The zoo of multiple Eisenstein series

Henrik Bachmann

Nagoya University

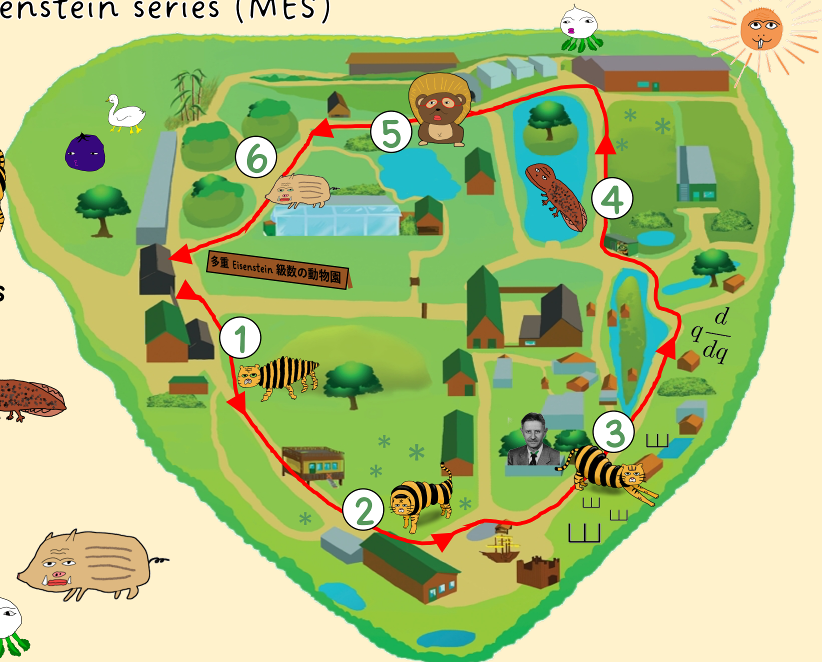
based on joint works with: A. Burmester, J.W. van Ittersum, U. Kühn. N. Matthes, K. Tasaka

多重ゼータ値の諸相, 2022年5月18日 (小野さんお誕生日おめでとう)

Slides are available on: **www.henrikbachmann.com**

The zoo of multiple Eisenstein series (MES)

1. Classical MES
2. Stuffle regularised MES
3. Shuffle regularized MES
4. Combinatorial MES
5. Formal MES
6. Modular MES & Outlook



Definition

For $k_1, \dots, k_r \geq 1, x \in \mathbb{C} \backslash \mathbb{Z}_{<0}$ and $N \geq 1$ we define the **(truncated) multiple Hurwitz zeta function**

$$\zeta_N(k_1, \dots, k_r; x) = \sum_{N > n_1 > \cdots > n_r > 0} \frac{1}{(x+n_1)^{k_1} \dots (x+n_r)^{k_r}},$$

and write $\zeta_N(k_1, \dots, k_r) = \zeta_N(k_1, \dots, k_r; 0)$ for the truncated multiple zeta values.

For $k_1 \geq 2$ the **multiple zeta values** are given by

$$\zeta(k_1, \dots, k_r) = \lim_{N \to \infty} \zeta_N(k_1, \dots, k_r).$$

- **depth** : r
- **weight** : $k_1 + \cdots + k_r$
- $\mathcal{Z}$: $\mathbb{Q}$-algebra of MZVs

(c.f. Komori & Ono talk)

Harmonic product (coming from the definition as iterated sums)

Example in depth two ($k_1, k_2 \geq 1$)

$$\zeta_N(k_1; x) \cdot \zeta_N(k_2; x) = \zeta_N(k_1, k_2; x) + \zeta_N(k_2, k_1; x) + \zeta_N(k_1 + k_2; x)\,.$$

Shuffle product (coming from the expression as iterated integrals)

Example in depth two ($k_1, k_2 \geq 2$)

$$\zeta(k_1) \cdot \zeta(k_2) = \sum_{j=2}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) \zeta(j, k_1 + k_2 - j)\,.$$

Example

$$\begin{aligned} \zeta(2) \cdot \zeta(3) &\overset{\text{harmonic}}{=} \zeta(2,3) + \zeta(3,2) + \zeta(5) \\ &\overset{\text{shuffle}}{=} \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1)\,. \end{aligned}$$

$$\Longrightarrow 2\zeta(3,2) + 6\zeta(4,1) \overset{\text{(finite) double shuffle}}{=} \zeta(5)\,.$$

- L: countable set (set of **letters**).
- $\diamond$: commutative and associative product on $\mathbb{Q}L$.
- **word**: monic monomial in the non-commutative polynomial ring $\mathbb{Q}\langle L\rangle$. ($\mathbf{1}$: empty word)

Definition

The **quasi-shuffle product** $*_\diamond$ on $\mathbb{Q}\langle L\rangle$ is defined as the $\mathbb{Q}$-bilinear product satisfying $\mathbf{1} *_\diamond w = w *_\diamond \mathbf{1} = w$ for any word $w \in \mathbb{Q}\langle L\rangle$ and

$$aw *_\diamond bv = a(w *_\diamond bv) + b(aw *_\diamond v) + (a \diamond b)(w *_\diamond v)$$

for any letters $a, b \in L$ and words $w, v \in \mathbb{Q}\langle L\rangle$.

Theorem (Hoffman)

$(\mathbb{Q}\langle L\rangle, *_\diamond)$ is a commutative $\mathbb{Q}$-algebra. Moreover, this algebra can be equipped with the structure of a Hopf algebra with the coproduct given by

$$\Delta(w) = \sum_{uv=w} u \otimes v\,.$$

- **Harmonic product $*$:** $L_z = \{z_k \mid k \geq 1\}$ and $z_{k_1} \diamond z_{k_2} = z_{k_1+k_2}$ for all $k_1, k_2 \geq 1$.

$$z_2 * z_3 = z_2 z_3 + z_3 z_2 + z_5 \,.$$

We set $\mathfrak{H}^1 = \mathbb{Q}\langle L_z \rangle$. (Compare with: $\zeta(2)\zeta(3) = \zeta(2,3) + \zeta(3,2) + \zeta(5)$)

- **Shuffle product $\sqcup\!\sqcup$:** $L_{xy} = \{x, y\}$ and $a \diamond b = 0$ for $a, b \in L_{xy}$. We write $\mathfrak{H} = \mathbb{Q}\langle x, y \rangle$.

$$xy \sqcup\!\sqcup\, xxy = xyxxy + 3xxyxy + 6xxxyy \,.$$

By identifying $z_k \leftrightarrow \overbrace{x \cdots x}^{k-1} y$ we can also equip $\mathfrak{H}^1$ with the shuffle product, e.g.

$$z_2 \sqcup\!\sqcup\, z_3 = z_2 z_3 + 3 z_3 z_2 + 6 z_4 z_1 \,.$$

(Compare with: $\zeta(2)\zeta(3) = \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1)$)

- **Index shuffle product $\overline{\sqcup\!\sqcup}$:** $L_z = \{z_k \mid k \geq 1\}$ and $z_{k_1} \diamond z_{k_2} = 0$ for all $k_1, k_2 \geq 1$

$$z_2 \,\overline{\sqcup\!\sqcup}\, z_3 = z_2 z_3 + z_3 z_2 \,.$$

- As usual (c.f. Machide talk) we write $\mathfrak{H}^0 = \mathbb{Q} + x\mathfrak{H}y \subset \mathfrak{H}^1$ for the space of **admissible words**.
- As not usual we also consider the following subspace of $\mathfrak{H}^0$

$$\mathfrak{H}^2 = \mathbb{Q} + \langle k_1, \dots, k_r \mid r \geq 1, k_1, \dots, k_r \geq 2 \rangle_{\mathbb{Q}}$$

Both $\mathfrak{H}^0$ and $\mathfrak{H}^2$ are closed under $*$ but only $\mathfrak{H}^0$ is closed under $\sqcup\!\sqcup$. We obtain the following inclusion of $\mathbb{Q}$-algebras

$$\mathfrak{H}^2_* \subset \mathfrak{H}^0_* \subset \mathfrak{H}^1_* \,,$$
$$\mathfrak{H}^0_{\sqcup\!\sqcup} \subset \mathfrak{H}^1_{\sqcup\!\sqcup} \subset \mathfrak{H}_{\sqcup\!\sqcup} \,.$$

- Often it is convinient to think of (variations of) multiple zeta values as maps.
- **By abuse of notation and the sanity of the audience: We use the same symbol for maps and the corresponding object and assume it is always clear from context what we mean.**

For example, for any $N \geq 1$ the truncated Hurwitz zeta function can be viewed as a $\mathbb{Q}$-linear map

$$\begin{aligned} \zeta_N(-;x) : \mathfrak{H}^1 &\longrightarrow \mathbb{C}(x), \\ w = z_{k_1} \dots z_{k_r} &\longmapsto \zeta_N(w;x) := \zeta_N(k_1, \dots, k_r; x) \,. \end{aligned}$$

For all maps we consider we always send the empty word to $\mathbf{1}$, e.g. $\zeta_N(\emptyset; x) = \zeta_N(\mathbf{1}; x) = 1$.

