(n^2-19)}{720}(1-\zeta_n)^4\,.$$

① z_n - Degenerated Bernoulli numbers

Carlitz (1956) introduced the **degenerated Bernoulli** numbers $b_k(n) \in \mathbb{Q}[\frac{1}{n}]$

$$\sum_{k=0}^{\infty} b_k(n) \frac{x^k}{k!} = \frac{x}{(1+\frac{x}{n})^n - 1} \,.$$

These numbers can be seen as a degeneration of the Bernoulli numbers

$$\lim_{n\to\infty} b_k(n) = B_k \,, \qquad \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{e^x - 1} \,.$$

For even $k \geq 2$ it is

$$\zeta(k) = -\frac{B_k}{k!} \frac{(-2\pi i)^k}{2} \,.$$

Similarly we have (for all $n, k \geq 1$)

$$z_n(k) = -\frac{b_k(n)}{k!} \left(n(1-\zeta_n)\right)^k \,.$$

With the same calculation as for q-analogues one checks that for $a, b \geq 1$

$$z_n(a) \cdot z_n(b) = z_n(a,b) + z_n(b,a) + z_n(a+b) + (1-\zeta_n) z_n(a+b-1)\,.$$

In particular for $a = 1$ it is

$$z_n(1) \cdot z_n(b) = z_n(1,b) + z_n(b,1) + z_n(b+1) + (1-\zeta_n) z_n(b)\,.$$

But since $z_n(1) = \frac{n-1}{2}(1-\zeta_n)$ we obtain for $n \neq 3$

$$(1-\zeta_n) z_n(b) = \frac{2}{n-3}\left(z_n(1,b) + z_n(b,1) + z_n(b+1)\right)\,,$$

which gives

$$\begin{aligned} z_n(a) \cdot z_n(b) = & z_n(a,b) + z_n(b,a) + z_n(a+b) \\ & + \frac{2}{n-3}\left(z_n(1,a+b-1) + z_n(a+b-1,1) + z_n(a+b)\right)\,. \end{aligned}$$

Lemma

For $n \gg k$ and all index sets $\mathbf{k}$ and $m \geq 0$ there exists $\alpha_{\mathbf{k}',n} \in \mathbb{Q}$ for index sets $\mathbf{k}'$ with

$$(1-\zeta_n)^m z_n(\mathbf{k}) = \sum_{\mathrm{wt}(\mathbf{k}')=\mathrm{wt}(\mathbf{k})+m} \alpha_{\mathbf{k}',n} \cdot z_n(\mathbf{k}')\,.$$

Corollary

- The product $z_n(\mathbf{k}_1) \cdot z_n(\mathbf{k}_2)$ can be written as a linear combination of $z_n(\mathbf{k}_3)$ with $\mathrm{wt}(\mathbf{k}_3) = \mathrm{wt}(\mathbf{k}_1) + \mathrm{wt}(\mathbf{k}_2)$.
- Every $z_n^\star$ can be written in terms of z_n and vice versa.

The second statement in depth 2 follows for example from

$$z_n^\star(a,b) = z_n(a,b) + z_n(a+b) + (1-\zeta_n) z_n(a+b-1)\,.$$

In the following we will just focus on linear relations between $z_n^\star$.

Definition (rough)

Every index set $\mathbf{k} = (k_1, \dots, k_r)$ can be written as

$$(k_1, \dots, k_r) = (\overbrace{1 + \cdots + 1}^{k_1}, \dots, \overbrace{1 + \cdots + 1}^{k_r})\,.$$

Define the **Hoffman dual** $\mathbf{k}^\vee$ by interchanging **,** and **+** in this representation.

Example The Hoffman dual of $\mathbf{k} = (3, 2)$ is given by

$$\mathbf{k}^\vee = (3,2)^\vee = (1+1+1, 1+1)^\vee = (1, 1, 1+1, 1) = (1, 1, 2, 1)\,.$$

For an index set $\mathbf{k} = (k_1, \ldots, k_r)$ we define its reverse by $\overline{\mathbf{k}} = (k_r, \ldots, k_1)$.

Theorem (B., Takeyama, Tasaka)

For all $n \geq 1$ and all index sets $\mathbf{k} = (k_1, \ldots, k_r)$ we have

$$z_n^\star(\mathbf{k}) = (-1)^{\mathrm{wt}(\mathbf{k})+1} z_n^\star(\overline{\mathbf{k}^\vee})\,.$$

Example: Since $(3,2)^\vee = (1,1,2,1)$ it is

$$z_n^\star(3,2) = (-1)^{5+1} z_n^\star(\overline{1,1,2,1}) = z_n^\star(1,2,1,1)\,.$$

The **duality** together with the **harmonic product** gives a large family of algebraic & linear relations between $z_n^\star$ or z_n.

