Homework 1

Deadline: 15th May (23:55 JST), 2022

You can submit your solution at NUCT, by email (henrik.bachmann@math.nagoya-u.ac.jp) or hand it in physically.

Exercise 1. Use the finite double shuffle relations to show $\zeta(6) = 6\zeta(3,3) - 3\zeta(4,2)$.

Exercise 2. Find a formula for the number of indices and admissible indices of a given weight (and depth), i.e. for $r, k \geq 0$ find explicit expressions for the following four integers

$$I_k = \sum_{r\geq 0} I_{k,r}\,, \qquad I_{k,r} = \left|\{\mathbf{k} \in \mathbb{Z}_{\geq 1}^r \mid \mathrm{wt}(\mathbf{k}) = k\}\right|,$$
$$I_k^0 = \sum_{r\geq 0} I_{k,r}^0\,, \qquad I_{k,r}^0 = \left|\{\mathbf{k} \in \mathbb{Z}_{\geq 1}^r \mid \mathrm{wt}(\mathbf{k}) = k, \mathbf{k} \text{ is admissible}\}\right|.$$

Exercise 3.

(i) Show that Conjecture 1.13 together with Proposition 1.12 would imply that all multiple zeta values (except for $\zeta(\emptyset) = 1$) are transcendental.

(ii) Show that Conjecture 1.17 (Hoffman) would imply Conjecture 1.15 (Zagier).

Exercise 4. Show that for any $k_1, \dots, k_r \in \mathbb{Z}$ we have $\zeta_{\mathcal{A}}(k_1, \dots, k_r) \in \mathcal{Z}^{\mathcal{A}}$, i.e. show that you can write $\zeta_{\mathcal{A}}(k_1, \dots, k_r)$ as a linear combination of finite multiple zeta values with positive entries.

Exercise 5. Show that for all $m \geq 1$ and $k_1, \dots, k_r \geq 1$ we have

$$S_m(k_1, \dots, k_r) = \sum_{j=0}^{r} (-1)^{k_1+\dots+k_j} H_m(k_j, k_{j-1}, \dots, k_1) H_m(k_{j+1}, \dots, k_r)\,.$$