Interpolated Schur multiple zeta values

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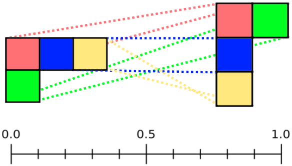


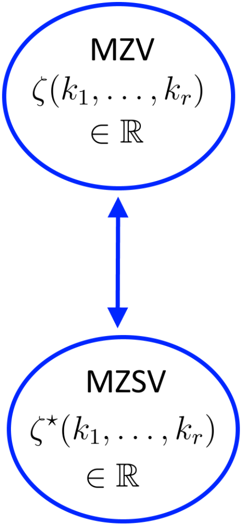
Tokyo Metropolitan University - Number theory seminar

13th December 2016

Slides are available here: **www.henrikbachmann.com**

Content of this talk

- Overview
- (harmonic) Interpolated multiple zeta values
- (harmonic) Interpolated Schur multiple zeta values
- Main result
- Lemma of Lindström, Gessel & Viennot
- t-Lattice Graph & Proof sketch of the main result



MZV

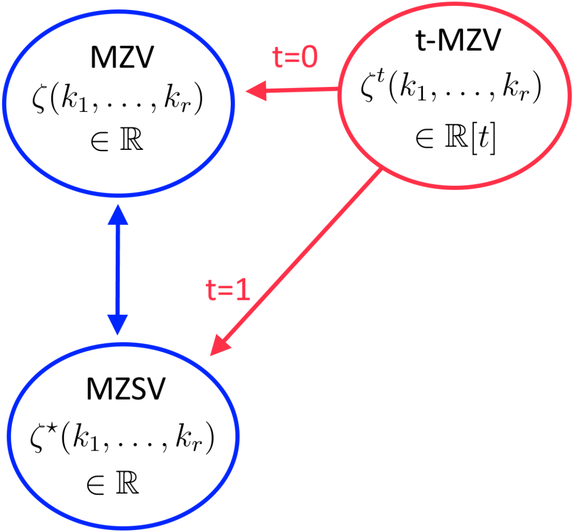
$\zeta(k_1, \ldots, k_r)$

$\in \mathbb{R}$

MZSV

$\zeta^\star(k_1, \ldots, k_r)$

$\in \mathbb{R}$



MZV

$\zeta(k_1, \ldots, k_r)$

$\in \mathbb{R}$

t-MZV

$\zeta^t(k_1, \ldots, k_r)$

$\in \mathbb{R}[t]$

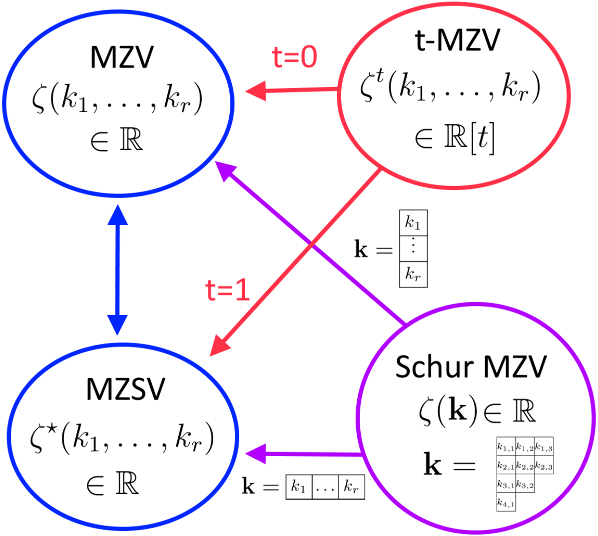
t=0

t=1

MZSV

$\zeta^\star(k_1, \ldots, k_r)$

$\in \mathbb{R}$



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t=0

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$\mathbf{k} = \begin{array}{|c|}\hline k_1 \\ \hline \vdots \\ \hline k_r \\ \hline\end{array}$

MZSV

$\zeta^\star(k_1,\ldots,k_r)$

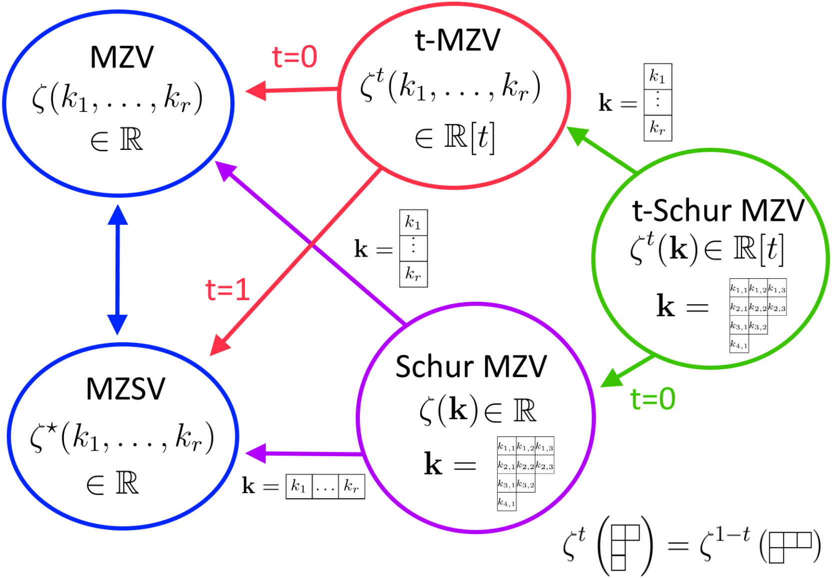
$\in \mathbb{R}$

$\mathbf{k} = \begin{array}{|c|c|c|}\hline k_1 & \ldots & k_r \\ \hline\end{array}$

Schur MZV

$\zeta(\mathbf{k}) \in \mathbb{R}$

$\mathbf{k} = \begin{array}{|c|c|c|}\hline k_{1,1} & k_{1,2} & k_{1,3} \\ \hline k_{2,1} & k_{2,2} & k_{2,3} \\ \hline k_{3,1} & k_{3,2} \\ \cline{1-2} k_{4,1} \\ \cline{1-1}\end{array}$



MZV

$\zeta(k_1,\dots,k_r)$

$\in \mathbb{R}$

t-MZV

$\zeta^t(k_1,\dots,k_r)$

$\in \mathbb{R}[t]$

t=0

t=1

$\mathbf{k} = \begin{array}{|c|}\hline k_1 \\ \hline \vdots \\ \hline k_r \\ \hline\end{array}$

t-Schur MZV

$\zeta^t(\mathbf{k}) \in \mathbb{R}[t]$

$\mathbf{k} = \begin{array}{|c|c|c|}\hline k_{1,1} & k_{1,2} & k_{1,3} \\ \hline k_{2,1} & k_{2,2} & k_{2,3} \\ \hline k_{3,1} & k_{3,2} \\ \cline{1-2} k_{4,1} \\ \cline{1-1}\end{array}$

$\mathbf{k} = \begin{array}{|c|}\hline k_1 \\ \hline \vdots \\ \hline k_r \\ \hline\end{array}$

t=0

MZSV

$\zeta^\star(k_1,\dots,k_r)$

$\in \mathbb{R}$

$\mathbf{k} = \begin{array}{|c|c|c|}\hline k_1 & \dots & k_r \\ \hline\end{array}$

Schur MZV

$\zeta(\mathbf{k}) \in \mathbb{R}$

$\mathbf{k} = \begin{array}{|c|c|c|}\hline k_{1,1} & k_{1,2} & k_{1,3} \\ \hline k_{2,1} & k_{2,2} & k_{2,3} \\ \hline k_{3,1} & k_{3,2} \\ \cline{1-2} k_{4,1} \\ \cline{1-1}\end{array}$