The multiple zeta values can be viewed as $\mathbb{Q}$-algebra homomorphism $\zeta : \mathfrak{H}^0_\bullet \to \mathcal{Z}$ for $\bullet \in \{*, ш\}$.
Since $\mathfrak{H}^1_* = \mathfrak{H}^0_*[z_1]$ and $\mathfrak{H}_ш = \mathfrak{H}^0_ш[x, y]$ there exist algebra homomorphisms

$$\zeta^* : \mathfrak{H}^1_* \to \mathcal{Z}[T]\,,$$
$$\zeta^ш : \mathfrak{H}_ш \to \mathcal{Z}[T, X]\,,$$

uniquely determined by

- $\zeta^*(z_1) = \zeta^ш(y) = T, \qquad \zeta^ш(x) = X,$
- $\zeta^*_{|\mathfrak{H}^0} = \zeta^ш_{|\mathfrak{H}^0} = \zeta.$

Theorem (Ihara-Kaneko-Zagier)

Define the $\mathbb{R}$-linear map $\rho : \mathbb{R}[T] \to \mathbb{R}[T]$ by

$$\cdots + \frac{1}{2}\rho(T^2)u^2 + \cdots = \rho(e^{Tu}) := \exp\left(Tu + \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\zeta(n)u^n\right) = \cdots + \frac{1}{2}\left(T^2 + \zeta(2)\right)u^2 + \ldots\,.$$

Then we have $\zeta^ш_{|\mathfrak{H}^1} = \rho \circ \zeta^*$. $\quad$ ($\rightsquigarrow$ **extended double shuffle relations**)

Let $A = (\mathbb{Q}\langle L \rangle, *_\diamond)$ be a quasi-shuffle algebra with the coproduct given by

$$\Delta(w) = \sum_{uv=w} u \otimes v \, .$$

For an $\mathbb{Q}$-algebra B with multiplication m and $f, g \in \mathrm{Hom}(A, B)$ the **convolution product** is defined by

$$f \star g = m \circ (f \otimes g) \circ \Delta \, .$$

Fact

If $f, g \in \mathrm{Hom}(A, B)$ then $f \star g \in \mathrm{Hom}(A, B)$.

The antipode $S : A \to A$ is the inverse of Id with respect to $\star$, i.e. $(S \star \mathrm{Id})(w) = \begin{cases} 1, w = \emptyset \\ 0, \text{else} \end{cases}$

For example, in the case $*_\diamond = ⧢$ the antipode is given by $S(a_1 \dots a_m) = (-1)^m a_m \dots a_1$.

(c.f. Komiyama talk)

Proposition

For $k_1, \dots, k_r \geq 1$ and $k = k_1 + \cdots + k_r$ we have

$$\sum_{\substack{1 \leq j \leq r \\ l_1 + \cdots + l_{j-1} + l_{j+1} + \cdots + l_r = k-1}} (-1)^{e_j} \prod_{\substack{1 \leq i \leq r \\ i \neq j}} \binom{l_i - 1}{k_i - 1} \zeta^{ш}(l_1, \dots, l_{j-1}) \zeta^{ш}(l_r, l_{r-1}, \dots, l_{j+1}) = 0\,,$$

where $e_j = l_1 + \cdots + l_{j-1} + k_j$.

Proof: Using

$$\zeta^{ш}(x^{k_1-1}y \dots x^{k_r-1}yx^n) = (-1)^n \sum_{l_1 + \cdots + l_r = k_1 + \cdots + k_r + n} \prod_{i=1}^{r} \binom{l_i - 1}{k_i - 1} \zeta^{ш}(l_1, \dots, l_r)$$

and the following antipode relation for $a_1 \dots a_m = x^{k_1-1}y \dots x^{k_r-1}$

$$\sum_{i=0}^{m} (-1)^i \zeta^{ш}(a_1 \dots a_i) \zeta^{ш}(a_m a_{m-1} \dots a_{i+1}) = 0\,.$$

Riemann zeta values also appear in the Fourier expansion of the **Eisenstein series** defined for even $k \geq 4$ by

$$\mathbb{G}(k;\tau) = \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k} = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \,,$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ is the divisor sum, $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ and $q = e^{2\pi i \tau}$.

Goal

Define a multiple version of $\mathbb{G}$, such that

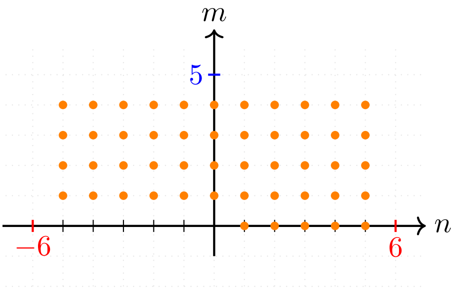
$$\mathbb{G}(k_1,\ldots,k_r;\tau) = \zeta(k_1,\ldots,k_r) + \sum_{n>0} a_n q^n \,.$$

For $M \geq 1$ set

$$\mathbb{Z}_M = \{m \in \mathbb{Z} \mid |m| < M\}.$$

and for $\tau \in \mathbb{H}$ define on $\mathbb{Z}\tau + \mathbb{Z}$ the **order** $\succ$ by

$$m_1\tau + n_1 \succ m_2\tau + n_2 \quad :\Leftrightarrow \quad (m_1 > m_2) \text{ or } (m_1 = m_2 \text{ and } n_1 > n_2).$$



All the points $\lambda \in \mathbb{Z}_5 i + \mathbb{Z}_6$ satisfying $\lambda \succ 0$.

For $M \geq 1$ set

$$\mathbb{Z}_M = \{m \in \mathbb{Z} \mid |m| < M\}\,.$$

and for $\tau \in \mathbb{H}$ define on $\mathbb{Z}\tau + \mathbb{Z}$ the **order** $\succ$ by

$$m_1\tau + n_1 \succ m_2\tau + n_2 \quad :\Leftrightarrow \quad (m_1 > m_2) \text{ or } (m_1 = m_2 \text{ and } n_1 > n_2)\,.$$

Definition

For integers $k_1, \dots, k_r \geq 1$, and $M, N \geq 1$ we define the **truncated multiple Eisenstein series** by

$$\mathbb{G}_{M,N}(k_1, \dots, k_r; \tau) = \sum_{\substack{\lambda_1 \succ \dots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}_M\tau + \mathbb{Z}_N}} \frac{1}{\lambda_1^{k_1} \dots \lambda_r^{k_r}}\,.$$

For $k_1, \dots, k_r \geq 2$ the **multiple Eisenstein series** are defined by

$$\mathbb{G}(k_1, \dots, k_r; \tau) = \lim_{M \to \infty} \lim_{N \to \infty} \mathbb{G}_{M,N}(k_1, \dots, k_r; \tau)\,.$$

For $k_1, \ldots, k_r \geq 2$ the multiple Eisenstein series is defined by

$$\mathbb{G}(k_1, \ldots, k_r; \tau) = \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \ldots \lambda_r^{k_r}} .$$

- These are holomorphic functions on the upper-half plane $\mathbb{H}$, but in general they are not modular.
- The product of multiple Eisenstein series can also be express by the **harmonic product** formula, e.g.

$$\mathbb{G}(4; \tau) \cdot \mathbb{G}(3; \tau) = \mathbb{G}(4, 3; \tau) + \mathbb{G}(3, 4; \tau) + \mathbb{G}(7; \tau) .$$

- We can view them as algebra homomorphisms

$$\begin{aligned} \mathbb{G} : \mathfrak{H}^2_* &\to \mathcal{O}(\mathbb{H}) \\ w = z_{k_1} \ldots z_{k_r} &\longmapsto \mathbb{G}(w; -) := \mathbb{G}(k_1, \ldots, k_r; -) . \end{aligned}$$

Definition

For $k_1, \dots, k_r \geq 1$ we define the q-series $g(k_1, \dots, k_r) \in \mathbb{Q}[[q]]$ by

$$g(k_1, \dots, k_r; \tau) = g(k_1, \dots, k_r) = \sum_{\substack{m_1 > \dots > m_r > 0 \\ n_1, \dots, n_r > 0}} \frac{n_1^{k_1 - 1}}{(k_1 - 1)!} \cdots \frac{n_r^{k_r - 1}}{(k_r - 1)!} q^{m_1 n_1 + \dots + m_r n_r} .$$

In the case $r = 1$ these are the generating series of divisor-sums $\sigma_{k-1}(n) = \sum_{d|n} n^{k-1}$

$$g(k) = \sum_{m,n>0} \frac{n^{k-1}}{(k-1)!} q^{mn} = \frac{1}{(k-1)!} \sum_{n>0} \sigma_{k-1}(n) q^n ,$$

and they can be viewed as ***q*-analogues of multiple zeta values**, since for $k_1 \geq 2, k_2, \dots, k_r \geq 1$ we have

$$\lim_{q \to 1} (1 - q)^{k_1 + \dots + k_r} g(k_1, \dots, k_r) = \zeta(k_1, \dots, k_r) .$$

$$\hat{g}(k_1,\dots,k_r) := (-2\pi i)^{k_1+\dots+k_r} g(k_1,\dots,k_r) \in \mathbb{Q}[\pi i][\![q]\!]\,.$$

