Sum/Some formulas for Schur multiple zeta values

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based on joint works (in progress) with

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$$\sum \zeta\left(\ydiagram{2+1,2+1,3}\right) = \binom{k-1}{r-1}\zeta(k)$$

$$E_{2k}\left(\ydiagram{2,1}\right) = 2E_{2k}\left(\ydiagram{1,1,1}\right) + \frac{3}{4}E_{2k}\left(\ydiagram{3}\right) - \frac{7}{4}E_{2k}\left(\ydiagram{1,1}\right)$$

$$\zeta\left(\ytableaushort{\none 1,13,31,13,3}\right) \in \mathbb{Q}\pi^{16}$$

$$E_{2k}\left(\ydiagram{2,1}\right) = \frac{3k-1}{4}\zeta(2k) - \frac{1}{2}\zeta(2)\zeta(2k-2)$$

Multiple zeta values and related topics, Kyushu University, 12th June 2019

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Overview

1
Schur multiple zeta values

$$\zeta\left(\begin{array}{cccc} & & & h \\ & e & f & g \\ b & d & & \\ a & c & & \end{array}\right)$$

2
Sum formulas

$$\zeta(1,4)+\zeta(2,3)+\zeta(3,2)=\zeta(5)$$

3 & 4
Symmetric & Even sum formulas

$$\zeta(2,4)+\zeta(4,2)=\frac{3}{4}\zeta(6)$$

5
1-3 Schur MZV

$$\zeta\left(\begin{array}{cccc} & & & 1 \\ & & \iddots & 3 \\ & 1 & \iddots & \\ 1 & 3 & & \end{array}\right)=\frac{2}{4^n}\zeta(4n+1)$$

Definition

For $k_1, \ldots, k_{r-1} \geq 1, k_r \geq 2$ define the **multiple zeta value** (MZV) by

$$\zeta(k_1, \ldots, k_r) = \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}$$

and the **multiple zeta-star value** (MZSV) by

$$\zeta^\star(k_1, \ldots, k_r) = \sum_{0<m_1\leq\cdots\leq m_r} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}.$$

weight: $k_1 + \cdots + k_r$, **depth**: r.

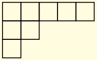
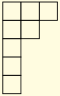
Today: Introduce a simultaneous generalization of MZV and MZSV, given by Schur MZV, and discuss their sum formulas.

① Schur MZV - Partitions

- By a **partition** we denote a tuple $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \cdots \geq \lambda_n \geq 1$.
- $|\lambda| = \lambda_1 + \cdots + \lambda_n$.
- Its **transpose** is denoted by $\lambda' = (\lambda'_1, \dots, \lambda'_m)$ and it is defined by transposing the corresponding Young diagram.

Example

A partition and its transpose visualized by Young diagrams

$\lambda = (5, 2, 1) =$  $\lambda' = (3, 2, 1, 1, 1) =$ 

① Schur MZV - Partitions & Young Tableaux

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition.

- For another partition $\mu = (\mu_1, \dots, \mu_r)$ we write $\mu \subset \lambda$ if $r \leq n$ and $\mu_j < \lambda_j$ for $j = 1, \dots, r$.
- For partitions λ, μ with $\mu \subset \lambda$ we define

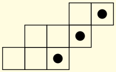
$$D(\lambda/\mu) = \left\{(i,j) \in \mathbb{Z}^2 \mid 1 \leq i \leq n \,, \mu_i < j \leq \lambda_i \right\} \,.$$

- We denote the set of all **corners** of λ/μ by $\mathrm{Cor}(\lambda/\mu) \subset D(\lambda/\mu)$.

Example When $\lambda/\mu = (5,4,3)/(3,1)$ we have

$$D(\lambda/\mu) = \{(1,4),(1,5),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3)\}\,,$$
$$\mathrm{Cor}(\lambda/\mu) = \{(1,5),(2,4),(3,3)\}\,,$$

which we visualize (Corners = ●) in the corresponding Young diagram:



- A (skew) **Young tableau** $\mathbf{k} = (k_{i,j})$ of shape λ/μ is a collection of $k_{i,j} \in \mathbb{N}$ for all $(i,j) \in D(\lambda/\mu)$.

Example When $\lambda/\mu = (5,4,3)/(3,1)$ we visualize this Young tableau by

$$\mathbf{k} = (k_{i,j}) = \begin{array}{|c|c|c|c|c|} \cline{4-5} \multicolumn{3}{c|}{} & k_{1,4} & k_{1,5} \\ \cline{2-5} \multicolumn{1}{c|}{} & k_{2,2} & k_{2,3} & k_{2,4} & \multicolumn{1}{c}{} \\ \cline{1-4} k_{3,1} & k_{3,2} & k_{3,3} & \multicolumn{2}{c}{} \\ \cline{1-3} \end{array} .$$

- A Young tableau $(m_{i,j})$ is called **semi-standard** if $m_{i,j} < m_{i+1,j}$ and $m_{i,j} \leq m_{i,j+1}$ for all i and j.
- The set of all Young tableaux and all semi-standard Young tableaux of shape λ/μ are denoted by $\mathrm{YT}(\lambda/\mu)$ and $\mathrm{SSYT}(\lambda/\mu)$, respectively.

① Schur MZV - Definition

A Young tableau $\mathbf{k} = (k_{i,j}) \in \mathrm{YT}(\lambda/\mu)$ is called **admissible** if $k_{i,j} \geq 2$ for $(i,j) \in \mathrm{Cor}(\lambda/\mu)$.