$\zeta^t\left(\begin{array}{|c|c|}\hline \; & \; \\ \hline \; \\ \cline{1-1} \; \\ \cline{1-1}\end{array}\right) = \zeta^{1-t}\left(\begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \; \\ \cline{1-1}\end{array}\right)$

Multiple zeta values

Definition

For $N \geq 1$ and $k_1, \ldots, k_r \in \mathbb{Z}$ define the (harmonic) multiple zeta value by

$$\zeta_N(k_1, \ldots, k_r) = \sum_{0<m_1<\cdots<m_r<N} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{Q}$$

and the (harmonic) multiple zeta-star value by

$$\zeta_N^\star(k_1, \ldots, k_r) = \sum_{0<m_1\leq\cdots\leq m_r<N} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{Q}\,.$$

For $k_1, \ldots, k_{r-1} \geq 1$ and $k_r \geq 2$ we have

$$\zeta(k_1, \ldots, k_r) = \lim_{N\to\infty} \zeta_N(k_1, \ldots, k_r) \in \mathbb{R}\,.$$

Interpolated multiple zeta values

For numbers $m_1 \leq m_2 \leq \cdots \leq m_r$ define the number of their equalities by

$$e(m_1, \ldots, m_r) = \sharp \left\{ 1 \leq i \leq r-1 \mid m_i = m_{i+1} \right\} .$$

Definition

For $N \geq 1$ and $k_1, \ldots, k_r \in \mathbb{Z}$ define the (harmonic) interpolated multiple zeta value by

$$\zeta_N^t(k_1, \ldots, k_r) = \sum_{0 < m_1 \leq \cdots \leq m_r < N} \frac{t^{e(m_1, \ldots, m_r)}}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{Q}[t].$$

For $t = 0$ and $t = 1$ it is $\zeta_N = \zeta_N^0$ and $\zeta_N^\star = \zeta_N^1$.

Interpolated multiple zeta values

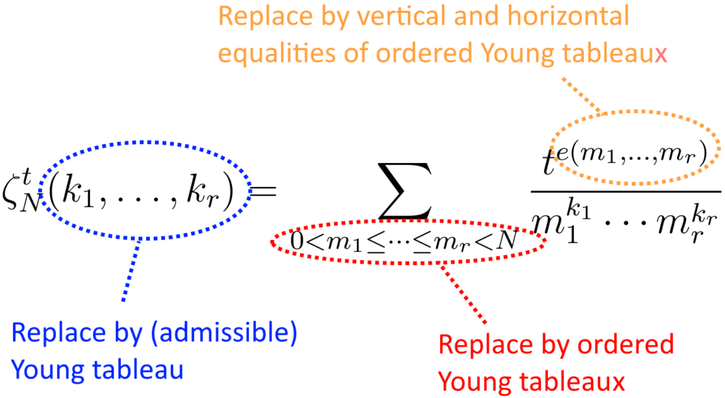
Every ζ_N^t can be written as a polynomial with coefficients given by ζ_N.

Example

For $a, b, c \in \mathbb{Z}$ and $N \geq 1$ we have

$$\begin{aligned}
\zeta_N^t(a) &= \zeta_N(a)\,, \\
\zeta_N^t(a,b) &= \zeta_N(a,b) + \zeta_N(a+b)t\,, \\
\zeta_N^t(a,b,c) &= \zeta_N(a,b,c) + \left(\zeta_N(a+b,c) + \zeta_N(a,b+c)\right)t \\
&\quad + \zeta_N(a+b+c)t^2\,.
\end{aligned}$$

The idea of interpolated Schur multiple zeta



Replace by vertical and horizontal equalities of ordered Young tableaux

$$\zeta_N^t(k_1,\ldots,k_r)=\sum_{0<m_1\leq\cdots\leq m_r<N}\frac{t^{e(m_1,\ldots,m_r)}}{m_1^{k_1}\cdots m_r^{k_r}}$$

Replace by (admissible) Young tableau

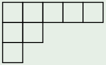
Replace by ordered Young tableaux

Partitions & (Ordered) Young Tableaux

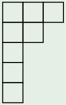
- By a **partition** (of $\lambda_1 + \cdots + \lambda_h$) we denote a tuple $\lambda = (\lambda_1, \ldots, \lambda_h)$ with $\lambda_1 \geq \cdots \geq \lambda_h \geq 1$.
- Its **conjugation** is denoted by $\lambda' = (\lambda'_1, \ldots, \lambda'_{h'})$ and it is defined by transposing the corresponding Young diagram.

Example: Partition of 8

$$\lambda = (5, 2, 1) =$$



$$\lambda' = (3, 2, 1, 1, 1) =$$



Partitions & (Ordered) Young Tableaux

- For a partition $\lambda = (\lambda_1, \ldots, \lambda_h)$ we define its **coordinates** by

$$D(\lambda) = \left\{ (i,j) \in \mathbb{Z}^2 \mid 1 \leq i \leq h \,, 1 \leq j \leq \lambda_i \right\} \,.$$

- By a **Young tableau** of shape λ we denote a tupel $\mathbf{k} = (\lambda, (k_{i,j}))$ with $k_{i,j} \in \mathbb{Z}$ for indices $(i,j) \in D(\lambda)$.
- Its **conjugation** is given by $\mathbf{k}' = (\lambda', (k_{j,i}))$.
- We visualize them in the usual way. For example for $\lambda = (3,3,2,1)$ we write

$$\mathbf{k} = (\lambda, (k_{i,j})) = \begin{array}{|c|c|c|} \hline k_{1,1} & k_{1,2} & k_{1,3} \\ \hline k_{2,1} & k_{2,2} & k_{2,3} \\ \hline k_{3,1} & k_{3,2} & \multicolumn{1}{c}{} \\ \cline{1-2} k_{4,1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \,.$$

- By $\mathrm{YT}(\lambda)$ we denote the **set of all Young tableaux** of shape λ.

Partitions & (Ordered) Young Tableaux

For a partition $\lambda = (\lambda_1, \ldots, \lambda_h)$ and $N \geq 1$ we define the set of **ordered Young tableaux** by

$$\mathrm{OYT}_N(\lambda) = \left\{ \begin{array}{c} \mathbf{m} \in \mathrm{YT}(\lambda) \\ \mathbf{m} = (\lambda, (m_{i,j})) \end{array} \;\middle|\; \begin{array}{c} 0 < m_{i,j} < N \\ m_{i,j} \leq m_{i+1,j} \\ m_{i,j} \leq m_{i,j+1} \\ m_{i,j} < m_{i+1,j+1} \end{array} \right\} .$$

Example for $N = 4$ and $\lambda = (2, 2)$

$$\mathrm{OYT}_4\left(\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array}\right) = \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 3 \\ \hline\end{array}, \ldots, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 2 & 3 \\ \hline\end{array} \right\} .$$

For the set of all ordered Young tableaux we write

$$\mathrm{OYT}(\lambda) = \bigcup_{N \geq 1} \mathrm{OYT}_N(\lambda) .$$

Partitions & (Ordered) Young Tableaux

Definition

For an ordered Young tableau $\mathbf{m} = (\lambda, (m_{i,j})) \in \mathrm{OYT}(\lambda)$ we define the number of **vertical equalities** by

$$v(\mathbf{m}) = \#\left\{(i,j) \in D(\lambda) \mid m_{i,j} = m_{i+1,j}\right\}$$

and the number of **horizontal equalities** by

$$h(\mathbf{m}) = \#\left\{(i,j) \in D(\lambda) \mid m_{i,j} = m_{i,j+1}\right\} \,.$$