Theorem (Gangl-Kaneko-Zagier 2006 ($r=2$), B. 2012 ($r\geq 2$))

For $k_1,\dots,k_r \geq 2$ *there exist explicit* $\alpha^{k_1,\dots,k_r}_{l_1,\dots,l_r,j} \in \mathbb{Z}$, *such that for* $q = e^{2\pi i \tau}$ *we have*

$$\mathbb{G}(k_1,\dots,k_r) = \zeta(k_1,\dots,k_r) + \sum_{\substack{0<j<r \\ l_1+\dots+l_r=k_1+\dots+k_r}} \alpha^{k_1,\dots,k_r}_{l_1,\dots,l_r,j}\, \zeta(l_1,\dots,l_j)\hat{g}(l_{j+1},\dots,l_r) + \hat{g}(k_1,\dots,k_r)\,.$$

In particular, $\mathbb{G}(k_1,\dots,k_r) = \zeta(k_1,\dots,k_r) + \sum_{n>0} a_{k_1,\dots,k_r}(n) q^n$ *for some* $a_{k_1,\dots,k_r}(n) \in \mathcal{Z}[\pi i]$.

Examples

$$\mathbb{G}(k;\tau) = \zeta(k) + \hat{g}(k)\,,$$

$$\mathbb{G}(3,2;\tau) = \zeta(3,2) + 3\zeta(3)\hat{g}(2) + 2\zeta(2)\hat{g}(3) + \hat{g}(3,2)\,.$$

Definition

For $k_1, \dots, k_r \geq 1$, $N \geq 1$ and $x \in \mathbb{C} \setminus \mathbb{Z}$ define the **(truncated) multitangent function** by

$$\Psi_N(k_1, \dots, k_r; x) := \sum_{\substack{N > n_1 > \dots > n_r > -N \\ n_i \in \mathbb{Z}}} \frac{1}{(x+n_1)^{k_1} \cdots (x+n_r)^{k_r}}.$$

For $k_1, k_r \geq 2$ the **multitangent function** is given by $\Psi(k_1, \dots, k_r; x) = \lim_{N \to \infty} \Psi_N(k_1, \dots, k_r; x)$.

In depth one we have for $k \geq 2$ the **Lipschitz formula** $(q = e^{2\pi i \tau})$

$$\Psi(k; \tau) = \sum_{n \in \mathbb{Z}} \frac{1}{(\tau + n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{d > 0} d^{k-1} q^d .$$

In particular for $k_1, \dots, k_r \geq 2$ we get

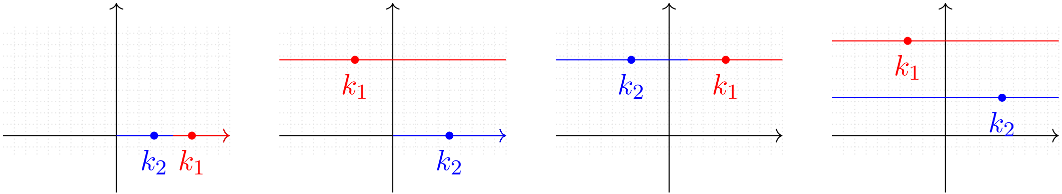
$$\hat{g}(k_1, \dots, k_r) = \sum_{m_1 > \dots > m_r > 0} \Psi_{k_1}(m_1 \tau) \cdots \Psi_{k_r}(m_r \tau) .$$

We can write $\mathbb{G}$ as sums over Ψ. For example, in depth two we have:

$$\mathbb{G}(k_1,k_2;\tau) = \sum_{m_1\tau+n_1\succ m_2\tau+n_2\succ 0} \frac{1}{(m_1\tau+n_1)^{k_1}(m_2\tau+n_2)^{k_2}}$$

$$= \left(\sum_{\substack{m_1=m_2=0\\ n_1>n_2>0}} + \sum_{\substack{m_1>m_2=0\\ n_1\in\mathbb{Z},n_2>0}} + \sum_{\substack{m_1=m_2>0\\ n_1>n_2}} + \sum_{\substack{m_1>m_2>0\\ n_1,n_2\in\mathbb{Z}}} \right) \frac{1}{(m_1\tau+n_1)^{k_1}(m_2\tau+n_2)^{k_2}}$$

$$= \zeta(k_1,k_2) + \sum_{m>0} \Psi(k_1;m\tau)\zeta(k_2) + \sum_{m>0} \Psi(k_1,k_2;m\tau) + \sum_{m_1>m_2>0} \Psi(k_1;m_1\tau)\Psi(k_2;m_2\tau)\,.$$



Theorem (Bouillot 2011)

For $k_1, \ldots, k_r \geq 1$ with $k_1, k_r \geq 2$ and $k = k_1 + \cdots + k_r$ the multitangent function can be written as

$$\Psi(k_1, \ldots, k_r; \tau) = \sum_{\substack{1 \leq j \leq r \\ l_1 + \cdots + l_r = k}} (-1)^{\bullet} \prod_{\substack{1 \leq i \leq r \\ i \neq j}} \binom{l_i - 1}{k_i - 1} \zeta(l_1, \ldots, l_{j-1}) \, \Psi(l_j; \tau) \, \zeta(l_r, l_{r-1}, \ldots, l_{j+1}) \, .$$

where $\bullet = l_1 + \cdots + l_{j-1} + k_j + k$. Moreover, the terms with $\Psi(1; \tau)$ vanish.

Proof: Partial fraction decomposition and antipote relation.

Sums over multitangent $\rightsquigarrow$ Sums over monotangent with MZV coefficients $\rightsquigarrow$ MZV-linear combination of $\hat{g}$

Corollary

For $w_1, \ldots, w_l \in \mathfrak{H}^2$ we have

$$\sum_{m_1 > \cdots > m_l > 0} \Psi(w_1; m_1\tau) \ldots \Psi(w_l; m_l\tau) \subset \mathcal{Z}[\pi i] \otimes \langle g(w) \mid w \in \mathfrak{H}^2 \rangle_{\mathbb{Q}} \, .$$

Multiple zeta values satisfy various relations. For example,

$$\zeta(2)^2 = \frac{5}{2}\zeta(4)\,, \qquad \zeta(5) = 2\zeta(3,2) + 6\zeta(4,1)\,.$$

Question

Do multiple Eisenstein series satisfy these relations?

Multiple zeta values satisfy various relations. For example,

$$\zeta(2)^2 = \frac{5}{2}\zeta(4)\,, \qquad \zeta(5) = 2\zeta(3,2) + 6\zeta(4,1)\,.$$

Question

Do multiple Eisenstein series satisfy these relations?

The first relation is clearly not satisfied, since setting $G_k = (-2\pi i)^{-k}\mathbb{G}(k;\tau)$ we have

$$G_2^2 = \frac{5}{2}G_4 - \frac{1}{2}q\frac{d}{dq}G_2\,.$$

The second relation can not be satisfied since $\mathbb{G}_{4,1}$ is not defined.

Question

1. Are there "natural" extensions of the algebra homomorphism $\mathbb{G} : \mathfrak{H}_*^2 \to \mathcal{O}(\mathbb{H})$ to $\mathfrak{H}_*^1$ or $\mathfrak{H}_{\sqcup}^1$?
2. How to include derivatives in our algebraic setup?

We saw that for $k_1, k_2 \geq 2$

$$\mathbb{G}(k_1,k_2;\tau) = \zeta(k_1,k_2) + \sum_{m>0} \Psi(k_1;m\tau)\zeta(k_2) + \sum_{m>0} \Psi(k_1,k_2;m\tau) + \sum_{m_1>m_2>0} \Psi(k_1;m_1\tau)\Psi(k_2;m_2\tau)\,.$$

Define the part coming from the sums parts in the upper half-plane by

$$\hat{g}^*(k_1) = \sum_{m>0} \Psi(k_1;m\tau),$$

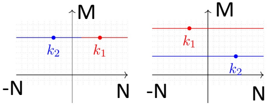
$$\hat{g}^*(k_1,k_2) = \sum_{m>0} \Psi(k_1,k_2;m\tau) + \sum_{m_1>m_2>0} \Psi(k_1;m_1\tau)\Psi(k_2;m_2\tau)\,.$$

Then we have for $k_1, k_2 \geq 2$

$$\mathbb{G}(k_1,k_2;\tau) = \zeta(k_1,k_2) + g^*(k_1)\zeta(k_2) + g^*(k_1,k_2)\,.$$

and in general $\mathbb{G} = g^* \star \zeta$ (as functions on $\mathfrak{H}^2$).