Definition (Yamasaki 2010, Nakasuji-Phuksuwan-Yamasaki 2018)

For an admissible $\mathbf{k} = (k_{i,j}) \in \mathrm{YT}(\lambda/\mu)$ the **Schur multiple zeta value** is defined by

$$\zeta(\mathbf{k}) = \sum_{(m_{i,j}) \in \mathrm{SSYT}(\lambda/\mu)} \prod_{(i,j) \in D(\lambda/\mu)} \frac{1}{m_{i,j}^{k_{i,j}}}.$$

These generalize MZV and MZSV in the following way.

$$\zeta(k_1, \dots, k_r) = \zeta\left(\begin{array}{|c|}\hline k_1 \\ \hline \vdots \\ \hline k_r \\ \hline \end{array}\right) \quad \text{and} \quad \zeta^\star(k_1, \dots, k_r) = \zeta\left(\begin{array}{|c|c|c|}\hline k_1 & \cdots & k_r \\ \hline \end{array}\right).$$

Example 1 For $a \geq 1$ and $b, c \geq 2$ we have

$$\zeta\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & \\ \cline{1-1}\end{array}\right) = \sum_{\substack{m_a \leq m_b \\ \wedge \\ m_c}} \frac{1}{m_a^a \cdot m_b^b \cdot m_c^c}\,.$$

Clearly every Schur MZV is just a linear combination of MZV, e.g.

$$\zeta\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & \\ \cline{1-1}\end{array}\right) = \zeta(a,c,b) + \zeta(a,b,c) + \zeta(a+b,c) + \zeta(a,b+c)\,.$$

① Schur MZV - Definition - Examples

Example 2

For $a, b, d \geq 1$ and $c, e, f \geq 2$ we have

$$\zeta\left(\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & \multicolumn{1}{|c}{} \\ \cline{1-2} f & \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) = \sum_{\substack{\begin{array}{ccccc} m_a & \leq & m_b & \leq & m_c \\ \wedge & & \wedge & & \\ m_d & \leq & m_e & & \\ \wedge & & & & \\ m_f & & & & \end{array}}} \frac{1}{m_a^a \cdot m_b^b \cdot m_c^c \cdot m_d^d \cdot m_e^e \cdot m_f^f}\,.$$

Example 3

For $b, d \geq 1$ and $c, e, f \geq 2$ we have

$$\zeta\left(\begin{array}{ccc} \cline{2-3} & \multicolumn{1}{|c|}{b} & \multicolumn{1}{c|}{c} \\ \hline \multicolumn{1}{|c|}{d} & \multicolumn{1}{c|}{e} & \\ \cline{1-2} \multicolumn{1}{|c|}{f} & & \\ \cline{1-1}\end{array}\right) = \sum_{\substack{\begin{array}{ccccc} & & m_b & \leq & m_c \\ & & \wedge & & \\ m_d & \leq & m_e & & \\ \wedge & & & & \\ m_f & & & & \end{array}}} \frac{1}{m_b^b \cdot m_c^c \cdot m_d^d \cdot m_e^e \cdot m_f^f}\,.$$

① Schur MZV - Arakawa-Kaneko zeta values & Ohno's identity

In Ohno's talk on monday we saw the following results.

Theorem (Kuba 2010, Yamamoto 2016+)

For $k_1,\dots,k_{r-1},m \geq 1$, $k_r \geq 2$ the Arakawa-Kaneko zeta value can be written as

$$\xi(k_1,\dots,k_{r-1},k_r-1;m) = \zeta\left(\underbrace{\begin{array}{|c|c|c|c|} \cline{4-4} \multicolumn{3}{c|}{} & k_1 \\ \cline{4-4} \multicolumn{3}{c|}{} & \vdots \\ \hline 1 & \dots & 1 & k_r \\ \hline \end{array}}_{m}\right).$$

Ohno's Identity: For $a,b \geq 1$ we have

$$\zeta\left(\underbrace{\left.\begin{array}{|c|c|c|c|} \cline{4-4} \multicolumn{3}{c|}{} & 1 \\ \cline{4-4} \multicolumn{3}{c|}{} & \vdots \\ \cline{4-4} \multicolumn{3}{c|}{} & 1 \\ \hline 1 & \dots & 1 & 2 \\ \hline \end{array}\right\}b}_{a+1}\right) = \zeta\left(\underbrace{\left.\begin{array}{|c|c|c|c|} \cline{4-4} \multicolumn{3}{c|}{} & 1 \\ \cline{4-4} \multicolumn{3}{c|}{} & \vdots \\ \cline{4-4} \multicolumn{3}{c|}{} & 1 \\ \hline 1 & \dots & 1 & 2 \\ \hline \end{array}\right\}a}_{b+1}\right).$$

Compared to multiple zeta values, the product of two arbitrary Schur multiple zeta values can be written quite easily.

Example The harmonic product formula of MZV is given by

$$\begin{aligned}\zeta(a)\cdot\zeta(b) &= \sum_{m>0}\frac{1}{m^a}\sum_{n>0}\frac{1}{n^b}\\ &= \sum_{0<m<n}\frac{1}{m^a n^b} + \sum_{0<n<m}\frac{1}{m^a n^b} + \sum_{m=n>0}\frac{1}{m^{a+b}}\\ &= \zeta(a,b) + \zeta(b,a) + \zeta(a+b)\,.\end{aligned}$$

Using the notion of Schur MZV this can be written as

$$\zeta\left(\boxed{a}\right)\zeta\left(\boxed{b}\right) = \sum_{0<m\leq n}\frac{1}{m^a n^b} + \sum_{0<n<m}\frac{1}{m^a n^b} = \zeta\left(\boxed{a}\boxed{b}\right) + \zeta\left(\begin{array}{|c|}\hline b\\ \hline a\\ \hline\end{array}\right).$$

① Schur MZV - Products

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Example The harmonic product formula of MZV is given by

$$\zeta(a)\cdot\zeta(b,c) = \zeta(a,b,c) + \zeta(b,a,c) + \zeta(b,c,a) + \zeta(a+b,c) + \zeta(b,a+c)\,.$$

Using the notion of Schur MZV this can be written as

$$\zeta\left(\,\boxed{a}\,\right)\zeta\left(\begin{array}{|c|}\hline b\\ \hline c\\ \hline\end{array}\right) = \zeta\left(\begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{b}\\ \hline \multicolumn{1}{|c|}{a} & \multicolumn{1}{c|}{c}\\ \hline\end{array}\right) + \zeta\left(\begin{array}{|c|}\hline b\\ \hline c\\ \hline a\\ \hline\end{array}\right).$$

① Schur MZV - Products

In general the product of two Schur MZV is always the sum of two Schur MZV.

Example

$$\zeta\left(\begin{ytableau} \none & e \\ b & d \\ a & c \end{ytableau}\right)\zeta\left(\begin{ytableau} \none & h \\ f & g \end{ytableau}\right) = \zeta\left(\begin{ytableau} \none & \none & h \\ \none & f & g \\ \none & e \\ b & d \\ a & c \end{ytableau}\right) + \zeta\left(\begin{ytableau} \none & \none & \none & h \\ \none & e & f & g \\ b & d \\ a & c \end{ytableau}\right) .$$

Some sum formulas...