Example:

$$v\left(\begin{array}{|c|c|c|}\hline 2&2&3\\\hline 2&3\\\cline{1-2} 2&4\\\cline{1-2} 2\\\cline{1-1}\end{array}\right) = 3 \quad , \quad h\left(\begin{array}{|c|c|c|}\hline 2&2&3\\\hline 2&3\\\cline{1-2} 2&4\\\cline{1-2} 2\\\cline{1-1}\end{array}\right) = 1\,.$$

Interpolated Schur multiple zeta values

Definition

For a Young tableau $\mathbf{k} = (\lambda, (k_{i,j})) \in \mathrm{YT}(\lambda)$ and $N \geq 1$ we define the (harmonic) **interpolated Schur multiple zeta value** by

$$\zeta_N^t(\mathbf{k}) = \sum_{\substack{\mathbf{m} \in \mathrm{OYT}_N(\lambda) \\ \mathbf{m} = (\lambda, (m_{i,j}))}} t^{v(\mathbf{m})} (1-t)^{h(\mathbf{m})} \prod_{(i,j) \in D(\lambda)} \frac{1}{m_{i,j}^{k_{i,j}}} \in \mathbb{Q}[t]\,.$$

Interpolated Schur multiple zeta values

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$$\zeta_N^t(\mathbf{k}) = \sum_{\substack{\mathbf{m} \in \mathrm{OYT}_N(\lambda) \\ \mathbf{m}=(\lambda,(m_{i,j}))}} t^{v(\mathbf{m})}(1-t)^{h(\mathbf{m})} \prod_{(i,j) \in D(\lambda)} \frac{1}{m_{i,j}^{k_{i,j}}} \in \mathbb{Q}[t]\,.$$

Example for $N = 3$, $\lambda = (2,1)$ and $a, b, c \in \mathbb{Z}$

$$\mathrm{OYT}_3\left(\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}\right) = \left\{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\right\}\,.$$

$$\zeta_3^t\left(\begin{array}{|c|c|}\hline a & b \\ \hline c \\ \cline{1-1}\end{array}\right) = t(1-t) + \frac{t}{2^b} + \frac{1-t}{2^c} + \frac{1}{2^{b+c}} + \frac{t(1-t)}{2^{a+b+c}}\,.$$

Interpolated Schur multiple zeta values

Interpolated Schur multiple zeta values generalize interpolated multiple zeta values, since for $\lambda = (1, \dots, 1)$ we have:

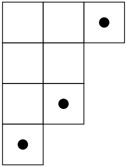
$$\zeta_N^t \left(\begin{array}{|c|}\hline k_1 \\ \hline \vdots \\ \hline k_r \\ \hline \end{array} \right) = \sum_{\mathbf{m} = \begin{array}{|c|}\hline m_1 \\ \hline \vdots \\ \hline m_r \\ \hline \end{array} \in \mathrm{OYT}_N \left(\begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline \end{array} \right)} \frac{t^{v(\mathbf{m})} (1-t)^{\overbrace{h(\mathbf{m})}^{=0}}}{m_1^{k_1} \cdots m_r^{k_r}}$$

$$= \sum_{0 < m_1 \leq \cdots \leq m_r < N} \frac{t^{e(m_1, \dots, m_r)}}{m_1^{k_1} \cdots m_r^{k_r}} = \zeta_N^t (k_1, \dots, k_r) \, .$$

Interpolated Schur multiple zeta values

Define the **corners** of λ by

$$\operatorname{Cor}(\lambda) = \{(i,j) \in D(\lambda) \mid (i,j+1) \notin D(\lambda) \text{ and } (i+1,j) \notin D(\lambda)\}\ .$$



A Young tableau $\mathbf{k} = (\lambda, (k_{i,j})) \in \operatorname{YT}(\lambda)$ with

$$k_{i,j} \geq 1 \text{ for } (i,j) \notin \operatorname{Cor}(\lambda) \text{ and}$$
$$k_{i,j} \geq 2 \text{ for } (i,j) \in \operatorname{Cor}(\lambda)$$

is called **admissible**.

Lemma

For an admissible Young tableau $\mathbf{k}$ the following limit exists:

$$\zeta^t(\mathbf{k}) := \lim_{N\to\infty} \zeta^t_N(\mathbf{k}) \in \mathbb{R}[t]\,.$$

(Interpolated) Schur multiple zeta values

Definition

For a partition λ the set of **semi-standard Young tableaux** is given by

$$\mathrm{SSYT}(\lambda) = \{\mathbf{m} \in \mathrm{OYT}(\lambda) \mid v(\mathbf{m}) = 0\} \ .$$

For an admissible Young tableau $\mathbf{k} = (\lambda, (k_{i,j})) \in \mathrm{YT}(\lambda)$ the **Schur multiple zeta value** (Y. Yamasaki) is defined by

$$\zeta(\mathbf{k}) = \sum_{\substack{\mathbf{m}\in\mathrm{SSYT}(\lambda) \\ \mathbf{m}=(\lambda,(m_{i,j}))}} \prod_{(i,j)\in D(\lambda)} \frac{1}{m_{i,j}^{k_{i,j}}}$$

(Interpolated) Schur multiple zeta values

Interpolated Schur multiple zeta values generalize Schur multiple zeta values, since for $t = 0$ we have

$$\begin{aligned}
\zeta^0(\mathbf{k}) &= \sum_{\substack{\mathbf{m} \in \mathrm{OYT}(\lambda) \\ \mathbf{m} = (\lambda, (m_{i,j}))}} 0^{v(\mathbf{m})} (1-0)^{h(\mathbf{m})} \prod_{(i,j) \in D(\lambda)} \frac{1}{m_{i,j}^{k_{i,j}}} \\
&= \sum_{\substack{\mathbf{m} \in \mathrm{OYT}(\lambda) \\ \mathbf{m} = (\lambda, (m_{i,j})) \\ v(\mathbf{m}) = 0}} \prod_{(i,j) \in D(\lambda)} \frac{1}{m_{i,j}^{k_{i,j}}} = \zeta(\mathbf{k})\,.
\end{aligned}$$

Interpolated Schur multiple zeta values

The transformation $t \to 1 - t$ corresponds to the conjugation $\mathbf{k} \to \mathbf{k}'$:

$$\begin{aligned}
\zeta_N^{1-t}(\mathbf{k}) &= \sum_{\substack{\mathbf{m}\in \mathrm{OYT}_N(\lambda)\\ \mathbf{m}=(\lambda,(m_{i,j}))}} (1-t)^{v(\mathbf{m})} t^{h(\mathbf{m})} \prod_{(i,j)\in D(\lambda)} \frac{1}{m_{i,j}^{k_{i,j}}} \\
&= \sum_{\substack{\mathbf{m}\in \mathrm{OYT}_N(\lambda)\\ \mathbf{m}=(\lambda,(m_{i,j}))}} (1-t)^{h(\mathbf{m}')} t^{v(\mathbf{m}')} \prod_{(i,j)\in D(\lambda)} \frac{1}{m_{i,j}^{k_{i,j}}} \\
&= \sum_{\substack{\mathbf{m}\in \mathrm{OYT}_N(\lambda')\\ \mathbf{m}=(\lambda,(m_{i,j}))}} (1-t)^{h(\mathbf{m})} t^{v(\mathbf{m})} \prod_{(i,j)\in D(\lambda')} \frac{1}{m_{i,j}^{k_{j,i}}} = \zeta_N^{t}(\mathbf{k}')
\end{aligned}$$

Interpolated Schur multiple zeta values

Summarizing Proposition

- In the case $\lambda = (1, \dots, 1)$ interpolated Schur multiple zeta values are interpolated multiple zeta values:

$$\zeta_N^t\left(\begin{array}{|c|}\hline k_1 \\ \hline \vdots \\ \hline k_r \\ \hline\end{array}\right) = \zeta_N^t(k_1, \dots, k_r)\,.$$

- For an admissible Young tableau $\mathbf{k} \in \mathrm{YT}(\lambda)$ we obtain the Schur multiple zeta values for $t = 0$:

$$\zeta^0(\mathbf{k}) = \zeta(\mathbf{k})\,.$$

- For all $N \geq 1$ and a Young tableau $\mathbf{k} \in \mathrm{YT}(\lambda)$ it is

$$\zeta_N^{1-t}(\mathbf{k}) = \zeta_N^t(\mathbf{k}')\,.$$

Interpolated Schur multiple zeta values

In the following $(a_i)_{i\in\mathbb{Z}}$ is a family of arbitrary integers $a_i \in \mathbb{Z}$.