The anatomy of classical (truncated) multiple Eisenstein series



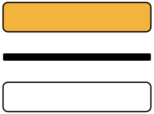
$\hat{g}_{M,N}(-;\tau)$

$\mathbb{G}_{M,N}$

ζ_N

Multiple zeta values

Multitangent function $\Psi_N(-;x)$

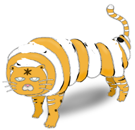


Multiple Hurwitz zeta $\zeta_N(-;x)$

$C(-;x)$

(reversed) Multiple Hurwitz zeta $\zeta_{\overline{N}}(-;x)$

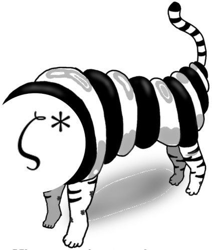
The construction of stuffle regularized multiple Eisenstein series



$$\hat{g}^*(w;\tau) := \lim_{M\to\infty} \hat{g}_M(w;\tau)$$

$$\hat{g}_M(-;\tau) = \mathop{\bigstar}_{m=1}^{M-1} \Psi^*(-;m\tau)$$

$$\mathbb{G}^* = \hat{g}^* \star \zeta^*$$



Stuffle regularized
Multiple zeta values

Regularized Multitangent function

$$\Psi^*(-;x) = \zeta^*(-;x) \star C(-;x) \star \zeta^{-,*}(-;x)$$

Regularized Multiple Hurwitz zeta $\zeta^*(-;x)$

$C(-;x)$

(reversed) Regularized Multiple Hurwitz zeta $\zeta^{-,*}(-;x)$

② Stuffle regularized MES - $\hat{g}_{M,N}$

For $M, N \geq 1$ define the map $\hat{g}_{M,N}(-;\tau) : \mathfrak{H}^1 \to \mathcal{O}(\mathbb{H})$ for $w \in \mathfrak{H}^1$ by

$$\hat{g}_{M,N}(w;\tau) = \sum_{\substack{j\geq 1 \\ w_1\dots w_j = w \\ w_1,\dots,w_j \neq \emptyset}} \sum_{M>m_1>\dots>m_j>0} \Psi_N(w_1;m_1\tau)\dots\Psi_N(w_j;m_j\tau)\,.$$

② Stuffle regularized MES - $\hat{g}_{M,N}$

For $M, N \geq 1$ define the map $\hat{g}_{M,N}(-;\tau) : \mathfrak{H}^1 \to \mathcal{O}(\mathbb{H})$ for $w \in \mathfrak{H}^1$ by

$$\hat{g}_{M,N}(w;\tau) = \sum_{\substack{j\geq 1\\ w_1\dots w_j = w\\ w_1,\dots,w_j\neq\emptyset}} \sum_{M>m_1>\dots>m_j>0} \Psi_N(w_1;m_1\tau)\dots\Psi_N(w_j;m_j\tau)\,.$$

This formula is ugly, but using the convolution product we can write it as

$$\hat{g}_{M,N}(-;\tau) = \mathop{\bigstar}_{m=1}^{M-1} \Psi_N(-;m\tau)\,,$$

where we write $\bigstar_{j=a}^{b} f_j = f_b \star f_{b-1} \star \dots \star f_a$.

We have

$$\mathbb{G}_{M,N} = \hat{g}_{M,N} \star \zeta_N\,.$$

Goal

Make sense of the limits $M, N \to \infty$ to obtain **Stuffle regularized multiple Eisenstein series** $\mathbb{G}^*$.

Define for $k_1, \ldots, k_r \geq 1$, $x \in \mathbb{H}$ and $N \geq 1$

$$C(k_1, \ldots, k_r; x) = \begin{cases} 1, & r = 0, \\ \frac{1}{x^{k_1}}, & r = 1, \\ 0, & r \geq 2 \end{cases},$$

$$\zeta_N^-(k_1, \ldots, k_r; x) = \sum_{0 > n_1 > \cdots > n_r > -N} \frac{1}{(x+n_1)^{k_1} \ldots (x+n_r)^{k_r}}.$$

These give algebra homomorphisms $C, \zeta_N^- : \mathfrak{H}_*^1 \to \mathcal{O}(\mathbb{H})$

Proposition

For $N \geq 1$ we have

$$\Psi_N(-; x) = \zeta_N(-; x) \star C(-; x) \star \zeta_N^-(-; x).$$

The limit $N \to \infty$ of

$$\Psi_N(-;x) = \zeta_N(-;x) \star C(-;x) \star \zeta_N^-(-;x)$$

does not exist, but the multiple hurwitz zeta function can be regularized (c.f. Bouillot, Kaneko-Xu-Yamamoto) to algebra homomorphism $\zeta^*(-;x) : \mathfrak{H}^1_* \to \mathcal{O}(\mathbb{H})$, such that

- For $k_1 \geq 2$ we have $\zeta^*(k_1,\ldots,k_r;x) = \lim_{N\to\infty} \zeta_N(k_1,\ldots,k_r;x)$,
- $\zeta^*(1;x) = \sum_{n>0}\left(\frac{1}{n+x} - \frac{1}{n}\right)$.

Definition

We define the algebra homomorphism $\Psi^* : \mathfrak{H}^1_* \to \mathcal{O}(\mathbb{H})$ by

$$\Psi^*(-;x) = \zeta^*(-;x) \star C(-;x) \star \zeta^{-,*}(-;x)\,,$$

where $\zeta^{-,*}$ is defined by ζ^* in the obvious way.

For $M \geq 1$ define the map $\hat{g}_M : \mathfrak{H}^1 \to \mathcal{O}(\mathbb{H})$

$$\hat{g}_M(-;\tau) = \mathop{\bigstar}_{m=1}^{M-1} \Psi^*(-;m\tau) = \mathop{\bigstar}_{m=1}^{M-1} \Big(\zeta^*(-;m\tau) \star C(-;m\tau) \star \zeta^{-,*}(-;m\tau)\Big).$$

Proposition (B.)

For all $w \in \mathfrak{H}^0$ the limit $\hat{g}^*(w;\tau) := \lim_{M\to\infty} \hat{g}_M(w;\tau)$ exists.

Define the algebra homomorphism $\hat{g}^* : \mathfrak{H}^1_* \to \mathcal{O}(\mathbb{H})$ by

- $\hat{g}^*(w;\tau) = \lim_{M\to\infty} \hat{g}_M(w;\tau)$ for $w \in \mathfrak{H}^0$.
- $\hat{g}^*(z_1;\tau) = \hat{g}(1;\tau)$.

Definition

Define the **stuffle regularized multiple Eisenstein series** as the algebra homomorphism $\mathbb{G}^* : \mathfrak{H}^1_* \to \mathcal{O}(\mathbb{H})$

$$\mathbb{G}^* = \hat{g}^* \star \zeta^*.$$

By construction we have $\mathbb{G}^*_{|\mathfrak{H}^2} = \mathbb{G}$.

Question

Is there a "natural" construction of an algebra homomorphism $\mathbb{G}^{\sqcup\!\sqcup} : \mathfrak{H}^1_{\sqcup\!\sqcup} \to \mathcal{O}(\mathbb{H})$?

Equipped with the **Goncharov coproduct** Δ_G the algebra $\mathfrak{H}^1_{\sqcup\!\sqcup}$ becomes a Hopf algebra.

There exist explicit formulas for Δ_G, e.g.

$$\Delta_G(z_3z_2) = z_3z_2 \otimes 1 + 3z_2 \otimes z_3 + 2z_3 \otimes z_2 + 1 \otimes z_3z_2\,.$$

Compare this to the Fourier expansion of $\mathbb{G}_{3,2}$:

$$\mathbb{G}(3,2;\tau) = \zeta(3,2) + 3\hat{g}(2)\zeta(3) + 2\hat{g}(3)\zeta(2) + \hat{g}(3,2)\,.$$

Question

Is there a "natural" construction of an algebra homomorphism $\mathbb{G}^{⧢} : \mathfrak{H}^1_{⧢} \to \mathcal{O}(\mathbb{H})$?

Equipped with the **Goncharov coproduct** Δ_G the algebra $\mathfrak{H}^1_{⧢}$ becomes a Hopf algebra.
There exist explicit formulas for Δ_G, e.g.

$$\Delta_G(z_3 z_2) = z_3 z_2 \otimes 1 + 3 z_2 \otimes z_3 + 2 z_3 \otimes z_2 + 1 \otimes z_3 z_2 \,.$$

Compare this to the Fourier expansion of $\mathbb{G}_{3,2}$:

$$\mathbb{G}(3,2;\tau) = \zeta(3,2) + 3\hat{g}(2)\zeta(3) + 2\hat{g}(3)\zeta(2) + \hat{g}(3,2)\,.$$

Write $f \star_G g = m \circ (f \otimes g) \circ \Delta_G$.