For all $k > r \geq 1$ we have

$$\sum_{\substack{k_1+\cdots+k_r=k \\ k_1,\ldots,k_{r-1}\geq 1, k_r\geq 2}} \zeta(k_1,\ldots,k_r) = \zeta(k)\,,$$

$$\sum_{\substack{k_1+\cdots+k_r=k \\ k_1,\ldots,k_{r-1}\geq 1, k_r\geq 2}} \zeta^\star(k_1,\ldots,k_r) = \binom{k-1}{r-1}\zeta(k)\,,$$

For $k \geq 4$ we have (Kaneko-Tsumura 2019, Kaneko's talk yesterday)

$$\sum_{\substack{a+b+c=k \\ a,b\geq 1, c\geq 2}} T(a,b,c) + \sum_{j=2}^{k-2} T(1,k-1-j,j) = \frac{2}{3}T(2)T(k-2)\,.$$

Key property of a "sum formula"

The number of terms on the right-hand side do not depend on the weight.

Goal of the project: For a fixed shape λ/μ we want to evaluate the following sum:

$$\sum_{\substack{\mathbf{k}\in \mathrm{YT}(\lambda/\mu)\\ \mathrm{wt}(\mathbf{k})=k \\ \mathbf{k} \text{ is admissible}}} \zeta(\mathbf{k})\,.$$

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Bad news: So far we just succeeded for a few shapes λ/μ.

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Bad news: So far we just succeeded for a few shapes λ/μ.

Good news: If we take the sum just over all $k_{i,j} \in \mathcal{N}$ for some subset $\mathcal{N} \subset \mathbb{Z}_{\geq 2}$ (symmetric sums), we can do it for all shapes.

② Sum formulas - for anti-hooks

Theorem (B.-Kadota-Suzuki-Yamamoto-Yamasaki, 2019+)

For $r \geq 1, s \geq 0$ and $k > r + s \geq 1$ we have

$$\sum_{\substack{k_1+\cdots+k_r+l_1+\cdots+l_s=k \\ k_1,\ldots,k_{r-1}\geq 1, k_r \geq 2 \\ l_1,\ldots,l_s \geq 1}} \zeta\left(\begin{array}{ccc} & & \boxed{l_1} \\ & & \boxed{\vdots} \\ & & \boxed{l_s} \\ \boxed{k_1} & \boxed{\ldots} & \boxed{k_r} \end{array}\right) = \binom{k-1}{r-1} \zeta(k)\,.$$

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Great!

Are all sum of Schur multiple zeta values rational multiples of $\zeta(k)$?

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Great!

Are all sum of Schur multiple zeta values rational multiples of $\zeta(k)$?

No

② Sum formulas - for hooks

Theorem (B.-Kadota-Suzuki-Yamamoto-Yamasaki, 2019+)

For $r, s \geq 1$ and $k > r+s$ we have

$$\sum_{\substack{k_1+\cdots+k_r+l_1+\cdots+l_s=k \\ k_1,\ldots,k_{r-1}\geq 1, k_r\geq 2 \\ l_1,\ldots,l_{s-1}\geq 1, l_s\geq 2}} \zeta\left(\begin{array}{|c|c|c|c|}\hline k_1 & l_1 & \ldots & l_s \\ \hline \vdots & \multicolumn{3}{c}{} \\ \cline{1-1} k_r & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}\right)$$

$$= \binom{k-2}{s}\zeta(k) - \sum_{a=1}^{s}\binom{k-a-2}{s-a}\zeta(a,k-a) - (-1)^s \sum_{a=s+1}^{r+s-1}\zeta(a,k-a)$$

$$- \sum_{a=2}^{s}(-1)^a\binom{k-a-2}{s-a}\zeta(a)\zeta(k-a) - \sum_{a=2}^{r}\binom{k-a-2}{s-1}\zeta(a)\zeta(k-a)$$

$$- \sum_{a=r+1}^{r+s-1}(-1)^{r-a}\binom{k-a-2}{r+s-a-1}\zeta(a)\zeta(k-a).$$

Question

Sum of Schur multiple zeta values are multiple zetas of depth = ”number of corners”?

② Sum formulas - for $\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}$

Question

Sum of Schur multiple zeta values are multiple zetas of depth = "number of corners"?

No...

Theorem (B.-Kadota-Suzuki-Yamamoto-Yamasaki, 2019+)

For $k \geq 5$ we have

$$\sum_{\substack{a+b+c+d=k \\ a,b,c \geq 1 \\ d \geq 2}} \zeta\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline \end{array}\right) = (k-4)\zeta(k) - (k-2)\zeta(1,k-1) + (2k-6)\zeta(2,k-2) - 2\zeta(k-3,3) + (k-2)\zeta(k-2,2)\,.$$

③ Symmetric sum formulas - ... because classical sums are hard

- In general it is difficult to evaluate sums over all admissible Young tableaux.
- For example: Summing the following over all $a \geq 1, b, c \geq 2$ with $a + b + c = k$

$$\zeta\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) = \zeta(a,b,c) + \zeta(a,c,b) + \zeta(a+b,c) + \zeta(a,b+c)\,,$$

we get the sum

$$\sum_{\substack{a+b+c=k \\ a \geq 1, b, c \geq 2}} \zeta(a,b,c) = \underbrace{\sum_{\substack{a+b+c=k \\ a,b \geq 1, c \geq 2}} \zeta(a,b,c)}_{=\zeta(k)} - \sum_{\substack{a+c=k-1 \\ a \geq 1, c \geq 2}} \zeta(a,1,c)\,.$$

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$$\zeta\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & \\ \cline{1-1}\end{array}\right) = \zeta(a,b,c) + \zeta(a,c,b) + \zeta(a+b,c) + \zeta(a,b+c)\,,$$

we get the sum

$$\sum_{\substack{a+b+c=k\\ a\geq 1, b,c\geq 2}} \zeta(a,b,c) = \underbrace{\sum_{\substack{a+b+c=k\\ a,b\geq 1, c\geq 2}} \zeta(a,b,c)}_{=\zeta(k)} - \sum_{\substack{a+c=k-1\\ a\geq 1, c\geq 2}} \zeta(a,1,c)\,.$$

Easier: Evaluate symmetric sums of the form

$$\sum_{a,b,c\in\mathcal{N}} \zeta\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & \\ \cline{1-1}\end{array}\right)$$

for some $\mathcal{N} \subset \mathbb{Z}_{\geq 2}$.