Theorem (B. 2016)

For $N \geq 1$ and a Young tableau $\mathbf{k} = (\lambda, (k_{i,j}))$ with $k_{i,j} = a_{j-i}$ we have

$$\zeta_N^t(\mathbf{k}) = \det(\, \zeta_N^t(a_{j-1}, a_{j-2}, \ldots, a_{j-(\lambda'_i - i + j)})\,)_{1\leq i,j\leq \lambda_1}\ ,$$

where we set the (i, j)-th entry to be $\begin{cases} 1 \text{ if } \lambda'_i - i + j = 0 \\ 0 \text{ if } \lambda'_i - i + j < 0 \end{cases}$.

The condition $k_{i,j} = a_{j-i}$ means that $\mathbf{k}$ looks like this:

$$\mathbf{k} = \begin{array}{|c|c|c|c|c}
\cline{1-4}
a_0 & a_1 & a_2 & a_3 & \cdots \\
\cline{1-4}
a_{-1} & a_0 & a_1 & \multicolumn{1}{c}{\cdots} & \\
\cline{1-3}
a_{-2} & a_{-1} & \multicolumn{1}{c}{\ddots} & & \\
\cline{1-2}
\multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{\vdots} & & &
\end{array}$$

Interpolated Schur multiple zeta values

Theorem (B. 2016)

For $N \geq 1$ and a Young tableau $\mathbf{k} = (\lambda, (k_{i,j}))$ with $k_{i,j} = a_{j-i}$ we have

$$\zeta_N^t(\mathbf{k}) = \det(\, \zeta_N^t(a_{j-1}, a_{j-2}, \ldots, a_{j-(\lambda_i'-i+j)})\,)_{1 \leq i,j \leq \lambda_1} \,,$$

where we set the (i,j)-th entry to be $\begin{cases} 1 \text{ if } \lambda_i' - i + j = 0 \\ 0 \text{ if } \lambda_i' - i + j < 0 \end{cases}$.

- The cases $t = 0$ and $t = 1$ of this Theorem were originally proven by M. Nakasuji and O. Phuksuwan (2016).
- Our definition of interpolated Schur multiple zeta values was motivated by their work.

Interpolated Schur multiple zeta values

Theorem (B. 2016)

For $N \geq 1$ and a Young tableau $\mathbf{k} = (\lambda, (k_{i,j}))$ with $k_{i,j} = a_{j-i}$ we have

$$\zeta_N^t(\mathbf{k}) = \det(\, \zeta_N^t(a_{j-1}, a_{j-2}, \dots, a_{j-(\lambda_i'-i+j)})\,)_{1 \leq i,j \leq \lambda_1} \;,$$

where we set the (i,j)-th entry to be $\begin{cases} 1 \text{ if } \lambda_i' - i + j = 0 \\ 0 \text{ if } \lambda_i' - i + j < 0 \end{cases}$.

Example

For $\lambda = (2,1,1)$ it is $\lambda' = (3,1)$ and the Theorem states that

$$\zeta_N^t \left(\begin{array}{|c|c|} \hline a_0 & a_1 \\ \hline a_{-1} & \\ \cline{1-1} a_{-2} & \\ \cline{1-1} \end{array} \right) = \det \begin{pmatrix} \zeta_N^t(a_0, a_{-1}, a_{-2}) & \zeta_N^t(a_1, a_0, a_{-1}, a_{-2}) \\ 1 & \zeta_N^t(a_1) \end{pmatrix} .$$

Interpolated Schur multiple zeta values

By the Proposition before we know that

$$\zeta_N^t\left(\begin{array}{|c|c|}\hline a_0 & a_1 \\ \hline a_{-1} & \multicolumn{1}{c}{} \\ \cline{1-1} a_{-2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) = \zeta_N^{1-t}\left(\begin{array}{|c|c|c|}\hline a_0 & a_{-1} & a_{-2} \\ \hline a_1 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\right).$$

For this we can use the Theorem again which gives

$$\det\begin{pmatrix} \zeta_N^{1-t}(a_0,a_1) & \zeta_N^{1-t}(a_{-1},a_0,a_1) & \zeta_N^{1-t}(a_{-2},a_{-1},a_0,a_1) \\ 1 & \zeta_N^{1-t}(a_{-1}) & \zeta_N^{1-t}(a_{-2},a_{-1}) \\ 0 & 1 & \zeta_N^{1-t}(a_{-2}) \end{pmatrix}$$
$$= \det\begin{pmatrix} \zeta_N^{t}(a_0,a_{-1},a_{-2}) & \zeta_N^{t}(a_1,a_0,a_{-1},a_{-2}) \\ 1 & \zeta_N^{t}(a_1) \end{pmatrix}$$

Interpolated Schur multiple zeta values

As a Corollary of the Theorem and the Proposition we obtain the following family of algebraic relations between interpolated multiple zeta values ζ_N^t and ζ_N^{1-t}:

Corollary

With the same Notation as before we have for every partition λ and $N \geq 1$

$$\det\left(\zeta_N^t(a_{j-1}, a_{j-2}, \dots, a_{j-(\lambda_i'-i+j)}) \right)_{1\leq i,j\leq \lambda_1}$$

$$=$$

$$\det\left(\zeta_N^{1-t}(a_{1-j}, a_{2-j}, \dots, a_{(\lambda_i-i+j)-j}) \right)_{1\leq i,j\leq \lambda_1'} .$$

- If $a_j \geq 2$ for $j \in \mathbb{Z}$ this gives algebraic relations between ζ^t and ζ^{1-t}.
- In particular for $t = \frac{1}{2}$ we get algebraic relations for $\zeta^{\frac{1}{2}}$.

Interpolated Schur multiple zeta values

Choosing $\lambda = (r)$ and setting $a_j = k_{j+1}$ we obtain for $k_1, \dots, k_r \geq 2$ the identity

$$\zeta^{1-t}(k_1, \dots, k_r) =$$

$$\det \begin{pmatrix} \zeta^t(k_1) & \zeta^t(k_2, k_1) & \zeta^t(k_3, k_2, k_1) & \cdots & \zeta^t(k_r, \dots, k_1) \\ 1 & \zeta^t(k_2) & \zeta^t(k_3, k_2) & & \zeta^t(k_r, \dots, k_2) \\ 0 & 1 & \zeta^t(k_3) & \zeta^t(k_4, k_3) & \vdots \\ \vdots & \ddots & \ddots & \ddots & \zeta^t(k_r, k_{r-1}) \\ 0 & \cdots & 0 & 1 & \zeta^t(k_r) \end{pmatrix} .$$

Lemma of Lindström, Gessel & Viennot

A graph is a tupel $G = (V, E)$ of vertices V and edges E.