Example

Using duality and the harmonic product one can show that

$$2z_n^\star(4,1) + z_n^\star(3,2) = \frac{(n^4-1)(n+5)}{1440}(1-\zeta_n)^5 + \frac{n+2}{3}(1-\zeta_n)^2 z_n^\star(2,1)$$

(Of course the right-hand side could also be written as a linear combination of $z_n^\star$.)

Observation

For a fixed n and a fixed weight $1 \leq k < n$, all linear relations between $z_n^\star(\mathbf{k})$ (resp. $z_n(\mathbf{k})$) in weight $\mathrm{wt}(\mathbf{k}) = k$ seem to follow from the duality and the harmonic product.

① z_n - Summary

- For a fixed n, the $z_n(\mathbf{k})$ are elements in $\mathbb{Q}(\zeta_n)$.
- They satisfy various linear relations, which conjecturally can all be described using duality and the harmonic product.
- We also have results on Sum formulas / Ohno-Zagier relations / etc.
- There exist explicit formulas for $z_n(k,\dots,k) \in (1-\zeta_n)^{k+\cdots+k} \cdot \mathbb{Q}$.

We now want to give the connection to finite multiple zeta values.

Definition

For an index $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{N}^r$ the **finite multiple zeta value** are defined by

$$\zeta_{\mathcal{A}}(\mathbf{k}) = \left(\sum_{p > m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \mod p \right)_{p \text{ prime}} \in \mathcal{A},$$

where $\mathcal{A}$ is given by

$$\mathcal{A} = \prod_{p \text{ prime}} \mathbb{F}_p \Big/ \bigoplus_{p \text{ prime}} \mathbb{F}_p .$$

Also define $\zeta^{\star}_{\mathcal{A}}(\mathbf{k})$ by using $p > m_1 \geq \cdots \geq m_r > 0$ in the definition.
Denote by $\mathcal{Z}_{\mathcal{A}} \subset \mathcal{A}$ the $\mathbb{Q}$-vector space spanned by all $\zeta_{\mathcal{A}}(\mathbf{k})$.

- $\mathcal{A}$ and $\mathcal{Z}_{\mathcal{A}}$ are $\mathbb{Q}$-algebras.
- Satisfy the same harmonic product formula as MZ(S)V.

For this part we will focus on $n = p$ prime.

Lemma

For p prime it is $z_p(\mathbf{k}) \in \mathbb{Z}[\zeta_p]$.

Proof: This follows from the fact that the q-integer $[m]_q$ at $q = \zeta_p$ is a cyclotomic unit, when m is coprime with p.

In this case there exists a t with $m \cdot t \equiv 1 \mod p$ and therefore

$$\frac{1}{[m]_{\zeta_p}} = \frac{1-\zeta_p}{1-\zeta_p^m} = \frac{1-\zeta_p^{tm}}{1-\zeta_p^m} = 1 + \zeta_p^m + \cdots + \zeta_p^{(t-1)m} \in \mathbb{Z}[\zeta_p]\,.$$

Let $\mathfrak{p} = (1-\zeta_p)$ be the prime ideal of $\mathbb{Z}[\zeta_p]$ generated by $1-\zeta_p$.

Lemma

- It holds that $\mathbb{Z}[\zeta_p]/\mathfrak{p} = \mathbb{F}_p$.
- For $p > m > 0$ we have $[m]_{\zeta_p} \equiv m \mod \mathfrak{p}$.

From this we get

Theorem (B., Takeyama, Tasaka)

For any primitive root of unity ζ_p, we have

$$(z_p(\mathbf{k}) \mod \mathfrak{p})_p = \zeta_{\mathcal{A}}(\mathbf{k})$$

and

$$(z_p^\star(\mathbf{k};\zeta_p) \mod \mathfrak{p})_p = \zeta_{\mathcal{A}}^\star(\mathbf{k}).$$

In particular since $z_n(k) \in (1-\zeta_n)^k \cdot \mathbb{Q}$ it is $\zeta_{\mathcal{A}}(k) = 0$.