- For a subset $\mathcal{N} \subset \mathbb{Z}_{\geq 2}$ we define

$$\mathrm{YT}(\lambda, \mathcal{N}) = \{(k_{i,j}) \in \mathrm{YT}(\lambda) \mid k_{i,j} \in \mathcal{N}\}$$

- The generating series for sums of Schur multiple zeta values with entries in $\mathcal{N}$ by

$$G_{\mathcal{N}}(\lambda, X) = \sum_{\mathbf{k} \in YT(\lambda,\mathcal{N})} \zeta(\mathbf{k}) X^{\mathrm{wt}(\mathbf{k})} \in \mathcal{Z}[[X]]\,.$$

The coefficient of X^k in these generating series are examples of ”symmetric sums”.

③ Symmetric sum formulas - Symmetric functions

- Let $\Lambda \subset \mathbb{Q}[[x_1, x_2, \dots]]$ be the $\mathbb{Q}$-algebra of **symmetric functions**.
- For a partition λ let s_λ be the **Schur function** defined by

$$s_\lambda(x_1, x_2, \dots) = \sum_{(m_{i,j}) \in \mathrm{SSYT}(\lambda)} \prod_{(i,j) \in D(\lambda)} x_{m_{i,j}} .$$

- The s_λ form a basis of Λ.
- Notation: $(1^r) = (\underbrace{1, \dots, 1}_{r}) = \begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline \end{array}$.
- For $r \geq 1$ let $e_r = s_{(1^r)}$ be the **elementary symmetric functions**.

$$e_r = s_{(1^r)} = s_{\begin{array}{|c|}\hline \\ \hline \cdot \\ \hline \\ \hline \end{array}} = \sum_{0 < m_1 < \cdots < m_r} x_{m_1} \dots x_{m_r} .$$

③ Symmetric sum formulas - the map $\phi_{\mathcal{N}}$

Lemma

For any subset $\mathcal{N} \subset \mathbb{Z}_{\geq 2}$ the linear map $\phi_{\mathcal{N}}$, defined by $\phi_{\mathcal{N}}(1) = 1$ and on the generators by

$$\begin{aligned} \phi_{\mathcal{N}} : \Lambda &\longrightarrow \mathcal{Z}[[X]] \\ s_\lambda &\longmapsto G_{\mathcal{N}}(\lambda, X) \end{aligned}$$

is a $\mathbb{Q}$-algebra homomorphism.

Proof sketch: Make a change of variables

$$x_m \mapsto \sum_{k \in \mathcal{N}} \frac{X^k}{m^k}\,.$$

This Lemma allows us to use linear & algebraic relations among s_λ to evaluate the $G_{\mathcal{N}}$.

③ Symmetric sum formulas - Jacobi-Trudi formula

Proposition (Jacobi-Trudi formula)

For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ with transpose $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$ we have

$$s_\lambda = \det(e_{\lambda'_i - i + j})_{1 \le i,j \le m}\,.$$

Example

$$s_{\ydiagram{2,2}} = \det \begin{pmatrix} e_2 & e_3 \\ e_1 & e_2 \end{pmatrix} = e_2^2 - e_1 e_3 = s_{\ydiagram{1,1}}^2 - s_{\ydiagram{1}}\, s_{\ydiagram{1,1,1}}\,.$$

$$Z_{\mathcal{N}}(r, X) = G_{\mathcal{N}}((1^r), X) = \sum_{k_1,\dots,k_r \in \mathcal{N}} \zeta(k_1, \dots, k_r) X^{k_1+\cdots+k_r} .$$

Corollary

For any subset $\mathcal{N} \subset \mathbb{Z}_{\geq 2}$ and a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ with transpose $\lambda' = (\lambda'_1, \dots, \lambda'_m)$ we have

$$G_{\mathcal{N}}(\lambda, X) = \det(Z_{\mathcal{N}}(\lambda'_i - i + j, X))_{1 \leq i,j \leq m} .$$

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$$G_{\mathcal{N}}(\lambda, X) = \det(Z_{\mathcal{N}}(\lambda'_i - i + j, X))_{1 \leq i,j \leq m}\,.$$

Using Hoffmans symmetric sum formulas (appearing in his talk yesterday) we get

Corollary

For any subset $\mathcal{N} \subset \mathbb{Z}_{\geq 2}$ we have

$$G_{\mathcal{N}}(\lambda, X) \in \mathbb{Q}[\zeta(2), \zeta(3), \zeta(4), \dots][[X]]\,.$$

③ Symmetric sum formulas - for $\boxplus$

Example

$$G_{\mathcal{N}}\left(\boxplus, X\right) = Z_{\mathcal{N}}(2,X)^2 - Z_{\mathcal{N}}(1,X)Z_{\mathcal{N}}(3,X)\,.$$

Considering the coefficient of X^k gives the following sum formula for all $k \geq 1$

$$\sum_{\substack{a,b,c,d\in\mathcal{N}\\ a+b+c+d=k}} \zeta\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}\right) = \sum_{\substack{m+n=k\\ m,n\geq 1}} \left(\sum_{\substack{m_1+m_2=m\\ m_1,m_2\in\mathcal{N}}} \zeta(m_1,m_2)\right)\left(\sum_{\substack{n_1+n_2=n\\ n_1,n_2\in\mathcal{N}}} \zeta(n_1,n_2)\right)$$

$$- \sum_{\substack{m+n=k\\ m\in\mathcal{N}\\ n\geq 1}} \zeta(m) \sum_{\substack{n_1+n_2+n_3=n\\ n_1,n_2,n_3\in\mathcal{N}}} \zeta(n_1,n_2,n_3)\,.$$

From now on we consider the case $\mathcal{N} = \mathrm{ev} := \{2, 4, 6, \dots\}$.

- A Young tableau $\mathbf{k} = (k_{i,j}) \in \mathrm{YT}(\lambda)$ is called **even**, if all $k_{i,j}$ are even.