Our graphs today are all:

- **finite**

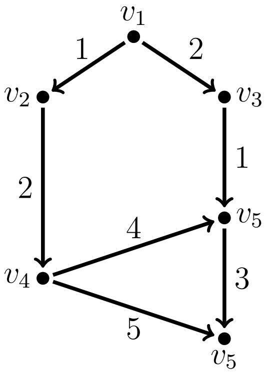
  V and E are finite.
- **directed**

  Edges have a direction.
- **acyclic**

  There are no cycles with respect to the direction.
- **weighted**

  We have a map

  $$w : E \to \mathbb{Q}[t]\,.$$

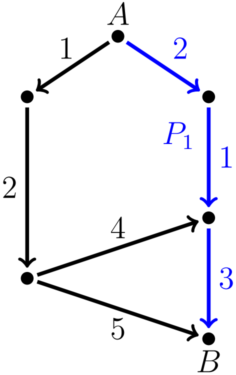


$$V = \{v_1, \dots, v_5\}$$

$$E = \{e_1, \dots, e_7\}$$

$$w(e_1) = 1,\ \dots,\ w(e_7) = 5$$

Lemma of Lindström, Gessel & Viennot



For vertices $A, B \in V$ we define the

- **weight of a path** $P : A \to B$ by

$$w(P) = \prod_{e \text{ edge in } P} w(e)\,.$$

- **weight from** A **to** B by

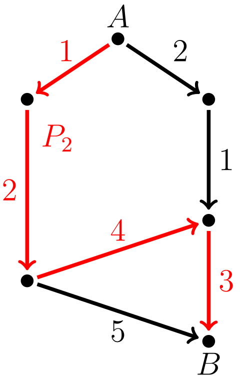
$$w(A,B) = \sum_{P:A\to B} w(P)\,.$$

Example:

$$w(P_1) = 2 \cdot 1 \cdot 3 = 6$$

$$w(A,B) = 6 + \dots$$

Lemma of Lindström, Gessel & Viennot



For vertices $A, B \in V$ we define the

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$$w(P) = \prod_{e \text{ edge in } P} w(e)\,.$$

- **weight from A to B by**

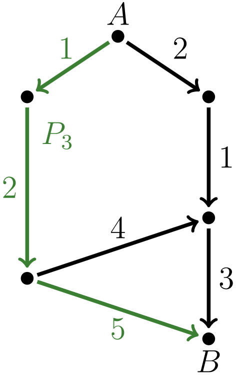
$$w(A,B) = \sum_{P:A\to B} w(P)\,.$$

Example:

$$w(P_1) = 2\cdot 1\cdot 3 = 6\,,\quad w(P_2) = 1\cdot 2\cdot 4\cdot 3 = 24$$

$$w(A,B) = 6 + 24 + \dots$$

Lemma of Lindström, Gessel & Viennot



For vertices $A, B \in V$ we define the

- **weight of a path** $P : A \to B$ by

$$w(P) = \prod_{e \text{ edge in } P} w(e)\,.$$

- **weight from** A **to** B by

$$w(A,B) = \sum_{P:A\to B} w(P)\,.$$

Example:

$$w(P_1) = 2 \cdot 1 \cdot 3 = 6\,, \quad w(P_2) = 1 \cdot 2 \cdot 4 \cdot 3 = 24$$

$$w(P_2) = 1 \cdot 2 \cdot 5 = 10$$

$$w(A,B) = 6 + 24 + 10 = 40\,.$$

Lemma of Lindström, Gessel & Viennot

Let $\mathcal{A} = \{A_1, \ldots, A_n\} \subset V$ and $\mathcal{B} = \{B_1, \ldots, B_n\} \subset V$ be two subsets of vertices with the same cardinality n.

Definition

A **vertex-disjoint path system** $\mathcal{P} : \mathcal{A} \to \mathcal{B}$ is a collection of vertex-disjoint paths

$$P_i : A_i \to B_{\sigma(i)} \,, \quad i = 1, \ldots, n$$

with $\sigma \in \Sigma_n$. We also write $\mathcal{P} = (P_1, \ldots, P_n)$.

- **Sign** of $\mathcal{P}$:

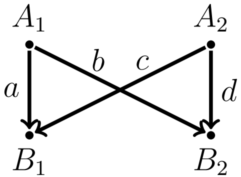
$$\operatorname{sign} \mathcal{P} := \operatorname{sign} \sigma \,.$$

- **Weight** of $\mathcal{P}$:

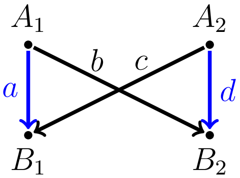
$$w(\mathcal{P}) := \prod_{j=1}^{n} w(P_i) \,.$$

Lemma of Lindström, Gessel & Viennot

Consider $\mathcal{A} = \{A_1, A_2\}$ and $\mathcal{B} = \{B_1, B_2\}$ in the following graph:



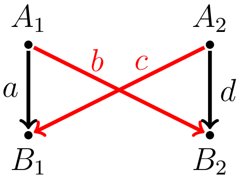
This graph hast two vertex-disjoint path systems $\mathcal{P} : \mathcal{A} \to \mathcal{B}$:



$$\mathcal{P} = (P_1, P_2)$$

$$w(\mathcal{P}) = w(P_1) \cdot w(P_2) = a \cdot d$$

$$\operatorname{sign}(\mathcal{P}) = 1$$



$$\mathcal{P} = (P_1, P_2)$$

$$w(\mathcal{P}) = w(P_1) \cdot w(P_2) = b \cdot c$$

$$\operatorname{sign}(\mathcal{P}) = -1$$

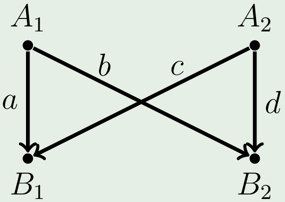
Lemma of Lindström, Gessel & Viennot

Lemma (Lindström, Gessel & Viennot)

For a finite, weighted, acyclic, directed graph $G = (V, E)$ and two subsets of V $\mathcal{A} = \{A_1, \ldots, A_n\}$ and $\mathcal{B} = \{B_1, \ldots, B_n\}$ it is

$$\sum_{\mathcal{P}\,:\,\mathcal{A}\to\mathcal{B}} \operatorname{sign}\mathcal{P} \cdot w(\mathcal{P}) = \det(w(A_i, B_j))_{1\le i,j\le n}\,,$$

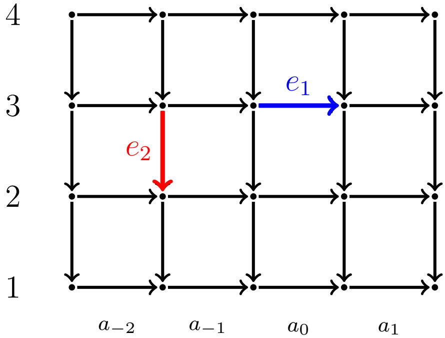
Example



$$ad - bc = \det\begin{pmatrix} w(A_1, B_1) & w(A_1, B_2) \\ w(A_2, B_1) & w(A_2, B_2) \end{pmatrix}$$
$$= \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Lemma of Lindström, Gessel & Viennot: Example

We now consider lattice graphs of the following form ($a_j \in \mathbb{Z}$)



where the weights of the marked edges are given by

$$w(e_1) = \frac{1}{3^{a_0}} \quad \text{and} \quad w(e_2) = 1\,.$$

Lemma of Lindström, Gessel & Viennot: Example

More precisely we consider the Lattice graph $G = (E, V)$ with

- Vertices:

$$V = \{(x, y) \mid -2 \leq x \leq 2\,, 1 \leq y \leq 4\}\,.$$

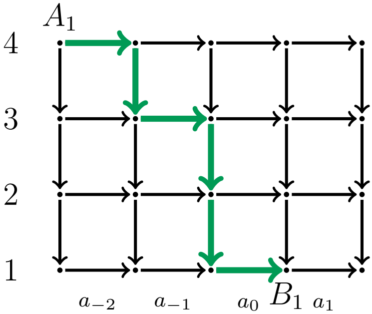
- Edges: $E = E^{hor} \cup E^{ver}$

$$E^{hor} = \{(x, y) \to (x+1, y) \mid -2 \leq x \leq 1\,, 1 \leq y \leq 4\}$$
$$E^{ver} = \{(x, y) \to (x, y-1) \mid -2 \leq x \leq 2\,, 2 \leq y \leq 4\}$$

- Weights:
  - For a horizontal edge $e \in E^{hor}$ from (x, y) to $(x+1, y)$ we define the weight by
$$w(e) = \frac{1}{y^{a_x}}\,.$$
  - For a vertical edge $e \in E^{ver}$ we set $w(e) = 1$.

