Homework 2

Deadline: 12th June (23:55 JST), 2022

You can submit your solution at NUCT, by email (henrik.bachmann@math.nagoya-u.ac.jp) or hand it in physically.

Exercise 7. Let ζ_p be a primitive p-th root of unity and consider the ideal $\mathfrak{p} = (1-\zeta_p)$ of $\mathbb{Z}[\zeta_p]$.

(i) Show that $\mathfrak{p}$ is prime.

(ii) Show that $\mathbb{Z}[\zeta_p]/\mathfrak{p} \cong \mathbb{Z}/p\mathbb{Z}$.

Exercise 8.

(i) Prove Proposition 1.38, i.e. show that for all $n, k \geq 1$ we have

$$H_{n-1}(k; e^{\frac{2\pi i}{n}}) = -\frac{b_k(n)}{k!}\left(n(1-e^{\frac{2\pi i}{n}})\right)^k .$$

(ii) Show that $b_k(n)\, n^k \in \mathbb{Q}[n]$.

(iii) Use (i) to give another proof of $\zeta_{\mathcal{A}}(k) = 0$ and $\zeta_{\mathcal{S}}(k) = 0$ by using Theorem 1.36 and 1.37.

Exercise 9. Show that the quasi-shuffle product $*_\diamond$ is associative.

Exercise 10.

(i) Show that for any $k \geq 1$ and $m \geq 1$ we have

$$1 + \sum_{n=1}^{\infty} H_m(\overbrace{k,\dots,k}^{n}) X^n = \exp\left(\sum_{n=1}^{\infty} (-1)^{n-1} H_m(nk) \frac{X^n}{n}\right) .$$

(ii) Prove that for $k \geq 1$ we have $\zeta_{\mathcal{A}}(k,\dots,k) = 0$ and $\zeta(2k,\dots,2k) \in \mathbb{Q}[\pi^2]$.