We saw before that for all p

$$2z_p^\star(4,1) + z_p^\star(3,2) = \frac{(p^4-1)(p+5)}{1440}(1-\zeta_p)^5 + \frac{p+2}{3}(1-\zeta_p)^2 z_p^\star(2,1)$$

The right-hand side vanishes for $p > 5$ in $\mathbb{Z}[\zeta_p]/\mathfrak{p}$ and therefore we obtain the relation

$$2\zeta_{\mathcal{A}}^\star(4,1) + \zeta_{\mathcal{A}}^\star(3,2) = 0\,.$$

Similar to the z_n the finite MZV also satisfy the duality relation

Theorem (Hoffman)

For all index sets $\mathbf{k}$ it is

$$\zeta^{\star}_{\mathcal{A}}(\mathbf{k}) = -\zeta^{\star}_{\mathcal{A}}(\mathbf{k}^{\vee})\,.$$

This follows from

$$z^{\star}_{n}(\mathbf{k}) = (-1)^{\mathrm{wt}(\mathbf{k})+1}\, z^{\star}_{n}(\overline{\mathbf{k}^{\vee}})\,.$$

and the easy to check relation $\zeta^{\star}_{\mathcal{A}}(\mathbf{k}) = (-1)^{\mathrm{wt}(\mathbf{k})}\zeta^{\star}_{\mathcal{A}}(\overline{\mathbf{k}})$.

Definition

For $k_1, \ldots, k_r \geq 1$ denote by $R_{k_1,\ldots,k_r}(T) \in \mathbb{R}[T]$ the regularized multiple zeta values, which are uniquely determined by

- $R_1(T) = T$,
- For $k_1 \geq 2$ it is $R_{k_1,\ldots,k_r}(T) = \zeta(k_1, \ldots, k_r)$,
- Their product can be expressed by the harmonic product formula.

Example: Since $R_1(T) \cdot R_2(T) = R_{1,2}(T) + R_{2,1}(T) + R_3(T)$ it is

$$R_{1,2}(T) = \zeta(2)T - \zeta(2,1) - \zeta(3)\,.$$

Definition

For an index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{N}^r$ define the **symmetrized multiple zeta values** by

$$\zeta_{\mathcal{S}}(\mathbf{k}) = \sum_{a=0}^{r} (-1)^{k_1+\dots+k_a} R_{k_a,k_{a-1},\dots,k_1}(T) R_{k_{a+1},k_{a+2},\dots,k_r}(T)$$

and

$$\zeta^{\star}_{\mathcal{S}}(\mathbf{k}) = \zeta^{\star}_{\mathcal{S}}(k_1, \dots, k_r) = \sum_{\substack{\square \text{ is either a comma ',' } \\ \text{or a plus '+'}}} \zeta_{\mathcal{S}}(k_1 \square \cdots \square k_r).$$

- One can check that the definition of $\zeta_{\mathcal{S}}$ is independent of T.
- In depth $r = 1$ it is

$$\zeta_{\mathcal{S}}(k) = R_k(T) + (-1)^k R_k(T) = \begin{cases} 2\zeta(k) & , k \text{ is even} \\ 0 & , k \text{ is odd} \end{cases}.$$

- Now we want to give the connection of z_n to the symmetrized multiple zeta values.
- For this part we will fix the primitive n-th root of unity $\zeta_n = e^{\frac{2\pi i}{n}}$.

We saw earlier that in depth one we have for all $k, n \geq 1$

$$z_n(k) = -\frac{b_k(n)}{k!}\left(n(1-\zeta_n)\right)^k .$$

Notice that

$$\lim_{n\to\infty} n(1-\zeta_n) = \lim_{n\to\infty} n\left(-\frac{2\pi i}{n} - \frac{1}{2}\left(\frac{2\pi i}{n}\right)^2 - \dots\right) = -2\pi i .$$

With $\lim_{n\to\infty} b_k(n) = B_k$ and $\zeta(k) = -\frac{B_k}{k!}\frac{(-2\pi i)^k}{2}$ for even k we obtain

$$\lim_{n\to\infty} z_n(k) = \begin{cases} -\pi i & (k=1) \\ 2\zeta(k) & (k \geq 2,\ k \text{ is even}) \\ 0 & (k \geq 3,\ k \text{ is odd}) \end{cases}$$

In particular $\mathrm{Re}\left(\lim_{n\to\infty} z_n(k)\right) = \zeta_{\mathcal{S}}(k)$.