Lemma of Lindström, Gessel & Viennot: Example

Now consider the following paths from A_1 to B_1:

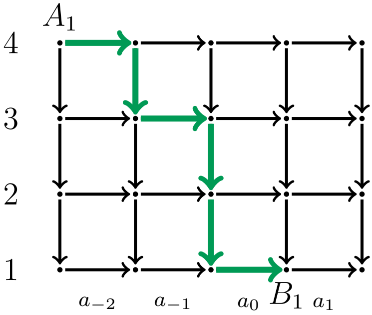


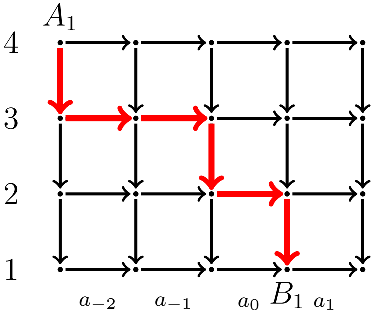
$$w(P) = \frac{1}{4^{a_{-2}}} \cdot 1 \cdot \frac{1}{3^{a_{-1}}} \cdot 1^2 \cdot \frac{1}{1^{a_0}}$$

$$= \frac{1}{1^{a_0} 3^{a_{-1}} 4^{a_{-2}}}$$

Lemma of Lindström, Gessel & Viennot: Example

Now consider the following paths from A_1 to B_1:





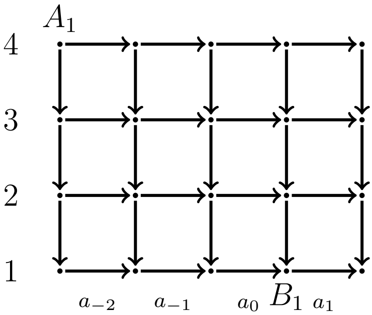
$$w(P) = \frac{1}{4^{a_{-2}}} \cdot 1 \cdot \frac{1}{3^{a_{-1}}} \cdot 1^2 \cdot \frac{1}{1^{a_0}} \qquad w(P) = 1 \cdot \frac{1}{3^{a_{-2}}} \cdot \frac{1}{3^{a_{-1}}} \cdot 1 \cdot \frac{1}{2^{a_0}} \cdot 1$$

$$= \frac{1}{1^{a_0} 3^{a_{-1}} 4^{a_{-2}}} \qquad = \frac{1}{2^{a_0} 3^{a_{-1}} 3^{a_{-2}}}$$

Lemma of Lindström, Gessel & Viennot: Example

It is easy to see that we have

$$w(A_1, B_1) = \sum_{P: A_1 \to B_1} w(P)$$
$$= \sum_{0<m_1\le m_2\le m_3<5} \frac{1}{m_1^{a_0} m_2^{a_{-1}} m_3^{a_{-2}}} = \zeta_5^1(a_0, a_{-1}, a_{-2})\,.$$

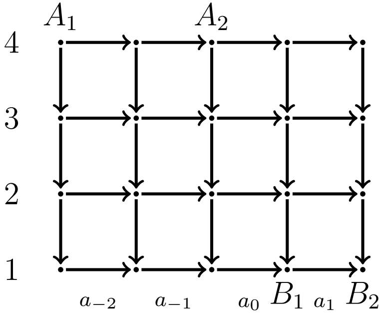


Lemma of Lindström, Gessel & Viennot: Example

With the same arguments we also obtain that

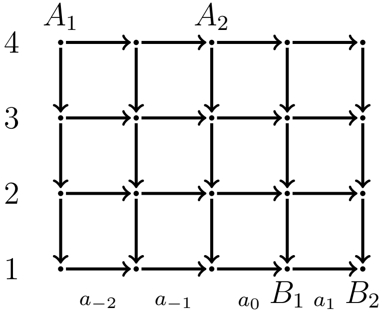
$$\begin{aligned} w(A_1,B_1) &= \zeta_5^1(a_0,a_{-1},a_{-2})\,, \quad & w(A_1,B_2) &= \zeta_5^1(a_1,a_0,a_{-1},a_{-2}) \\ w(A_2,B_1) &= \zeta_5^1(a_0)\,, & w(A_2,B_2) &= \zeta_5^1(a_1,a_0) \end{aligned}$$

for $\mathcal{A} = \{A_1, A_2\}$ and $\mathcal{B} = \{B_1, B_2\}$ given by



Lemma of Lindström, Gessel & Viennot: Example

Question: How do vertex-disjoint path systems $\mathcal{P}$ from $\mathcal{A} = \{A_1, A_2\}$ to $\mathcal{B} = \{B_1, B_2\}$ look like?



Claim

$$\sum_{\mathcal{P}\,:\,\mathcal{A}\to\mathcal{B}} \operatorname{sign}\mathcal{P}\cdot w(\mathcal{P}) = \zeta_5^1\left(\begin{array}{|c|c|}\hline a_0 & a_1 \\ \hline a_{-1} & a_0 \\ \hline a_{-2} & \\ \cline{1-1}\end{array}\right).$$

Lemma of Lindström, Gessel & Viennot: Example

Notice that in this example we always have $\operatorname{sign}\mathcal{P} = 1$, i.e. we want to show

$$\sum_{\mathcal{P}\,:\,\mathcal{A}\to\mathcal{B}} w(\mathcal{P}) \overset{!}{=} \zeta_5^1 \left(\begin{ytableau} a_0 & a_1 \\ a_{-1} & a_0 \\ a_{-2} \end{ytableau} \right)$$

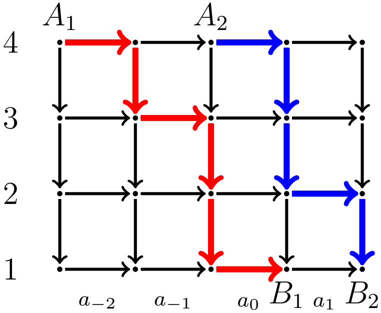
$$= \sum_{\mathbf{m}\in\mathrm{OYT}_5\left(\ydiagram{2,2,1}\right)} 1^{v(\mathbf{m})}(1-1)^{h(\mathbf{m})} \prod_{(i,j)\in D\left(\ydiagram{2,2,1}\right)} \frac{1}{m_{i,j}^{a_{j-i}}}$$

$$= \sum_{\substack{\mathbf{m}\in\mathrm{OYT}_5\left(\ydiagram{2,2,1}\right)\\ h(\mathbf{m})=0}} \prod_{(i,j)\in D\left(\ydiagram{2,2,1}\right)} \frac{1}{m_{i,j}^{a_{j-i}}}$$

We will do this by illustrating the following 1:1 correspondence

$$\left\{ \begin{array}{c} \text{Vertex-disjoint path systems} \\ \mathcal{P}\,:\,\mathcal{A}\to\mathcal{B} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Ordered Young tableaux} \\ \mathbf{m}\in\mathrm{OYT}_5\left(\ydiagram{2,2,1}\right) \text{ with } h(\mathbf{m}) = 0 \end{array} \right\}$$