With the notation from before we have

$$G_{\mathrm{ev}}(\lambda, X) = \sum_{\substack{\mathbf{k} \in YT(\lambda) \\ \mathbf{k} \text{ even}}} \zeta(\mathbf{k}) X^{\mathrm{wt}(\mathbf{k})} =: \sum_{k \geq 1} E_{2k}(\lambda) X^{2k} \,.$$

In the following we are therefore interested in evaluating

$$E_{2k}(\lambda) = \sum_{\substack{\mathbf{k} \in \mathrm{YT}(\lambda) \\ \mathbf{k} \text{ even} \\ \mathrm{wt}(\mathbf{k}) = 2k}} \zeta(\mathbf{k}) \,.$$

Theorem (Hoffman 2017, but already presented tomorrow 7 years ago in Hamburg)

For $k \geq r \geq 1$ we have

$$
\begin{aligned}
E_{2k}((1^r)) = & \frac{1}{2^{2(r-1)}} \binom{2r-1}{r} \zeta(2k) \\
& - \sum_{j=1}^{\lfloor \frac{r-1}{2} \rfloor} \frac{1}{2^{2r-3}(2j+1)B_{2j}} \binom{2r-2j-1}{r} \zeta(2j)\zeta(2k-2j) \,.
\end{aligned}
$$

Example

$$
\begin{aligned}
E_{2k}\left(\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}\right) &= \frac{3}{4}\zeta(2k) \,, \\
E_{2k}\left(\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array}\right) &= \frac{5}{8}\zeta(2k) - \frac{1}{4}\zeta(2)\zeta(2k-2) \,, \\
E_{2k}\left(\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}\right) &= \frac{35}{64}\zeta(2k) - \frac{5}{16}\zeta(2)\zeta(2k-2) \,, \\
E_{2k}\left(\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}\right) &= \frac{63}{128}\zeta(2k) - \frac{21}{64}\zeta(2)\zeta(2k-2) + \frac{3}{64}\zeta(4)\zeta(2k-4) \,.
\end{aligned}
$$

④ Even sum formulas - for rows $\square\square\square$ (MZSV)

Lemma (Hoffman, 2017)

For $k \geq r \geq 1$ have

$$E_{2k}\left(\square\cdots\square\right) = E_{2k}((r)) = \sum_{j=1}^{r} \binom{k-j}{r-j} E_{2k}((1^j))\,.$$

Example

$$\begin{aligned}
E_{2k}\left(\square\square\right) &= \frac{4k-1}{4}\zeta(2k)\,,\\
E_{2k}\left(\square\square\square\right) &= \frac{4k^2-6k+1}{8}\zeta(2k) - \frac{1}{4}\zeta(2)\zeta(2k-2)\,,\\
E_{2k}\left(\square\square\square\square\right) &= \frac{32k^3-120k^2+112k-15}{192}\zeta(2k) - \frac{4k-7}{16}\zeta(2)\zeta(2k-2)\,.
\end{aligned}$$

④ Even sum formulas - for all shapes

Theorem (B.-Kadota-Suzuki-Yamamoto-Yamasaki, 2019+)

For any partition $\lambda = (\lambda_1, ..., \lambda_n)$ there exist unique polynomials

$$p_{\lambda,j}(x) \in \mathbb{Q}[x]\,, \qquad j = 0, \ldots, \left\lfloor \frac{|\lambda| - 1}{2} \right\rfloor$$

of degree $\deg(p_{\lambda,j}) < \min(|\lambda| - 2j, \lambda_1)$, such that we have for all $k \geq |\lambda|$

$$E_{2k}(\lambda) = \sum_{j=0}^{\lfloor \frac{|\lambda|-1}{2} \rfloor} p_{\lambda,j}(k)\zeta(2j)\zeta(2(k-j))\,.$$

In particular we have $E_{2k}(\lambda) \in \pi^{2k}\mathbb{Q}$. $\left(\zeta(0) = -\frac{1}{2}\right)$

Remark

The Theorem also holds for skew type young diagrams λ/μ.

④ Even sum formulas - a few examples

Example

$$
\begin{aligned}
E_{2k}\left(\ydiagram{2,1}\right) &= \frac{3k-1}{4}\zeta(2k) - \frac{1}{2}\zeta(2)\zeta(2k-2)\,,\\
E_{2k}\left(\ydiagram{2,2}\right) &= -\frac{2k+1}{32}\zeta(2k) + \frac{2k-5}{8}\zeta(2)\zeta(2k-2)\,,\\
E_{2k}\left(\ydiagram{2,2,1}\right) &= -\frac{10k+5}{128}\zeta(2k) + \frac{8k-29}{64}\zeta(2)\zeta(2k-2) + \frac{15}{64}\zeta(4)\zeta(2k-4)\,,\\
E_{2k}\left(\ydiagram{2,2,2}\right) &= -\frac{10k+5}{512}\zeta(2k) - \frac{5k-5}{64}\zeta(2)\zeta(2k-2) + \frac{40k-55}{256}\zeta(4)\zeta(2k-4)\,.
\end{aligned}
$$

④ Even sum formulas - for $\begin{array}{|c|c|c|}\hline \, & \, & \, \\ \hline \, & \, & \, \\ \hline \, & \, & \, \\ \hline \end{array}$

Example For $k \geq 9$ we have

$$
\begin{aligned}
E_{2k}\left(\begin{array}{|c|c|c|}\hline \, & \, & \, \\ \hline \, & \, & \, \\ \hline \, & \, & \, \\ \hline \end{array}\right) = & -\frac{(2k+1)(k+1)}{2^{14}}\zeta(2k) \\
& + \frac{28k^2 - 78k + 35}{2^{13}}\zeta(2)\zeta(2k-2) \\
& + \frac{148k^2 - 798k + 989}{2^{13}}\zeta(4)\zeta(2k-4) \\
& - \frac{196k^2 - 882k + 749}{2^{13}}\zeta(6)\zeta(2k-6) \\
& + \frac{35}{2^{13}}\zeta(8)\zeta(2k-8)\,.
\end{aligned}
$$

④ Even sum formulas - proof sketch

Hoffman gives an explicit formula for the generating series for $E_{2k}((1^r))$

$$1 + \sum_{k \geq r \geq r} E_{2k}((1^r)) X^{2k} Y^r = \frac{\sin(\pi X \sqrt{1-Y})}{\sqrt{1-Y}\,\sin(\pi X)}\,,$$

from which we can deduce

$$Z_{\mathrm{ev}}(r, X) = \left(\frac{1}{r!} \left(\frac{d}{dY} \right)^r \frac{\sin(\pi X \sqrt{1-Y})}{\sqrt{1-Y}\,\sin(\pi X)} \right)_{\big| Y=0}\,.$$

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Doing messy calculations with the cotangents and its derivatives yields the result and an explicit construction of the polynomials $p_{\lambda,j}(x)$.