Theorem (B.-Tasaka 2017)

We have

$$\mathbb{G} = (\hat{g} \star_G \zeta)_{|\mathfrak{H}^2}\,.$$

Proposition (B.-Tasaka 2017)

There exists an algebra homomorphism $\hat{g}^{ш} : \mathfrak{H}^1_{ш} \to \mathcal{O}(\mathbb{H})$ with $\hat{g}^{ш}_{|\mathfrak{H}^2} = \hat{g}$.

Definition

Define the **shuffle regularized multiple Eisenstein series** as the algebra homomorphism $\mathbb{G}^{ш} : \mathfrak{H}^1_{ш} \to \mathcal{O}(\mathbb{H})$

$$\mathbb{G}^{ш} = \hat{g}^{ш} \star \zeta^{ш} \,.$$

By the previous mentioned results we have

$$\mathbb{G}^{ш}_{|\mathfrak{H}^2} = \mathbb{G} = \mathbb{G}^{*}_{|\mathfrak{H}^2} \,.$$

Corollary

The shuffle regularized multiple Eisenstein series satisfy the **restricted double shuffle relations**, i.e.

$$\mathbb{G}^{ш}(w ш v - w * v) = 0 \qquad (w, v \in \mathfrak{H}^2) \,.$$

But one can check that $\mathbb{G}^{ш}$ satisfy more relations than the restricted double shuffle relations, e.g.

$$\mathbb{G}^{ш}(z_2 ш z_2 z_1 - z_2 * z_2 z_1) = 0\,.$$

But they satisfy less relations than MZV, e.g. we have $\zeta(3) - \zeta(2,1) = 0$, but for $\bullet \in \{*, ш\}$

$$\mathbb{G}^{\bullet}(3) - \mathbb{G}^{\bullet}(2,1) = \frac{(2\pi i)^2}{2} q \frac{d}{dq} \mathbb{G}^{\bullet}(1)\,.$$

Goal

- Introduce an algebraic setup which can deal with derivatives $\rightsquigarrow$ **double shuffle relations for functions**
- The letters z_k will be replaces by z_d^k for $d \geq 0$ and above equations becomes (roughly)

$$z_0^3 - z_0^2 z_0^1 = z_1^2$$

- More precisely the operator $q\frac{d}{dq}$ corresponds to a derivation δ given by

$$\delta z_{d_1}^{k_1} \dots z_{d_r}^{k_r} = \sum_{j=1}^{r} k_j z_{d_1}^{k_1} \dots z_{d_j+1}^{k_j+1} \dots z_{d_r}^{k_r}$$

Let $\mathcal{A}$ be a $\mathbb{Q}$-algebra.

Definition

1. A **mould** with values in $\mathcal{A}$ is a family $Z = (Z^{(r)})_{r\geq 0}$ with $Z^{(r)} \in \mathcal{A}[[X_1, \dots, X_r]]$.
2. For a mould Z with

$$Z^{(r)}(X_1, \dots, X_r) = \sum_{k_1, \dots, k_r \geq 1} z(k_1, \dots, k_r) X_1^{k_1 - 1} \dots X_r^{k_r - 1}$$

we define its **coefficient map** as the $\mathbb{Q}$-linear map given by $\varphi_Z(\mathbf{1}) = Z^{(0)}$ and on the generators by

$$\begin{aligned} \varphi_Z : \mathbb{Q}\langle L_z \rangle &\longrightarrow \mathcal{A} \\ z_{k_1} \dots z_{k_r} &\longmapsto z(k_1, \dots, k_r) \,. \end{aligned}$$

(c.f. Komiyama & Kimura talk)

Definition

1. A mould Z is called **$\diamond$-symmetril** if its coefficient map φ_Z gives an algebra homomorphism

$$\varphi_Z : (\mathbb{Q}\langle L_z\rangle, *_\diamond) \longrightarrow \mathcal{A}.$$

2. If $\diamond$ is given by $z_{k_1} \diamond z_{k_2} = z_{k_1+k_2}$ then we call a $\diamond$-symmetril mould **symmetril**. ($\leftrightsquigarrow$ harmonic product)
3. If $\diamond$ is given by $z_{k_1} \diamond z_{k_2} = 0$ then we call a $\diamond$-symmetril mould **symmetral**. ($\leftrightsquigarrow$ index shuffle product)

Example: The mould of **harmonic regularized multiple zeta values** $\mathfrak{z}$, whose depth r part is defined by

$$\mathfrak{z}(X_1,\dots,X_r) = \sum_{k_1,\dots,k_r \geq 1} \zeta^*(k_1,\dots,k_r) X_1^{k_1-1} \dots X_r^{k_r-1}.$$

is symmetril.

Let Z be a mould with $Z^{(1)}(X) = \sum_{k\geq 1} z(k)X^{k-1}$. Define the elements $\gamma_k^Z \in \mathcal{A}$ by

$$\sum_{k=0}^{\infty} \gamma_k^Z X^k := \exp\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n} z(n) X^n\right) .$$

With this we define the mould Z_γ by

$$Z_\gamma^{(r)}(X_1, \ldots, X_r) = \sum_{j=0}^{r} \gamma_j^Z Z^{(r-j)}(X_1 + \cdots + X_{r-j}, \ldots, X_1 + X_2, X_1).$$

Definition

We say a mould Z **satisfies the double shuffle relations** if Z is symmetril and Z_γ is symmetral.

Definition

We say a mould Z **satisfies the double shuffle relations** if Z is symmetril and Z_γ is symmetral.

In lowest depth, this means that if Z satisfies the double shuffle relations, then (c.f. Kimura talk)

$$
\begin{aligned}
Z(X_1)Z(X_2) &= Z(X_1, X_2) + Z(X_2, X_1) + \frac{Z(X_1) - Z(X_2)}{X_1 - X_2}\,, \\
Z_\gamma(X_1)Z_\gamma(X_2) &= Z_\gamma(X_1, X_2) + Z_\gamma(X_2, X_1) \\
&= Z(X_1 + X_2, X_1) + Z(X_1 + X_2, X_2) + \gamma_2^Z\,.
\end{aligned}
$$

Theorem (Ecalle, Ihara-Kaneko-Zagier, Racinet, . . .)

The mould of harmonic regularized multiple zeta values $\mathfrak{z}$ satisfies the double shuffle relations.

Theorem (Drinfeld + Furusho, Racinet)

There exists a mould $\mathfrak{b}$ with values in $\mathbb{Q}$, with the following properties.

1. *$\mathfrak{b}$ satisfies the double shuffle relations.*
2. *For all $r \geq 1$, $\mathfrak{b}(-X_1, \dots, -X_r) = (-1)^r \mathfrak{b}(X_1, \dots, X_r)$.*
3. *In depth one $\mathfrak{b}$ is given by*

$$\mathfrak{b}(X) = -\sum_{k \geq 2} \frac{B_k}{2k!} X^{k-1} = \sum_{m \geq 1} \frac{\zeta(2m)}{(2\pi i)^{2m}} X^{2m-1}.$$

This mould is not unique, but in the following, we will fix one choice of such a mould $\mathfrak{b}$ with coefficients β, i.e.

$$\mathfrak{b}(X_1, \dots, X_r) = \sum_{k_1, \dots, k_r \geq 1} \beta(k_1, \dots, k_r) X_1^{k_1 - 1} \dots X_r^{k_r - 1}.$$

Let $\mathcal{A}$ be a $\mathbb{Q}$-algebra, define $L_z^{\mathrm{bi}} = \{z_d^k \mid k \geq 1, d \geq 0\}$ and write $* = *_\diamond$ for $z_{d_1}^{k_1} \diamond z_{d_2}^{k_2} = z_{d_1+d_2}^{k_1+k_2}$.

Definition

1. A **bimould** with values in $\mathcal{A}$ is a family $B = (B^{(r)})_{r \geq 0}$ with $B^{(r)} \in \mathcal{A}[[X_1, \dots, X_r, Y_1, \dots, Y_r]]$.
2. For a bimould B with

$$B\binom{X_1, \dots, X_r}{Y_1, \dots, Y_r} = \sum_{\substack{k_1,\dots,k_r \geq 1 \\ d_1,\dots,d_r \geq 0}} b\binom{k_1, \dots, k_r}{d_1, \dots, d_r} X_1^{k_1-1} \cdots X_r^{k_r-1} \frac{Y_1^{d_1}}{d_1!} \cdots \frac{Y_1^{d_r}}{d_r!}$$

we define its **coefficient map** as the $\mathbb{Q}$-linear map given by $\varphi_B(\mathbf{1}) = B^{(0)}$ and on the generators by

$$\begin{aligned} \varphi_B : \mathbb{Q}\langle L_z^{\mathrm{bi}} \rangle &\longrightarrow \mathcal{A} \\ z_{d_1}^{k_1} \cdots z_{d_r}^{k_r} &\longmapsto b\binom{k_1, \dots, k_r}{d_1, \dots, d_r}. \end{aligned}$$

Definition

1. A bimould B is called **$\diamond$-symmetril** if its coefficient map φ_B gives an algebra homomorphism

$$\varphi_B : (\mathbb{Q}\langle L_z^{\mathrm{bi}}\rangle, *_\diamond) \longrightarrow \mathcal{A}.$$

2. If $\diamond$ is given by $z_{d_1}^{k_1} \diamond z_{d_2}^{k_2} = z_{d_1+d_2}^{k_1+k_2}$ then we call a $\diamond$-symmetril bimould **symmetril**.