Theorem (B., Takeyama, Tasaka)

For any index set $\mathbf{k} = (k_1, \dots, k_r)$ the limit $\lim_{n\to\infty} z_n(\mathbf{k}; e^{\frac{2\pi i}{n}})$ exists and we set

$$\xi(\mathbf{k}) := \lim_{n\to\infty} z_n(\mathbf{k}; e^{\frac{2\pi i}{n}}) \in \mathbb{C}$$

and

$$\xi^\star(\mathbf{k}) := \lim_{n\to\infty} z_n^\star(\mathbf{k}; e^{\frac{2\pi i}{n}}) \in \mathbb{C}\,.$$

As we saw before in depth one it is

$$\xi(k) = \begin{cases} -\pi i & (k=1) \\ 2\zeta(k) & (k \geq 2,\ k \text{ is even}) \\ 0 & (k \geq 3,\ k \text{ is odd}) \end{cases}$$

Theorem (B., Takeyama, Tasaka)

For any index set $\mathbf{k} = (k_1, \ldots, k_r)$ we have

$$\xi(\mathbf{k}) = \sum_{a=0}^{r} (-1)^{k_1+\cdots+k_a} R_{k_a,k_{a-1},\ldots,k_1}\Big(\frac{\pi i}{2}\Big) R_{k_{a+1},k_{a+2},\ldots,k_r}\Big(-\frac{\pi i}{2}\Big).$$

From this explicit expression we obtain

Theorem (B., Takeyama, Tasaka)

For any index set $\mathbf{k} = (k_1, \ldots, k_r)$ we have

$$\mathrm{Re}\,(\xi(\mathbf{k})) \equiv \zeta_{\mathcal{S}}(\mathbf{k}) \mod \zeta(2)\mathcal{Z}\,.$$

There are similar statements for $\xi^\star(\mathbf{k}) := \lim_{n\to\infty} z_n^\star(\mathbf{k}; e^{\frac{2\pi i}{n}})$ and $\zeta^\star_{\mathcal{S}}$.

We have seen before that for all n

$$2z_n^\star(4,1) + z_n^\star(3,2) = \frac{(n^4-1)(n+5)}{1440}(1-\zeta_n)^5 + \frac{n+2}{3}(1-\zeta_n)^2 z_n^\star(2,1)$$

Taking the limit $n \to \infty$ we obtain

$$2\xi_n^\star(4,1) + \xi_n^\star(3,2) = \frac{(-2\pi i)^5}{1440}$$

and in particular

$$2\zeta_{\mathcal{S}}^\star(4,1) + \zeta_{\mathcal{S}}^\star(3,2) \equiv 0 \mod \zeta(2)\mathcal{Z}.$$

From $z_n^\star(\mathbf{k}) = (-1)^{\mathrm{wt}(\mathbf{k})+1} z_n^\star(\overline{\mathbf{k}^\vee})$ we also obtain duality for $\xi^\star$ and $\zeta_{\mathcal{S}}^\star$

Theorem (B., Takeyama, Tasaka)

For any index $\mathbf{k}$, the following relations hold.

- $\xi^\star(\mathbf{k}^\vee) = -\,\overline{\xi^\star(\mathbf{k})}\,.$
- $\xi^\star(\overline{\mathbf{k}}) = (-1)^{\mathrm{wt}(\mathbf{k})}\,\overline{\xi^\star(\mathbf{k})}\,,$

Here the bar on the right-hand sides denotes complex conjugation.

Taking the real part gives:

Theorem (B., Takeyama, Tasaka)

For any index $\mathbf{k}$ we have

$$\zeta_{\mathcal{S}}^\star(\mathbf{k}) \equiv -\zeta_{\mathcal{S}}^\star(\mathbf{k}^\vee) \quad \text{and} \quad \zeta_{\mathcal{S}}^\star(\mathbf{k}) \equiv (-1)^{\mathrm{wt}(\mathbf{k})}\zeta_{\mathcal{S}}^\star(\overline{\mathbf{k}}) \mod \zeta(2)\mathcal{Z}.$$

So far we considered z_n for a fixed n.

Observation

For a fixed weight k, the number of linear relations between $z_n(\mathbf{k})$ with $\mathrm{wt}(\mathbf{k}) = k$ seem to be the same for all $n \gg k$.

We therefore want to introduce now a "global object" $Z(\mathbf{k})$.

