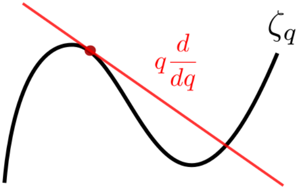
Derivatives of q-analogues of multiple zeta values

Henrik Bachmann
名古屋大学



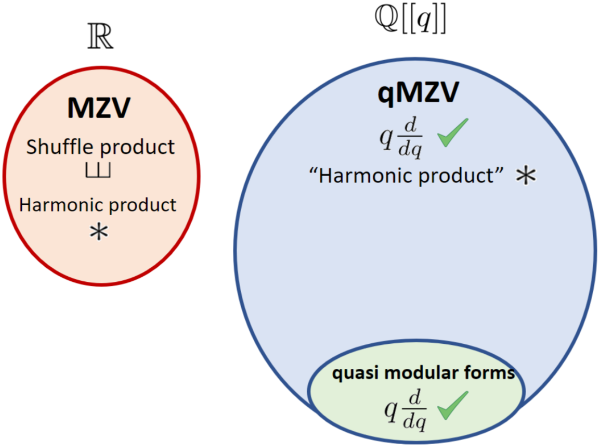
第10回多重ゼータ研究集会

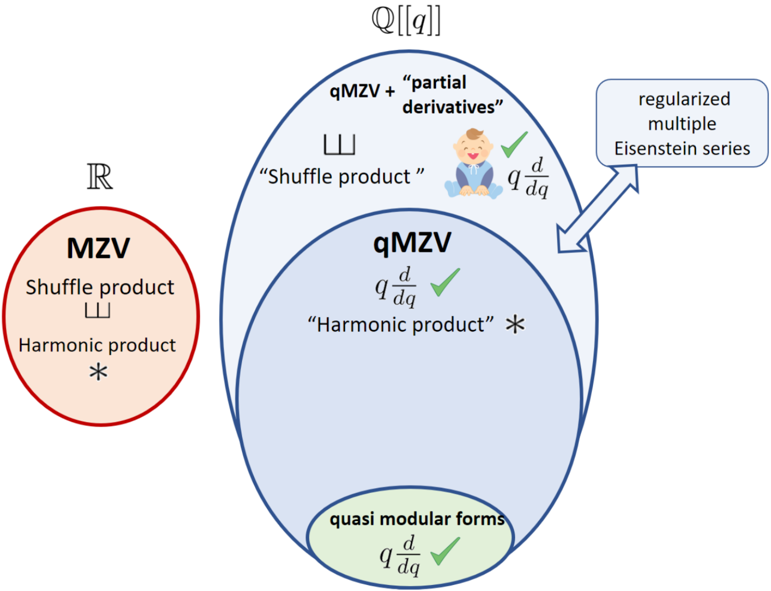
18th February 2017

Slides are available here: **www.henrikbachmann.com**

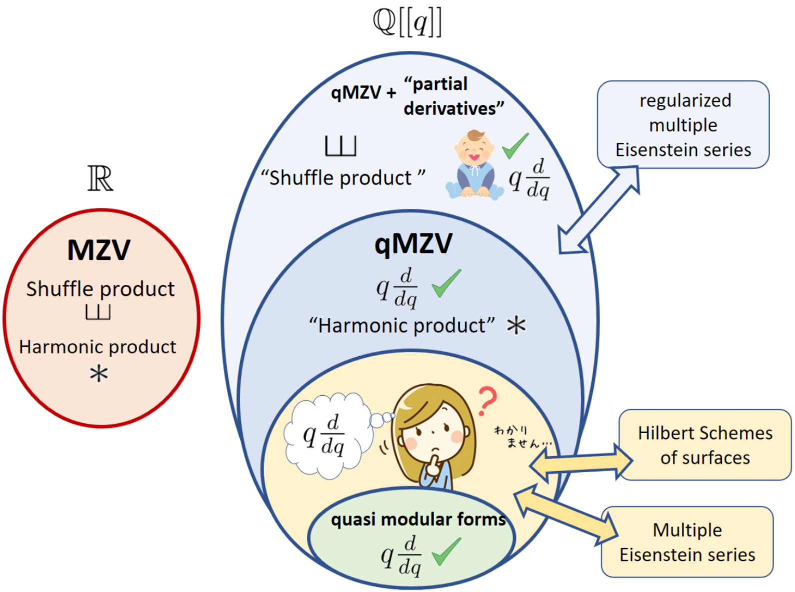
Content of this talk

- Overview & Multiple zeta values
- Different models of q-analogues of multiple zeta values
- Harmonic & Shuffle product
- Shuffle product $\leftrightarrow$ Derivatives
- Certain interesting subspaces and their derivatives