If B is symmetril then it satisfies in lowest depth

$$B\binom{X_1}{Y_1}B\binom{X_2}{Y_2} = B\binom{X_1, X_2}{Y_1, Y_2} + B\binom{X_2, X_1}{Y_2, Y_1} + \frac{B\binom{X_1}{Y_1+Y_2} - B\binom{X_2}{Y_1+Y_2}}{X_1 - X_2},$$

which is similar to the relation satisfied by a symmetril mould Z

$$Z(X_1)Z(X_2) = Z(X_1, X_2) + Z(X_2, X_1) + \frac{Z(X_1) - Z(X_2)}{X_1 - X_2}.$$

Let B and C two bimoulds with values in $\mathcal{A}$. The **mould product** $B \times C$ is the bimould given by

$$(B \times C)\begin{pmatrix} X_1, \dots, X_r \\ Y_1, \dots, Y_r \end{pmatrix} = \sum_{j=0}^{r} B\begin{pmatrix} X_1, \dots, X_j \\ Y_1, \dots, Y_j \end{pmatrix} C\begin{pmatrix} X_{j+1}, \dots, X_r \\ Y_{j+1}, \dots, Y_r \end{pmatrix}.$$

Proposition

If B and C are $\diamond$-symmetril then $B \times C$ is $\diamond$-symmetril.

Proof: The coefficient map of $B \times C$ is the convolution product of φ_B and φ_C, i.e.

$$\varphi_{B\times C} = m \circ (\varphi_B \otimes \varphi_C) \circ \Delta\,,$$

where $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the multiplication on $\mathcal{A}$ and Δ is the deconcatination coproduct on $\mathbb{Q}\langle L_z^{\mathrm{bi}}\rangle$. $\square$

Definition

A bimould B is called **swap invariant** if for all $r \geq 1$

$$B\begin{pmatrix} X_1, \dots, X_r \\ Y_1, \dots, Y_r \end{pmatrix} = B\begin{pmatrix} Y_1 + \dots + Y_r, Y_1 + \dots + Y_{r-1}, \dots, Y_1 + Y_2, Y_1 \\ X_r, X_{r-1} - X_r, \dots, X_2 - X_3, X_1 - X_2 \end{pmatrix}$$

Example: If B is swap invariant we have $B\binom{X}{Y} = B\binom{Y}{X}$, which gives, for example, $b\binom{1}{1} = b\binom{2}{0}$.

Definition

For a mould Z, we define the bimould B^Z by

$$B^Z\begin{pmatrix} X_1, \dots, X_r \\ Y_1, \dots, Y_r \end{pmatrix} = \sum_{j=0}^{r} Z_\gamma(Y_1, \dots, Y_j) Z(X_{j+1}, \dots, X_r) \,.$$

Recall that by definition

$$Z_\gamma^{(r)}(Y_1, \dots, Y_r) = \sum_{j=0}^{r} \gamma_j^Z Z^{(r-j)}(Y_1 + \cdots + Y_{r-j}, \dots, Y_1 + Y_2, Y_1) \,.$$

Proposition

- For any mould Z the bimould B^Z is swap invariant,
- If Z satisfies the double shuffle relations then B^Z is symmetril.

Z satisfies the double shuffle relations $\Rightarrow B^Z$ is swap invariant & symmetril.

Question ("$\Leftarrow$" ?)

Does a swap invariant & symmetril bimould B give a mould Z which satisfies the double shuffle relations by setting

$$Z(X_1,\dots,X_r) = B\binom{X_1,\dots,X_r}{0,\dots,0}?$$

No, not in general: Let B swap invariant & symmetril bimould. Then one can show that its coefficient satisfy

$$b\binom{2}{0}^2 = \frac{5}{2}b\binom{4}{0} - b\binom{3}{1}.$$

Compare this to

$$G_2^2 = \frac{5}{2}G_4 - \frac{1}{2}q\frac{d}{dq}G_2, \qquad \text{and} \qquad \zeta(2)^2 = \frac{5}{2}\zeta(4).$$

$\longrightarrow$ The coefficients of an swap invariant & symmetril bimould "behave like Eisenstein series".

Theorem ((B.-Burmester (2022+))

There exist a swap invariant & symmetril bimould $\mathfrak{G}$ *with values in* $\mathbb{Q}[[q]]$

$$\mathfrak{G}\binom{X_1,\ldots,X_r}{Y_1,\ldots,Y_r} = \sum_{\substack{k_1,\ldots,k_r\geq 1\\ d_1,\ldots,d_r\geq 0}} G\binom{k_1,\ldots,k_r}{d_1,\ldots,d_r} X_1^{k_1-1}\cdots X_r^{k_r-1}\frac{Y_1^{d_1}}{d_1!}\cdots\frac{Y_1^{d_r}}{d_r!}$$

such that the coefficients in depth one are given by Eisenstein series and their derivatives $(k > d \geq 0)$

$$G\binom{k}{d} = \frac{(k-d-1)!}{(k-1)!}\left(q\frac{d}{dq}\right)^d G_{k-d}\,.$$

Define the **combinatorial multiple Eisenstein series** for $k_1,\ldots,k_r \geq 1$ by

$$G(k_1,\ldots,k_r) := G\binom{k_1,\ldots,k_r}{0,\ldots,0}\,.$$

Denote the space spanned by all combinatorial multiple Eisenstein by

$$\mathcal{G} = \mathbb{Q} + \big\langle G(k_1,\dots,k_r) \mid r \geq 1, k_1,\dots,k_r \geq 1 \big\rangle_{\mathbb{Q}} \subset \mathbb{Q}[[q]]\,.$$

Theorem (B.-Burmester (2022+))

1. *The space $\mathcal{G}$ is a $\mathbb{Q}$-algebra which contains the space of (quasi-)modular forms with rational coefficients.*
2. *The combinatorial multiple Eisenstein series give an algebra homomorphism*

$$\begin{aligned} G : (\mathbb{Q}\langle L_z\rangle, *) &\longrightarrow \mathcal{G} \\ w = z_{k_1} \dots z_{k_r} &\longmapsto G(w) := G(k_1,\dots,k_r)\,. \end{aligned}$$

3. $\mathcal{G}$ *is closed under* $q\frac{d}{dq}$ *and for any* $w \in \mathbb{Q}\langle L_z\rangle$ *we have*

$$q\frac{d}{dq}G(w) = G(z_2 * w - z_2 \sqcup\!\sqcup\, w)\,.$$

The combinatorial multiple Eisenstein series have the form

$$G(k_1,\dots,k_r) = \beta(k_1,\dots,k_r) + \text{products of } \beta \text{ and } g \text{ in lower depths} + g(k_1,\dots,k_r)\,.$$

Example: $$G(3,2) = \beta(3,2) + 3\beta(3)g(2) + 2\beta(2)g(3) + g(3,2)$$

Therefore they can be seen as an interpolation between the harmonic regularized multiple zeta values and the rational solutions to double shuffle equations: For all $k_1,\dots,k_r \geq 1$ we have

$$\begin{aligned}\lim_{q\to 1}^{*}(1-q)^{k_1+\dots+k_r}G(k_1,\dots,k_r) &= \zeta^*(k_1,\dots,k_r)\\ \lim_{q\to 0} G(k_1,\dots,k_r) &= \beta(k_1,\dots,k_r)\,.\end{aligned}$$

Here $\lim^*_{q\to 1}$ means that for $k_1 = 1$ one needs to use a regularized limit (B.-van-Ittersum 2022+)

(Rough) Let S be the ideal in $(\mathbb{Q}\langle L_z^{\mathrm{bi}}\rangle, *)$ generated by the "swap invariance relations", e.g. $z_1^1 - z_0^2 \in S$.

Definition

The algebra of **formal multiple Eisenstein series** is defined by

$$\mathcal{G}^{\mathfrak{f}} = {}^{\mathbb{Q}\langle L_z^{\mathrm{bi}}\rangle}\!\big/_{S}$$

and we denote the class of a word $z_{d_1}^{k_1} \dots z_{d_r}^{k_r}$ by $G_{\mathfrak{f}}\binom{k_1,\dots,k_r}{d_1,\dots,d_r}$ and set $G_{\mathfrak{f}}(k_1,\dots,k_r) := G_{\mathfrak{f}}\binom{k_1,\dots,k_r}{0,\dots,0}$.