Lemma of Lindström, Gessel & Viennot: Example

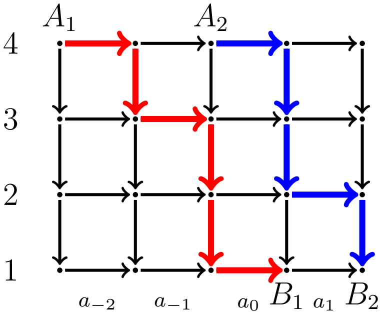


$$\mathcal{P} = (P_1, P_2)$$

$$w(P_1) = \frac{1}{1^{a_0} 3^{a_{-1}} 4^{a_{-2}}}$$

$$w(P_2) = \frac{1}{2^{a_1} 4^{a_0}}$$

Lemma of Lindström, Gessel & Viennot: Example



The vertex-disjoint path system $\mathcal{P}$ corresponds to

$$\mathbf{m} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 4 & \\ \cline{1-1} \end{array} \in \mathrm{OYT}_5\left(\ydiagram{2,2,1}\right)$$

It is $h(\mathbf{m}) = 0$ and we have

$$\mathcal{P} = (P_1, P_2)$$

$$w(P_1) = \frac{1}{1^{a_0}3^{a_{-1}}4^{a_{-2}}}$$

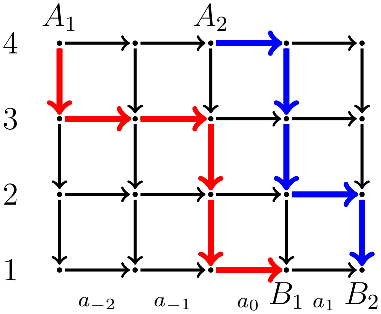
$$w(P_2) = \frac{1}{2^{a_1}4^{a_0}}$$

$$\prod_{(i,j)\in D\left(\ydiagram{2,2,1}\right)} \frac{1}{m_{i,j}^{a_{j-i}}} = \frac{1}{1^{a_0}3^{a_{-1}}4^{a_{-2}}2^{a_1}4^{a_0}}$$

$$= w(P_1) \cdot w(P_2)$$

$$= w(\mathcal{P})$$

Lemma of Lindström, Gessel & Viennot: Example



The vertex-disjoint path system $\mathcal{P}$ corresponds to

$$\mathbf{m} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 3 & \\ \cline{1-1}\end{array} \in \mathrm{OYT}_5\left(\ydiagram{2,2,1}\right)$$

It is $h(\mathbf{m}) = 0$ and we have

$$\mathcal{P} = (P_1, P_2)$$

$$w(P_1) = \frac{1}{1^{a_0} 3^{a_{-1}} 3^{a_{-2}}}$$

$$w(P_2) = \frac{1}{2^{a_1} 4^{a_0}}$$

$$\prod_{(i,j)\in D\left(\ydiagram{2,2,1}\right)} \frac{1}{m_{i,j}^{a_{j-i}}} = \frac{1}{1^{a_0} 3^{a_{-1}} 3^{a_{-2}} 2^{a_1} 4^{a_0}}$$

$$= w(P_1) \cdot w(P_2)$$

$$= w(\mathcal{P})$$

Lemma of Lindström, Gessel & Viennot: Example

We therefore have

$$\sum_{\mathcal{P}\,:\,\mathcal{A}\to\mathcal{B}} \operatorname{sign}\mathcal{P}\cdot w(\mathcal{P}) = \zeta_5^1\left(\begin{array}{|c|c|}\hline a_0 & a_1 \\ \hline a_{-1} & a_0 \\ \hline \multicolumn{1}{|c|}{a_{-2}} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right)$$

and by the Lemma of Lindström, Gessel & Viennot this equals

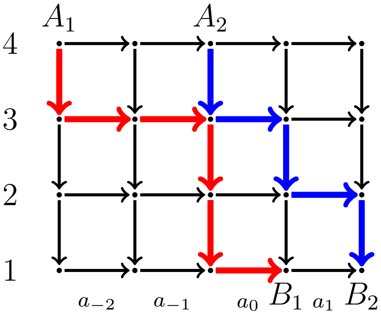
$$\det(w(A_i,B_j))_{1\le i,j\le 2} = \begin{pmatrix} \zeta_5^1(a_0,a_{-1},a_{-2}) & \zeta_5^1(a_1,a_0,a_{-1},a_{-2}) \\ \zeta_5^1(a_0) & \zeta_5^1(a_1,a_0) \end{pmatrix},$$

which is the statement of the Theorem for $\lambda=(2,2,1)$, $t=1$ and $N=5$.

Proof sketch of the Theorem

Question

How to include the parameter t and allow Young tableaux with horizontal equalities?



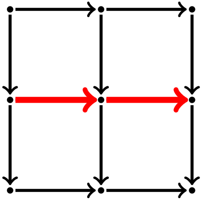
This path system (P_1, P_2) is **not** vertex-disjoint!

The "path system" (P_1, P_2) "corresponds" to

$$\mathbf{m} = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 3 \\ \hline 3 & \\ \cline{1-1} \end{array} \in \mathrm{OYT}_5\left(\begin{smallmatrix}\square&\square\\\square&\square\\\square&\end{smallmatrix}\right)$$

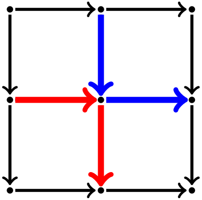
It is $h(\mathbf{m}) = 1$ and $v(\mathbf{m}) = 1$.

Proof sketch of the Theorem



If a path goes to the right twice we want to obtain a factor t.

$\rightsquigarrow$ vertical equality of $\mathbf{m}$



If two paths "touch" we want to obtain a factor $1 - t$.

$\rightsquigarrow$ horizontal equality of $\mathbf{m}$

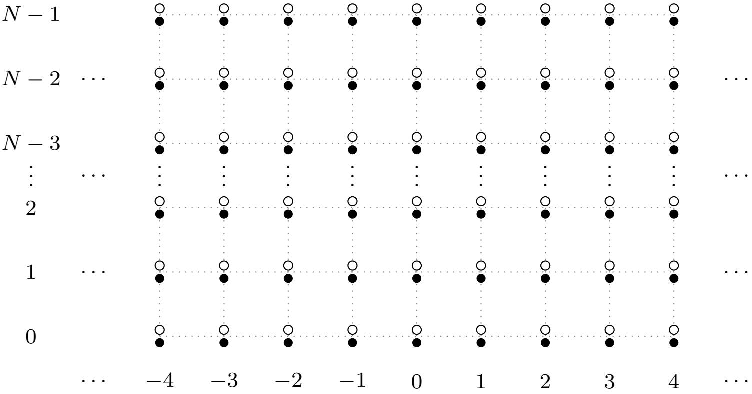
Solution

Blow up the vertices and replace each vertex by two new vertices and then use the same combinatorics and the Lemma of Lindström, Gessel & Viennot as before.