Remark

One can also prove our Theorem by using a result of Guo-Lei-**Zhao** (2015). In their work they show that one can also write the sum

$$\sum_{m+n=k} m^a n^b \zeta(2m)\zeta(2n)$$

as a sum of $p_j(k)\zeta(2j)\zeta(2(k-j))$ for some $p_j(x) \in \mathbb{Q}[x]$.

Due to the Theorem we have a map

$$\begin{aligned} \phi_{\mathrm{ev}} : \Lambda &\longrightarrow \mathbb{Q}[\pi^2][[X]] \\ s_\lambda &\longmapsto G_{\mathrm{ev}}(\lambda, X) \,. \end{aligned}$$

Question

Is ϕ_{ev} injective or are there "relations among sum formulas"?

4 Even sum formulas - relations among sum formulas

Due to the Theorem we have a map

$$\begin{aligned} \phi_{\mathrm{ev}} : \Lambda &\longrightarrow \mathbb{Q}[\pi^2][[X]] \\ s_\lambda &\longmapsto G_{\mathrm{ev}}(\lambda, X)\,. \end{aligned}$$

Question

Is ϕ_{ev} injective or are there "relations among sum formulas"?

Answer: There seem to be no relation among $G_{\mathrm{ev}}(\lambda, X)$ in a fixed depth $|\lambda|$, but allowing mixed depth we have for example for all $k \geq 2$

$$E_{2k}\left(\ydiagram{2,1}\right) + \frac{7}{4} E_{2k}\left(\ydiagram{1,1}\right) = 2E_{2k}\left(\ydiagram{1,1,1}\right) + \frac{3}{4} E_{2k}\left(\ydiagram{2}\right)$$

and for all $k \geq 4$

$$E_{2k}\left(\ydiagram{2,2}\right) + \frac{1}{2} E_{2k}\left(\ydiagram{3,1}\right) + \frac{15}{32} E_{2k}\left(\ydiagram{2}\right) = \frac{1}{2} E_{2k}\left(\ydiagram{2,1}\right) + \frac{3}{8} E_{2k}\left(\ydiagram{3}\right)\,.$$

④ 1-3-MZV - $\zeta(1,3,\ldots,1,3)$

Theorem (Borwein-Bradley-Broadhurst-Lisonek)

For all $n \geq 1$ we have

$$\zeta(1,3,\ldots,1,3) = \zeta(\{1,3\}^n) = \frac{2\pi^{4n}}{(4n+2)!} = \frac{1}{4^n}\zeta(\{4\}^n)\,.$$

Using the notion of Schur MZV the identity $\zeta(\{1,3\}^n) = \frac{1}{4^n}\zeta(\{4\}^n)$ reads

$$\zeta\left(\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \vdots \\ \hline 1 \\ \hline 3 \\ \hline \end{array}\right) = \frac{1}{4^n}\zeta(\{4\}^n)\,.$$

④ 1-3-MZV - 1-3-Stairs Formula

$$\zeta\left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline \vdots\\\hline 1\\\hline 3\\\hline\end{array}\right) = \frac{1}{4^n}\zeta(\{4\}^n)\,.$$

Theorem (B.-Yamasaki, 2018)

For any $n \geq 1$ we have

$$\zeta\left(\begin{array}{cccc} & & & 1\\ & & \iddots & 3\\ & 1 & \iddots & \\ 1 & 3 & & \end{array}\right) = \frac{2}{4^n}\zeta(4n+1)\,,\quad \zeta\left(\begin{array}{cccc} & & 1 & 3\\ & \iddots & 3 & \\ 1 & \iddots & & \\ 3 & & & \end{array}\right) = \frac{1}{4^n}\zeta(4n+3)\,,$$

$$\zeta\left(\begin{array}{ccc} & & 1\\ & \iddots & 3\\ 1 & \iddots & \\ 3 & & \end{array}\right) = \frac{1}{4^n}\zeta^{\star}(\{4\}^n)\,,\quad \zeta\left(\begin{array}{cccc} & & 1 & 3\\ & \iddots & \iddots & \\ 1 & 3 & & \end{array}\right) = \sum_{k=0}^{n}\frac{\zeta^{\star}(\{4\}^k)\zeta(\{4\}^{n-k})}{4^k}\,,$$

where n is the number of $\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 3\\\hline\end{array}$ respectively.

Consequence of the Theorem: Every Checkerboard Schur MZV is a polynomial in odd single zetas and π^4.

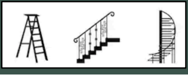
Theorem (B.-Yamasaki, B.-Charlton)

Schur MZV with alternating entries in 1 and 3 are elements in $\mathbb{Q}[\pi^4, \zeta(3), \zeta(5), \dots]$.

- We can give explicit formulas for a lot of shapes as determinants in odd zeta values and powers of π^4.

Example

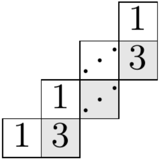
$$\zeta\left(\begin{array}{|c|c|c|}\hline 3 & 1 & 3 \\ \hline 1 & 3 & 1 \\ \hline 3 & 1 & 3 \\ \hline\end{array}\right) = \frac{1}{32}\begin{vmatrix} \zeta(3) & \frac{\pi^4}{180} & \zeta(7) \\ \frac{\pi^4}{72} & \zeta(5) & \frac{17\pi^8}{90720} \\ \zeta(7) & \frac{13\pi^8}{226800} & \zeta(11) \end{vmatrix}$$



1-3 Stair types

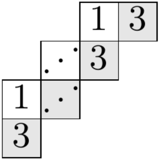
odd

$$\frac{2}{4^n}\zeta(4n+1)$$

$\zeta\Big($  $\Big)$

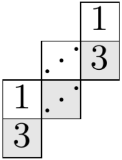
(1,1)-stair

$$\frac{1}{4^n}\zeta(4n+3)$$

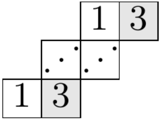
$\zeta\Big($  $\Big)$

(3,3)-stair

even

$\zeta\Big($  $\Big) \in \mathbb{Q}\pi^{4n}$

(3,1)-stair

$\zeta\Big($  $\Big) \in \mathbb{Q}\pi^{4n}$

(1,3)-stair

④ 1-3-MZV - - Refinement of the 1-3 Theorem

The Theorem from before can be refined in the following way:

Theorem (rough statement) (B.-Charlton, 2019+)

If a Young diagram $\mathbf{k}$ can be glued together by (a,b)-stairs, for fixed $a, b \in \{1,3\}$, then $\zeta(\mathbf{k})$ is a polynomial in (a,b)-stairs.