$\mathbb{R}$
MZV
Shuffle product
$ш$
Harmonic product
$*$
$\mathbb{Q}[[q]]$
qMZV
$q\frac{d}{dq}$
"Harmonic product" $*$
quasi modular forms
$q\frac{d}{dq}$



$\mathbb{Q}[[q]]$
$\mathbb{R}$
MZV
Shuffle product
ш
Harmonic product
$*$
qMZV + "partial derivatives"
ш
"Shuffle product "
$q\frac{d}{dq}$
regularized multiple Eisenstein series
qMZV
$q\frac{d}{dq}$
"Harmonic product" $*$
quasi modular forms
$q\frac{d}{dq}$



$\mathbb{Q}[[q]]$
$\mathbb{R}$
MZV
Shuffle product
ш
Harmonic product
*
qMZV + "partial derivatives"
ш
"Shuffle product"
$q\frac{d}{dq}$
regularized multiple Eisenstein series
qMZV
$q\frac{d}{dq}$
"Harmonic product" *
$q\frac{d}{dq}$
わかりません…
Hilbert Schemes of surfaces
quasi modular forms
$q\frac{d}{dq}$
Multiple Eisenstein series

Multiple zeta values

Definition

For $k_1, \dots, k_{r-1} \geq 1, k_r \geq 2$ define the **multiple zeta value** (MZV) by

$$\zeta(k_1, \dots, k_r) = \sum_{0<m_1<\dots<m_r} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}.$$

By r we denote its **depth** and $k_1 + \dots + k_r$ will be called its **weight**. For the $\mathbb{Q}$-vector space spanned by all multiple zeta values we write $\mathcal{Z}$.

- The product of two MZV can be expressed as a linear combination of MZV with the same weight (**harmonic product**). e.g:

$$\zeta(k_1) \cdot \zeta(k_2) = \zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1 + k_2)\,.$$

- MZV can be expressed as iterated integrals. This gives another way (**shuffle product**) to express the product of two MZV as a linear combination of MZV.
- These two products give a number of $\mathbb{Q}$-relations (double shuffle relations) between MZV.

q-analogues

"Roughly speaking, in mathematics, specifically in the areas of combinatorics and special functions, a q-analogue of a theorem, identity or expression is a generalization involving a new parameter q that returns the original theorem, identity or expression in the limit as $q \to 1$. "

ウィキペディア先生

- The easiest example is the q-analogue of a natural number m given by

$$[m]_q = \frac{1-q^m}{1-q} = 1 + q + \cdots + q^{m-1}\,, \qquad \lim_{q\to 1}[m]_q = m\,.$$

- One approach to get an q-analogue of multiple zeta values is to replace $\frac{1}{m^k}$ by $\frac{q^{(k-1)m}}{[m]_q^k}$.

Definition (Bradley, Zhao)

For $k_1, \ldots, k_{r-1} \geq 1, k_r \geq 2$ define the q-multiple zeta value by

$$\zeta_q(k_1, \ldots, k_r) = \sum_{0<m_1<\cdots<m_r} \frac{q^{(k_1-1)m_1} \ldots q^{(k_r-1)m_r}}{[m_1]_q^{k_1} \cdots [m_r]_q^{k_r}}.$$

- Clearly it is $\lim\limits_{q\to 1} \zeta_q(k_1, \ldots, k_r) = \zeta(k_1, \ldots, k_r)$.
- The $\mathbb{Q}$-vector space spanned by these series form a $\mathbb{Q}[q]$-algebra.

$$\zeta_q(2)\zeta_q(3) = \zeta_q(2,3) + \zeta_q(3,2) + \zeta_q(5) + (1-q)\zeta_q(4)\,.$$

q-analogues of multiple zeta values

The algebraic description of q-analogues become easier by removing the factors $(1-q)^k$.

Definition (Okuda-Takeyama)

- For $k_1, \dots, k_{r-1} \geq 1, k_r \geq 2$ define the (modified) q-multiple zeta value by

$$\overline{\zeta}_q(k_1, \dots, k_r) = \sum_{0<m_1<\dots<m_r} \frac{q^{(k_1-1)m_1} \dots q^{(k_r-1)m_r}}{(1-q^{m_1})^{k_1} \cdots (1-q^{m_r})^{k_r}}$$
$$= (1-q)^{-k} \zeta_q(k_1, \dots, k_r) \in \mathbb{Q}[[q]] \,.$$

  with $k = k_1 + \dots + k_r$.