Theorem (B.-Matthes-van-Ittersum (2022+))

The following map gives a derivation on $\mathcal{G}^{\mathfrak{f}}$

$$\partial G_{\mathfrak{f}}\binom{k_1,\dots,k_r}{d_1,\dots,d_r} = \sum_{j=1}^{r} k_j G_{\mathfrak{f}}\binom{k_1,\dots,k_j+1,\dots,k_r}{d_1,\dots,d_j+1,\dots,d_r}.$$

As an analogue of $G_2^2 = \frac{5}{2}G_4 - \frac{1}{2}q\frac{d}{dq}G_2$ we get $G_{\mathfrak{f}}(2)^2 = \frac{5}{2}G_{\mathfrak{f}}(4) - \frac{1}{2}\partial G_{\mathfrak{f}}(2)$.

Theorem (B.-Matthes-van-Ittersum (2022+))

1. *The space of formal modular forms $\mathcal{M}^{\mathfrak{f}} = \mathbb{Q}[G_{\mathfrak{f}}(4), G_{\mathfrak{f}}(6)]$ is isomorphic to the space of modular forms.*
2. *The space of formal quasi-modular forms $\widetilde{\mathcal{M}}^{\mathfrak{f}} = \mathbb{Q}[G_{\mathfrak{f}}(2), G_{\mathfrak{f}}(4), G_{\mathfrak{f}}(6)]$ is isomorphic to the space of quasi-modular forms as differential algebras.*
3. *There exist an ideal N, such that the algebra $\mathcal{Z}^{\mathfrak{f}} = \mathcal{G}^{\mathfrak{f}} / N$ is isomorphic to the algebra of* **formal multiple zeta values** *(defined by Racinet).*

Conjecture ($\mathfrak{sl}_2$-action)

There exist a unique derivation $\mathfrak{d}$ on $\mathcal{G}^{\mathfrak{f}}$ such that the triple $(\partial, W, \mathfrak{d})$ is an $\mathfrak{sl}_2$-triple, i.e.

$$[W, \partial] = 2\partial, \quad [W, \mathfrak{d}] = -2\mathfrak{d}, \qquad [\mathfrak{d}, \partial] = W\,,$$

where W is the weight operator.

We have an explicit conjectured construction of the derivation $\mathfrak{d}$. This $\mathfrak{sl}_2$-action would generalize the classical $\mathfrak{sl}_2$-action on the space of quasi-modular forms.

There are still some undiscovered species of multiple Eisenstein series.

1. Higher level analogues (cf. Kaneko-Tasaka 2013, Yuan-Zhao 2016).
2. Analytic realization of the formal multiple Eisenstein series.
3. Extension of the Kronecker realization (B.-Kühn-Matthes 2021) to higher depths. $\rightsquigarrow$ "Modular MES".
4. Connection to the Goncharov coproduct (cf. B.-Tasaka 2017).
5. Possible definition of q-Associators.
6. Basis & Dimension formulas (cf. B.-Kühn 2020).
7. Interpretation of the Broadhurst-Kreimer conjecture & exotic relations in this setup.
8. Adaptation of this setup for finite multiple zeta values (cf. Kaneko-Zagier, B.-Tasaka-Takeyama 2018).

Thank you for your attention.

The zoo of multiple Eisenstein series (MES)

1 **Classical MES**

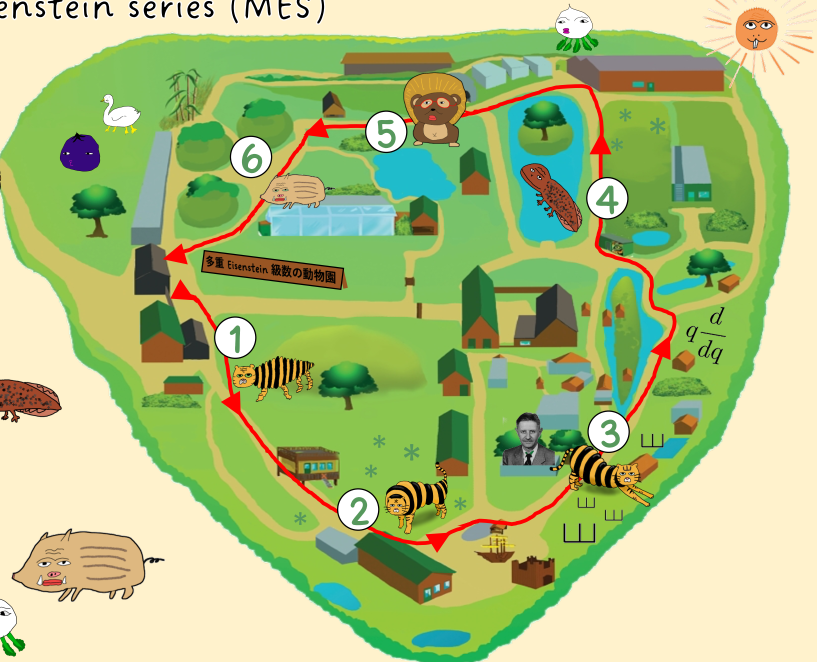
2 **Stuffle regularised MES**

3 **Shuffle regularized MES**

4 **Combinatorial MES**

5 **Formal MES**

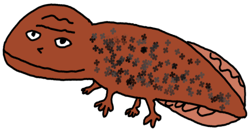
6 **Modular MES & Outlook**



Construction of combinatorial multiple Eisenstein series

Symmetril & swap invariant bimould

$$G\binom{k_1,\dots,k_r}{d_1,\dots,d_r} \qquad \mathfrak{G}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r}$$



Mould product

$$\mathfrak{G} = \mathfrak{g}^* \times \mathfrak{b}$$

Symmetril & swap invariant bimould

$$\mathfrak{b}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r}$$

Sums of multiple version of L: Symmetril bimould $\mathfrak{g}^*$

$$\mathfrak{g}^*\binom{X_1,X_2}{Y_1,Y_2} = \sum_{m_1>m_2>0} \mathfrak{L}_{m_1}\binom{X_1}{Y_1}\mathfrak{L}_{m_2}\binom{X_2}{Y_2} + \sum_{m>0}\mathfrak{L}_m\binom{X_1,X_2}{Y_1,Y_2}$$

B^Z

Rational solution for double shuffle equations

$$\mathfrak{b}(X_1,\dots,X_r)$$

Define multiple version of L by b and single version of L

$$\mathfrak{L}_m\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} = \sum_{j=1}^{r}\mathfrak{b}\binom{X_1-X_j,\dots,X_{j-1}-X_j}{Y_1,\dots,Y_{j-1}} L_m\binom{X_j}{Y_1+\dots+Y_r}\tilde{\mathfrak{b}}\binom{X_r-X_j,\dots,X_{j+1}-X_j}{Y_r,\dots,Y_{j+1}}$$

Symmetril bimould

$$\mathfrak{L}_m$$

The series g (sums of single version of L)

$$\mathfrak{g}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} = \sum_{m_1>\dots>m_r>0} L_{m_1}\binom{X_1}{Y_1}\dots L_{m_r}\binom{X_r}{Y_r}$$

Bonus - Construction of the bimould $\mathfrak{G}$

With $L_m\binom{X}{Y} = \frac{e^{X+mY}q^m}{1-e^X q^m}$ define the bimould $\mathfrak{g}$ with values in $\mathbb{Q}[[q]]$ by

$$\mathfrak{g}\begin{pmatrix} X_1, \dots, X_r \\ Y_1, \dots, Y_r \end{pmatrix} = \sum_{m_1 > \dots > m_r > 0} L_{m_1}\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \dots L_{m_r}\begin{pmatrix} X_r \\ Y_r \end{pmatrix}.$$

Theorem (B. 2013)

The bimould $\mathfrak{g}$ is swap invariant.

The coefficients generalize the q-series g. This bimould is not symmetril, but satisfies, for example,

$$\mathfrak{g}\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}\mathfrak{g}\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = \mathfrak{g}\begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix} + \mathfrak{g}\begin{pmatrix} X_2, X_1 \\ Y_2, Y_1 \end{pmatrix} + \frac{\mathfrak{g}\binom{X_1}{Y_1+Y_1} - \mathfrak{g}\binom{X_2}{Y_1+Y_1}}{X_1 - X_2}$$
$$+ \left(2\mathfrak{b}(X_2 - X_1) - \frac{1}{2}\right)\mathfrak{g}\begin{pmatrix} X_1 \\ Y_1 + Y_1 \end{pmatrix} + \left(2\mathfrak{b}(X_1 - X_2) - \frac{1}{2}\right)\mathfrak{g}\begin{pmatrix} X_2 \\ Y_1 + Y_1 \end{pmatrix}.$$

Using the swap invariance of $\mathfrak{g}$, the above relationship between $\mathfrak{g}$ and $\mathfrak{b}$ and the fact that $\mathfrak{b}$ satisfies the double shuffle relation, one can given an explicit (but complicated) construction of $\mathfrak{G}$.

Bonus - Construction of the bimould - $\mathfrak{L}_m$

Recall $L_m\binom{X}{Y} = \frac{e^{X+mY}q^m}{1-e^X q^m}$ and set $\tilde{\mathfrak{b}}\binom{X_1,\ldots,X_r}{Y_1,\ldots,Y_r} = \sum_{i=0}^r \frac{(-1)^i}{2^i i!}\mathfrak{b}\binom{X_{i+1},\ldots,X_r}{-Y_1,\ldots,-Y_{r-i}}$.