Example

$$\begin{array}{|c|c|c|}\hline 3 & 1 & 3 \\ \hline 1 & 3 & \\ \cline{1-2} 3 & & \\ \cline{1-1} \end{array} = \begin{array}{|c|}\hline 3 \\ \hline \end{array} \to \begin{array}{ccc} & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} & \\ \cline{1-2} \multicolumn{1}{|c|}{3} & & \\ \cline{1-1} \end{array}$$

$$\zeta\left(\begin{array}{|c|c|c|}\hline 3 & 1 & 3 \\ \hline 1 & 3 & \\ \cline{1-2} 3 & & \\ \cline{1-1} \end{array}\right) = \frac{1}{4^2}\begin{vmatrix} \zeta(3) & \zeta(7) \\ \zeta(7) & \zeta(11) \end{vmatrix}.$$

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If a Young diagram $\mathbf{k}$ can be glued together by (a,b)-stairs, for fixed $a, b \in \{1,3\}$, then $\zeta(\mathbf{k})$ is a polynomial in (a,b)-stairs.

Example

$$\begin{array}{cc} & 1 \\ 1 & 3 \\ 3 & 1 \\ 1 & 3 \\ 3 & \end{array} = \begin{array}{c} 1 \\ 3 \\ 1 \\ 3 \end{array} \to \begin{array}{c} 1 \\ 3 \\ 1 \\ 3 \end{array} \qquad \zeta\left(\begin{array}{cc} & 1 \\ 1 & 3 \\ 3 & 1 \\ 1 & 3 \\ 3 & \end{array}\right) \in \mathbb{Q}\pi^{16}$$

Summary

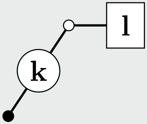
- Schur MZV generalize MZV and MZSV into one object.
- The algebraic structure of Schur MZV is easy to describe.
- We have sum formulas for a few shapes.
- Symmetric & Even sum formulas are known for all shapes.
- In the 1-3-case, one can write Schur MZV in terms of single zeta values.
- There are more results (e.g. Ohno-type relations for certain ribbons).
- There are various further open problems regarding Schur MZV
  (e.g. integral representation)

Thank you very much for your attention !

Theorem (Kaneko-Yamamoto)

For every indexsets $\mathbf{k} = (k_1, \dots, k_r), \mathbf{l} = (l_1, \dots, l_s), M = k_r + l_s$ we have

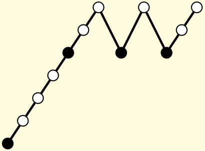
$$\zeta\left(\begin{array}{|c|c|c|c|}\hline \multicolumn{3}{c|}{} & k_1 \\ \cline{4-4} \multicolumn{3}{c|}{} & \vdots \\ \cline{4-4} \multicolumn{3}{c|}{} & k_{r-1} \\ \hline l_1 & \dots & l_{s-1} & M \\ \hline \end{array}\right) =: \zeta(\mathbf{k} \circledast \mathbf{l}^\star) = I\left(\right),$$



where the right-hand side is given by a Yamamoto 2-poset integral.

Example $\mathbf{k} = (4, 1), \mathbf{l} = (3, 2, 2)$:

$$\zeta\left(\begin{array}{|c|c|c|}\cline{3-3} \multicolumn{2}{c|}{} & 4 \\ \hline 3 & 2 & 3 \\ \hline \end{array}\right) = I\left(\right)$$



The result of Kaneko-Yamamoto can be generalized to arbitrary **ribbons**.

$$
\begin{array}{ccccccc}
 & & & & \boxed{h} & \boxed{i} & \boxed{j} \\
 & & & & \boxed{g} & & \\
 & & & \boxed{e} & \boxed{f} & & \\
\boxed{a} & \boxed{b} & \boxed{c} & \boxed{d} & & &
\end{array}
$$

Theorem (Nakasuji-Phuksuwan-Yamasaki)

Every Schur MZV of ribbon shape can be written as a Yamamoto 2-poset integral.

Open question

Can an arbitrary Schur MZV be written as a 2-poset integral?

$$
\zeta\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}\right) = \sum_{\substack{m_a \leq m_b \\ \wedge \qquad \wedge \\ m_c \leq m_d}} \frac{1}{m_a^a \cdot m_b^b \cdot m_c^c \cdot m_d^d} = I\left(\text{?} \right).
$$

⑥ Bonus - Special types of Young tableux & Regularized MZV

To state the Jacobi-Trudi formula we need the following notations.

- Let $T^{\mathrm{diag}}(\lambda/\mu)$ be the subset of $T(\lambda/\mu)$ consisting of all Young tableaux with the **same entries on the diagonal**.

Example

$$\begin{array}{|c|c|c|c|c|} \cline{2-5} \multicolumn{1}{c|}{} & 2 & 1 & 6 & 8 \\ \hline 9 & 5 & 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-4} 3 & 9 & 5 & 2 & \multicolumn{1}{c}{} \\ \cline{1-4} 1 & 3 & \multicolumn{3}{c}{} \\ \cline{1-2} 5 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} \in T^{\mathrm{diag}}((5,4,4,2,1)/(1))\,.$$

- Denote for $k_1, \dots, k_r \geq 1$ by $\zeta^*(k_1, \dots, k_r)$ the **stuffle regularized multiple zeta value** (with $\zeta^*(1) = 0$).

Example

$$\zeta^*(1) \cdot \zeta^*(2) = \zeta^*(1,2) + \zeta^*(2,1) + \zeta^*(3)\,,$$
$$\zeta^*(2,1) = -\zeta(1,2) - \zeta(3) = -2\zeta(3)\,.$$

⑥ Bonus - Regularized Jacobi-Trudi formula

Let $\lambda = (\lambda_1, \ldots, \lambda_h)$ and $\mu = (\mu_1, \ldots, \mu_r)$ be partitions with $\mu \subset \lambda$.