Proof sketch of the Theorem: t-Lattice Graph

For a fixed $N \geq 1$ we consider the following set of Vertices

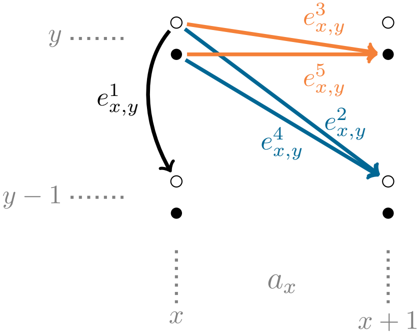
$$V = \left\{ v^{*}_{x,y} \mid x \in \mathbb{Z},\, 0 \leq y < N,\, * \in \{\bullet, \circ\} \right\} .$$



Proof sketch of the Theorem: t-Lattice Graph

For each position (x, y) we will have five different types of outgoing edges

$$E := \left\{ e^i_{x,y} \mid i = 1, \ldots, 5\,, x \in \mathbb{Z},\, 0 < y < N \right\}\,.$$



Weights:

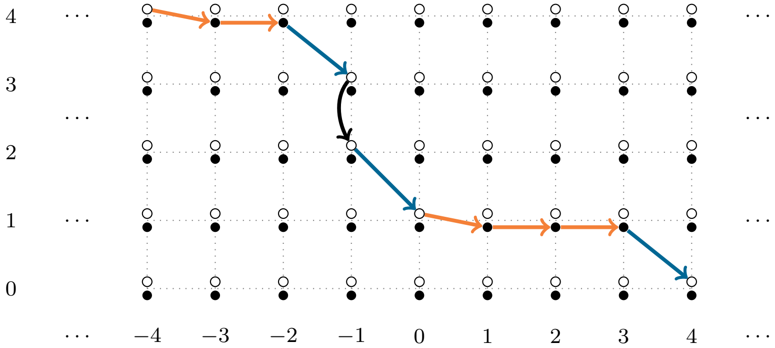
$$w(e^1_{x,y}) = 1$$

$$w(e^2_{x,y}) = w(e^4_{x,y}) = \frac{1}{y^{a_x}}$$

$$w(e^3_{x,y}) = w(e^5_{x,y}) = t \cdot \frac{1}{y^{a_x}}$$

Proof sketch of the Theorem: t-Lattice Graph

As an example consider the following path P from $A = v^{\circ}_{-4,4}$ to $B = v^{\circ}_{4,0}$.



$$w(P) = t\frac{1}{4^{a_{-4}}} \cdot t\frac{1}{4^{a_{-3}}} \cdot \frac{1}{4^{a_{-2}}} \cdot 1 \cdot \frac{1}{2^{a_{-1}}} \cdot t\frac{1}{1^{-a_0}} \cdot t\frac{1}{1^{-a_1}} \cdot t\frac{1}{1^{-a_2}} \cdot \frac{1}{1^{-a_3}}$$

$$= \frac{t^5}{2^{a_{-1}}4^{a_{-2}}4^{a_{-3}}4^{a_{-4}}}.$$

Proof sketch of the Theorem: t-Lattice Graph

This leads to the following Lemma:

Lemma 1

For integers $i \leq j$ the sum of all Paths weights of paths between the vertex $A = v^{\circ}_{i,N-1}$ and $B = v^{\circ}_{j+1,0}$ is given by

$$w(A,B) = \zeta^t_N(a_j, a_{j-1}, \ldots, a_i)\,.$$

- This gives the "ingredients" for the RHS of the Theorem.
- For a given partition λ we now need to define sets $\mathcal{A}$ and $\mathcal{B}$ to explain the LHS of the Theorem.

Proof sketch of the Theorem: t-Lattice Graph

For a partition $\lambda = (\lambda_1, \ldots, \lambda_h)$ with conjugate $\lambda' = (\lambda'_1, \ldots, \lambda'_{h'})$ we define sets $\mathcal{A}_\lambda := \{A_1, \ldots, A_{\lambda_1}\}$ and $\mathcal{B}_\lambda := \{B_1, \ldots, B_{\lambda_1}\}$, by

$$A_j = v^\circ_{j-\lambda'_j, N-1} \quad \text{and} \quad B_j = v^\circ_{j,0}\,.$$

With this notation we can state the following Lemma.

Lemma2

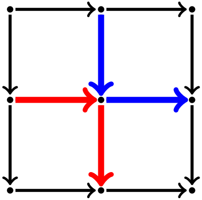
The interpolated Schur multiple zeta value $\zeta^t_N(\mathbf{k})$ for the young tableau $\mathbf{k} = (\lambda, k_{i,j})$ with $k_{i,j} = a_{j-i}$ is given by

$$\zeta^t_N(\mathbf{k}) = \sum_{\mathcal{P}\,:\,\mathcal{A}_\lambda \to \mathcal{B}_\lambda} \operatorname{sign} \mathcal{P} \cdot w(\mathcal{P})\,.$$

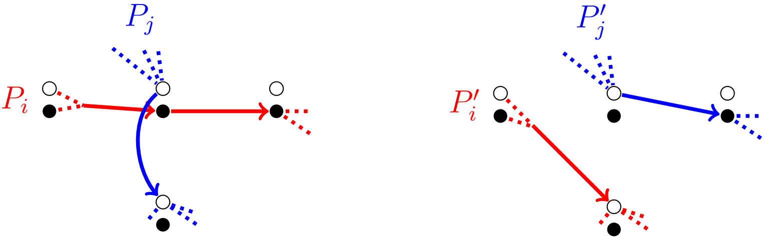
Lemma 1, Lemma 2 and the Lemma of Lindström, Gessel & Viennot imply the Theorem.

Proof sketch of Lemma 2

- The proof uses similar arguments as in the $t = 1$ example except that we need to take care of crossings.
- One of these crossings gives the factor $(1 - t)$ and corresponds to the "touching of paths", which we described before:



Proof sketch of Lemma 2



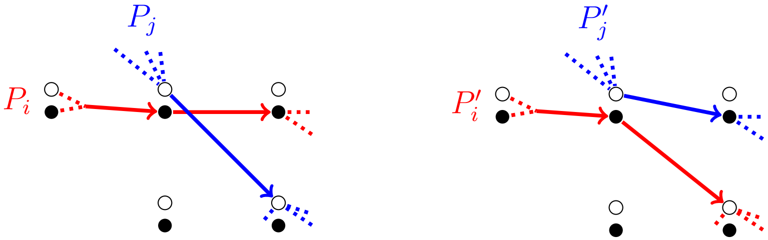
For every vertex-disjoint path system $\mathcal{P} : \mathcal{A}_\lambda \to \mathcal{B}_\lambda$ containing P_i and P_j there is a vertex-disjoint path system $\mathcal{P}' : \mathcal{A}_\lambda \to \mathcal{B}_\lambda$ containing P'_i and P'_j and which just differ as illustrated above. It is

$$\operatorname{sign}(\mathcal{P})w(\mathcal{P}) = -t\operatorname{sign}(\mathcal{P}')w(\mathcal{P}')$$

and therefore their sum gives

$$(1-t)\operatorname{sign}(\mathcal{P}')w(\mathcal{P}')\,.$$

Proof sketch of Lemma 2



For every vertex-disjoint path system $\mathcal{P} : \mathcal{A}_\lambda \to \mathcal{B}_\lambda$ containing P_i and P_j there is a vertex-disjoint path system $\mathcal{P}' : \mathcal{A}_\lambda \to \mathcal{B}_\lambda$ containing P_i' and P_j' and which just differ as illustrated above. It is

$$\operatorname{sign}(\mathcal{P})w(\mathcal{P}) = -\operatorname{sign}(\mathcal{P}')w(\mathcal{P}')$$

and therefore their sum gives 0.

Summarize

- Interpolated Schur multiple zeta values are polynomials which generalize Schur multiple zeta values and t-multiple zeta values.
- Interpolated Schur multiple zeta values of a certain type can be written as the determinant whose entries are given by t-multiple zeta values.
- The proof also works for other interpolated sums such as t-q-analogues of multiple zeta values (defined by N. Wakabayashi).
- The main Theorem can probably also be proven by using the harmonic product of t-multiple zeta values introduced by S. Yamamoto.

Thank you very much for your attention!

Slides are available here: **www.henrikbachmann.com**