Definition

For $m \geq 1$ we define the bimould $\mathfrak{L}_m$ by defining $\mathfrak{L}_m\binom{X_1,\ldots,X_r}{Y_1,\ldots,Y_r}$ as

$$\sum_{j=1}^r \mathfrak{b}\begin{pmatrix} X_1 - X_j, \ldots, X_{j-1} - X_j \\ Y_1, \ldots, Y_{j-1} \end{pmatrix} L_m\begin{pmatrix} X_j \\ Y_1 + \cdots + Y_r \end{pmatrix} \tilde{\mathfrak{b}}\begin{pmatrix} X_r - X_j, \ldots, X_{j+1} - X_j \\ Y_r, \ldots, Y_{j+1} \end{pmatrix}.$$

The $L_m\binom{X}{Y}$ can be seen as the generating series of the "(bi-)combinatorial version" of the monotangent function $\Psi_k^{\text{comb}}(\tau) = \frac{1}{(k-1)!}\sum_{d>0} d^{k-1}q^d$ (defined by the Lipschitz formula instead of nested sum), since

$$\sum_{k\geq 1} \Psi_k^{\text{comb}}(m\tau)X^{k-1} = \sum_{k\geq 1}\frac{1}{(k-1)!}\sum_{d>0} d^{k-1}q^{md}X^{k-1} = \sum_{d>0} e^{dX}q^{md} = \frac{e^X q^m}{1-e^X q^m} = L_m\begin{pmatrix} X \\ 0 \end{pmatrix}.$$

The $\mathfrak{L}_m$ can then be seen as the generating series of (bi-)combinatorial version of the multitangent functions.

(compare with the flexion units in the talk of Komiyama)

Lemma

Let B_m be a family of bimoulds which are $\diamond$-symmetril for all $m \geq 1$. Then the bimould C_M defined by

$$C_M\begin{pmatrix} X_1, \dots, X_r \\ Y_1, \dots, Y_r \end{pmatrix} = \sum_{\substack{1\leq j\leq r \\ 0=r_0<r_1<\cdots<r_{j-1}<r_j=r \\ M>m_1>\cdots>m_j>0}} \prod_{i=1}^{j} B_{m_i}\begin{pmatrix} X_{r_{i-1}+1}, \dots, X_{r_i} \\ Y_{r_{i-1}+1}, \dots, Y_{r_i} \end{pmatrix}$$

is $\diamond$-symmetril for all $M \geq 1$. **Proof:** *Show $C_{M+1} = B_M \times C_M$ and do induction on M.*

Definition

We define the bimould $\mathfrak{g}^*$ by

$$\mathfrak{g}^*\begin{pmatrix} X_1, \dots, X_r \\ Y_1, \dots, Y_r \end{pmatrix} = \sum_{\substack{1\leq j\leq r \\ 0=r_0<r_1<\cdots<r_{j-1}<r_j=r \\ m_1>\cdots>m_j>0}} \prod_{i=1}^{j} \mathfrak{L}_{m_i}\begin{pmatrix} X_{r_{i-1}+1}, \dots, X_{r_i} \\ Y_{r_{i-1}+1}, \dots, Y_{r_i} \end{pmatrix}.$$

Lemma $\Longrightarrow$ if the $\mathfrak{L}_m$ are symmetril for all m then $\mathfrak{g}^*$ is symmetril.

Definition (B.-Burmester (2022+))

The bimould of combinatorial (bi)-multiple Eisenstein series is defined by $\mathfrak{G} = \mathfrak{g}^* \times \mathfrak{b}$.

Definition (B.-Burmester (2022+))

For $j \geq 0$ we define the bimould $\mathfrak{G}_j = (\mathfrak{G}_j^{(r)})_{r\geq 0}$ as follows. In the case $j = 0$ we set $\mathfrak{G}_0 = \mathfrak{b}$ and $\mathfrak{G}_j^{(r)} = 0$ for $r < j$. If $1 \leq j \leq r$ we define

$$\mathfrak{G}_j\begin{pmatrix} X_1, \dots, X_r \\ Y_1, \dots, Y_r \end{pmatrix} = \sum_{\substack{0=r_0<r_1<\cdots<r_j\leq r \\ m_1>\cdots>m_j>0}} \prod_{i=1}^{j} \mathfrak{L}_{m_i}\begin{pmatrix} X_{r_{i-1}+1}, \dots, X_{r_i} \\ Y_{r_{i-1}+1}, \dots, Y_{r_i} \end{pmatrix} \mathfrak{b}\begin{pmatrix} X_{r_j+1}, \dots, X_r \\ Y_{r_j+1}, \dots, Y_r \end{pmatrix}.$$

Theorem (B.-Burmester (2022+))

The bimould $\mathfrak{G}_j$ is swap invariant for any $j \geq 0$ and we have $\mathfrak{G} = \sum_{j=0}^{r} \mathfrak{G}_j$, i.e. $\mathfrak{G}$ is swap invariant.

Bonus - Example of the bimould $\mathfrak{G}$

Let $\mathfrak{b} = B^{\mathfrak{b}}$ denote the bimould coming from the mould $\mathfrak{b}$, which satisfies the double shuffle relation. (i.e. the bimould $\mathfrak{b}$ is symmetril and swap invariant)

Example: In depth one and two the bimould $\mathfrak{G}$ is given by

$$\mathfrak{G}\begin{pmatrix}X_1\\Y_1\end{pmatrix} = \mathfrak{b}\begin{pmatrix}X_1\\Y_1\end{pmatrix} + \mathfrak{g}\begin{pmatrix}X_1\\Y_1\end{pmatrix},$$

$$\begin{aligned}\mathfrak{G}\begin{pmatrix}X_1, X_2\\Y_1, Y_2\end{pmatrix} &= \mathfrak{b}\begin{pmatrix}X_1, X_2\\Y_1, Y_2\end{pmatrix} - \mathfrak{b}\begin{pmatrix}X_1 - X_2\\Y_2\end{pmatrix}\mathfrak{g}\begin{pmatrix}X_1\\Y_1 + Y_2\end{pmatrix} - \frac{1}{2}\mathfrak{g}\begin{pmatrix}X_1\\Y_1 + Y_2\end{pmatrix}\\ &\quad + \mathfrak{b}\begin{pmatrix}X_2\\Y_2\end{pmatrix}\mathfrak{g}\begin{pmatrix}X_1\\Y_1\end{pmatrix} + \mathfrak{b}\begin{pmatrix}X_1 - X_2\\Y_1\end{pmatrix}\mathfrak{g}\begin{pmatrix}X_2\\Y_1 + Y_2\end{pmatrix} + \mathfrak{g}\begin{pmatrix}X_1, X_2\\Y_1, Y_2\end{pmatrix}.\end{aligned}$$

Bonus - Analogue for the double shuffle relation in small depth

As a consequence of the swap invariance the formal (and therefore also the combinatorial) bi-multiple Eisenstein series satisfy for $k_1, k_2 \geq 1, d_1, d_2 \geq 0$

$$G_{\mathfrak{f}}\binom{k_1}{d_1} G_{\mathfrak{f}}\binom{k_2}{d_2} = G_{\mathfrak{f}}\binom{k_1, k_2}{d_1, d_2} + G_{\mathfrak{f}}\binom{k_2, k_1}{d_2, d_1} + G_{\mathfrak{f}}\binom{k_1 + k_2}{d_1 + d_2}$$

$$= \sum_{\substack{l_1+l_2=k_1+k_2 \\ e_1+e_2=d_1+d_2 \\ l_1,l_2\geq 1, e_1,e_2\geq 0}} \left(\binom{l_1-1}{k_1-1}\binom{d_1}{e_1}(-1)^{d_1-e_1} + \binom{l_1-1}{k_2-1}\binom{d_2}{e_1}(-1)^{d_2-e_1} \right) G_{\mathfrak{f}}\binom{l_1, l_2}{e_1, e_2}$$

$$+ \frac{d_1! d_2!}{(d_1+d_2+1)!}\binom{k_1+k_2-2}{k_1-1} G_{\mathfrak{f}}\binom{k_1+k_2-1}{d_1+d_2+1}.$$

Example The $k_1 = 2, k_2 = 3, d_1 = d_2 = 0$ case gives

$$\begin{aligned} G_{\mathfrak{f}}(2) G_{\mathfrak{f}}(3) &= G_{\mathfrak{f}}(2,3) + G_{\mathfrak{f}}(3,2) + G_{\mathfrak{f}}(5) \\ &= G_{\mathfrak{f}}(2,3) + 3G_{\mathfrak{f}}(3,2) + 6G_{\mathfrak{f}}(4,1) + \partial G_{\mathfrak{f}}(3). \end{aligned}$$

Compare this to $\zeta(2) \cdot \zeta(3) = \zeta(2,3) + \zeta(3,2) + \zeta(5) = \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1)$.