Regularized Jacobi-Trudi formula (Nakasuji-Phuksuwan-Yamasaki, B.-Charlton)

For an admissible Young tableau $\mathbf{k} = (k_{i,j}) \in T^{\mathrm{diag}}(\lambda/\mu)$ and $d_{i-j} = k_{i,j}$ we have

$$\zeta(\mathbf{k}) = \det\left(\zeta^*(d_{-\mu'_j+j-1}, d_{-\mu'_j+j-2}, \ldots, d_{-\mu'_j+j-(\lambda'_i-\mu'_j-i+j)})\right)_{1\leq i,j\leq\lambda_1},$$

where we set $\zeta^*(\,\cdots) = \begin{cases} 1 \text{ if } \lambda'_i - \mu'_j - i + j = 0 \\ 0 \text{ if } \lambda'_i - \mu'_j - i + j < 0 \end{cases}.$

⑥ Bonus - Jacobi-Trudi formula - Example

Example

$$\zeta\left(\begin{ytableau} d_0 & d_1 & d_2 \\ d_{-1} & d_0 \\ d_{-2} \end{ytableau}\right) = \begin{vmatrix} \zeta(d_{-2}, d_{-1}, d_0) & \zeta(d_{-2}, \ldots, d_1) & \zeta(d_{-2}, \ldots, d_2) \\ \zeta(d_0) & \zeta(d_0, d_1) & \zeta(d_0, d_1, d_2) \\ 0 & 1 & \zeta(d_2) \end{vmatrix}$$

$$\zeta\left(\begin{ytableau} \none & d_1 & d_2 \\ d_{-1} & d_0 \\ d_{-2} \end{ytableau}\right) = \begin{vmatrix} \zeta(d_{-2}, d_{-1}) & \zeta(d_{-2}, \ldots, d_1) & \zeta(d_{-2}, \ldots, d_2) \\ 1 & \zeta(d_0, d_1) & \zeta(d_0, d_1, d_2) \\ 0 & 1 & \zeta(d_2) \end{vmatrix}$$

⑥ Bonus - Thick stairs for $(a,b)=(1,3)$

Consequence of Jacobi-Trudi formula and the formula for stairs: The thick stairs in the case $(a,b)=(1,3)$ are Hankel-determinants in odd zeta values.

Example

$$\zeta\left(\begin{array}{|c|c|c|}\hline 3 & 1 & 3 \\ \hline 1 & 3 & \multicolumn{1}{|c}{} \\ \cline{1-2} 3 & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array}\right) = \frac{1}{4^2}\begin{vmatrix} \zeta(3) & \zeta(7) \\ \zeta(7) & \zeta(11) \end{vmatrix},$$

$$\zeta\left(\begin{array}{|c|c|c|c|c|}\hline 3 & 1 & 3 & 1 & 3 \\ \hline 1 & 3 & 1 & 3 & \multicolumn{1}{|c}{} \\ \cline{1-4} 3 & 1 & 3 & \multicolumn{2}{|c}{} \\ \cline{1-3} 1 & 3 & \multicolumn{3}{|c}{} \\ \cline{1-2} 3 & \multicolumn{4}{|c}{} \\ \cline{1-1} \end{array}\right) = \frac{1}{4^6}\begin{vmatrix} \zeta(3) & \zeta(7) & \zeta(11) \\ \zeta(7) & \zeta(11) & \zeta(15) \\ \zeta(11) & \zeta(15) & \zeta(19) \end{vmatrix}.$$

⑥ Bonus - Thick stairs for $(a, b) = (1, 3)$

Consequence of Jacobi-Trudi formula and the formula for stairs: The thick stairs in the case $(a, b) = (1, 3)$ are Hankel-determinants in odd zeta values.

Example

$$\zeta\left(\begin{array}{ccccc} & 1 & 3 & 1 & 3 \\ 1 & 3 & 1 & 3 & \\ 3 & 1 & 3 & & \\ 1 & 3 & & & \\ 3 & & & & \end{array}\right) = \frac{1}{4^6}\begin{vmatrix} \zeta(11) & \zeta(15) \\ \zeta(15) & \zeta(19) \end{vmatrix},$$

$$\zeta\left(\begin{array}{ccccc} & & 3 & 1 & 3 \\ & 3 & 1 & 3 & \\ 3 & 1 & 3 & & \\ 1 & 3 & & & \\ 3 & & & & \end{array}\right) = -\frac{1}{4^4}\begin{vmatrix} 0 & 0 & \zeta(3) & \zeta(7) \\ 0 & \zeta(3) & \zeta(7) & \zeta(11) \\ \zeta(3) & \zeta(7) & \zeta(11) & \zeta(15) \\ \zeta(7) & \zeta(11) & \zeta(15) & \zeta(19) \end{vmatrix}.$$

We also have 1-3-Formulas for the (non-admissible) stuffle regularized MZV:

Theorem (B.-Yamasaki, B.-Charlton)

For $n \geq 0$ we have

$$\zeta^*(\{1,3\}^n,1) = \frac{1}{2^{2n-1}} \sum_{j=1}^{n} (-1)^j \zeta(4j+1)\zeta(\{4\}^{n-j}),$$

$$\zeta^*(\{3,1\}^n) = \frac{1}{2^{2n-3}} \sum_{\substack{1 \leq j \leq n-1 \\ 0 \leq k \leq n-1-j}} (-1)^{j+k} \zeta(4j+1)\zeta(4k+3)\zeta(\{4\}^{n-j-1-k})$$

$$+ (-1)^n \sum_{k=0}^{n} \frac{1}{4^k} \zeta^\star(\{4\}^k)\zeta(\{4\}^{n-k}).$$

⑥ Bonus - 1-2-Stairs

We have

$$A_{1,2}(n) = \zeta\left(\begin{array}{cccc} &&&\boxed{1}\\ &&\boxed{\iddots}&\boxed{2}\\ &\boxed{1}&\boxed{\iddots}&\\ \boxed{1}&\boxed{2}&& \end{array}\right) = 3\zeta(3n+1)$$

but in general it is

$$B_{1,2}(n) = \zeta\left(\begin{array}{cccc} &&\boxed{1}&\boxed{2}\\ &\boxed{\iddots}&\boxed{2}&\\ \boxed{1}&\boxed{\iddots}&&\\ \boxed{2}&&& \end{array}\right) \notin \mathbb{Q}[\zeta(k) \mid k \geq 2]\,.$$

Also easy to check:

$$\zeta\left(\begin{array}{ccc} &&\boxed{1}\\ &\boxed{\iddots}&\boxed{2}\\ \boxed{1}&\boxed{\iddots}&\\ \boxed{2}&& \end{array}\right) = \zeta^\star(\{3\}^n)\,.$$