We will now collect for all prime p the values z_p and define for this

$$\mathcal{A}^{\text{cyc}} = \left(\prod_{p:\text{prime}} \mathbb{Z}[\zeta_p]/(p) \right) \Big/ \left(\bigoplus_{p:\text{prime}} \mathbb{Z}[\zeta_p]/(p) \right),$$

- $\mathcal{A}^{\text{cyc}}$ is a $\mathbb{Q}$-algebra.
- It is independet of the choice of ζ_p, since $\mathbb{Z}[\zeta_p] = \mathbb{Z}[\zeta_p']$ for any other p-th primitive root of unity ζ_p'.

Definition

For an index $\mathbf{k} \in \mathbb{N}^r$ we define

$$Z(\mathbf{k}) = (z_p(\mathbf{k}) \mod p)_p \in \mathcal{A}^{\text{cyc}}$$

and

$$Z^\star(\mathbf{k}) = (z_p^\star(\mathbf{k}) \mod p)_p \in \mathcal{A}^{\text{cyc}}.$$

Let $\mathcal{Z}_k^{\mathrm{cyc}}$ be the $\mathbb{Q}$-vector space spanned by all $Z(\mathbf{k})$ of weight k.

Proposition

- The product $Z(\mathbf{k}_1) \cdot Z(\mathbf{k}_2)$ can be written as a linear combination of $Z(\mathbf{k}_3)$ with $\mathrm{wt}(\mathbf{k}_3) = \mathrm{wt}(\mathbf{k}_1) + \mathrm{wt}(\mathbf{k}_2)$.
- For any index $\mathbf{k}$ we also have the duality

$$Z^\star(\mathbf{k}) = (-1)^{\mathrm{wt}(\mathbf{k})+1} Z^\star(\overline{\mathbf{k}^\vee}).$$

Combining this with the product gives again a large family of linear relations and we obtain the following upper bounds for the dimensions:

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\dim_{\mathbb{Q}} \mathcal{Z}_k^{\text{cyc}} \leq$	1	1	1	2	2	4	5	8	12	17	27	38	57	84

Since $(p) = (1 - \zeta_p)^{p-1}$ we have a projection $\varphi : \mathcal{A}^{\mathrm{cyc}} \to \mathcal{A}$ sending $(a_p \mod (p))_p \in \mathcal{A}^{\mathrm{cyc}}$ to $(a_p \mod \mathfrak{p})_p \in \mathcal{A}$.

This give a $\mathbb{Q}$-algebra homomorphism

$$\begin{aligned} \varphi_{\mathcal{A}} : \mathcal{Z}^{\mathrm{cyc}} &\longrightarrow \mathcal{Z}_{\mathcal{A}} \\ Z(\mathbf{k}) &\longmapsto \zeta_{\mathcal{A}}(\mathbf{k}). \end{aligned}$$

The ideal $\ker \varphi_{\mathcal{A}}$ can be written as follows.

Proposition

We have $\ker \varphi_{\mathcal{A}} = (1 - \zeta_p)\mathcal{A}^{\mathrm{cyc}} \cap \mathcal{Z}^{\mathrm{cyc}}$.

We saw that relations between $z_n(\mathbf{k})$ gives relations between $\zeta_{\mathcal{A}}(\mathbf{k})$ and $\zeta_{\mathcal{S}}(\mathbf{k})$ $\bmod \zeta(2)\mathcal{Z}$.

This supports the following conjecture:

Conjecture (Kaneko-Zagier)

The map φ_{KZ}, defined by

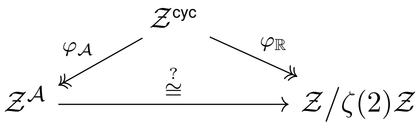
$$\begin{aligned} \varphi_{KZ} : \mathcal{Z}_{\mathcal{A}} &\longrightarrow \mathcal{Z}/\zeta(2)\mathcal{Z} \\ \zeta_{\mathcal{A}}(\mathbf{k}) &\longmapsto \zeta_{\mathcal{S}}(\mathbf{k}) \mod \zeta(2)\mathcal{Z} \end{aligned}$$

is a $\mathbb{Q}$-algebra isomorphism.

Conjecture

- The map $\varphi_{\mathbb{R}} : \mathcal{Z}^{\text{cyc}} \to \mathcal{Z}/\zeta(2)\mathcal{Z}$ that sends $Z(\mathbf{k})$ to $\operatorname{Re}(\xi(\mathbf{k}))$ is a $\mathbb{Q}$-algebra homomorphism.
- It holds $\ker \varphi_{\mathcal{A}} \overset{?}{=} \ker \varphi_{\mathbb{R}}$.

This Conjecture would imply the Kaneko-Zagier conjecture. We expect the following commutative diagram:



Thank you for your attention!