- For the $\mathbb{Q}$-vector space spanned by these series we write

$$\mathcal{Z}_q := \left\langle \overline{\zeta}_q(k_1, \dots, k_r) \mid r \geq 0, k_1, \dots, k_{r-1} \geq 1 \,, k_r \geq 2 \right\rangle_{\mathbb{Q}} \,,$$

  and set $\overline{\zeta}_q(k_1, \dots, k_r) = 1$ for $r = 0$.

- Clearly it is $\lim_{q \to 1} (1-q)^k \overline{\zeta}_q(k_1, \dots, k_r) = \zeta(k_1, \dots, k_r)$.
- The $\mathbb{Q}$-vector space $\mathcal{Z}_q$ is a $\mathbb{Q}$-algebra.

q-analogues of multiple zeta values

More generally: Given a family of polynomials $Q_k(X)$ for $k \geq 1$ with $Q_k(1) = 1$ one can define a q-analogue of multiple zeta values by

$$\sum_{0<m_1<\dots<m_r} \frac{Q_{k_1}(q^{m_1}) \dots Q_{k_r}(q^{m_r})}{(1-q^{m_1})^{k_1} \cdots (1-q^{m_r})^{k_r}}$$

- In the case $\overline{\zeta}_q$ the polynomials $Q_k(X) = X^{k-1}$ are used.
- To study the connection to modular forms the following polynomials are more useful

$$\frac{Q_k(X)}{(1-X)^k} := \sum_{d>0} \frac{d^{k-1}}{(k-1)!} X^d \,.$$

- In special cases (all $k_j \geq 2$) both models are basically the same.

q-analogues of multiple zeta values

Definition

- For $k_1, \dots, k_r \geq 1$ we define the following q-series in $\mathbb{Q}[[q]]$

$$g_{k_1,\dots,k_r}(q) := \sum_{\substack{0<u_1<\dots<u_r \\ 0<v_1,\dots,v_r}} \frac{v_1^{k_1-1} \dots v_l^{k_r-1}}{(k_1-1)! \dots (k_r-1)!} \cdot q^{u_1 v_1 + \dots + u_r v_r} .$$

  By $k_1 + \dots + k_r$ we denote its weight and by r its depth.

- For the $\mathbb{Q}$-vector space spanned by these series we write

$$\mathcal{G} := \left\langle g_{k_1,\dots,k_r}(q) \mid r \geq 0, k_1, \dots, k_r \geq 1 \right\rangle_{\mathbb{Q}} ,$$

  where we also set $g_{k_1,\dots,k_r}(q) = 1$ for $r = 0$.

q-analogues of multiple zeta values

Definition

- For $k_1, \dots, k_r \geq 1$ we define the following q-series in $\mathbb{Q}[[q]]$

$$g_{k_1,\dots,k_r}(q) := \sum_{\substack{0<u_1<\dots<u_r \\ 0<v_1,\dots,v_r}} \frac{v_1^{k_1-1} \dots v_l^{k_r-1}}{(k_1-1)! \dots (k_r-1)!} \cdot q^{u_1 v_1 + \dots + u_r v_r} .$$

  By $k_1 + \dots + k_r$ we denote its weight and by r its depth.

- For the $\mathbb{Q}$-vector space spanned by these series we write

$$\mathcal{G} := \left\langle g_{k_1,\dots,k_r}(q) \mid r \geq 0, k_1, \dots, k_r \geq 1 \right\rangle_{\mathbb{Q}} ,$$

  where we also set $g_{k_1,\dots,k_r}(q) = 1$ for $r = 0$.

In depth one these are just the generating series of the divisor-sum $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$:

$$g_k(q) = \sum_{\substack{0<u_1 \\ 0<v_1}} \frac{v_1^{k-1}}{(k-1)!} q^{u_1 v_1} = \frac{1}{(k-1)!} \sum_{n>0} \sigma_{k-1}(n) q^n .$$

q-analogues of multiple zeta values

For the generating function of the q-series $g_{k_1,\dots,k_r}$ we write

$$\mathfrak{g}(x_1,\dots,x_r) := \sum_{k_1,\dots,k_r\geq 1} g_{k_1,\dots,k_r}(q)\, x_1^{k_1-1}\dots x_r^{k_r-1}\,.$$

Lemma

The series $\mathfrak{g}$ can be written as

$$\mathfrak{g}(x_1,\dots,x_r) = \sum_{0<u_1<\dots<u_r} L_{u_1}(x_1)\dots L_{u_r}(x_r)\,,$$

where

$$L_u(x) = \frac{q^u e^x}{1-q^u e^x}\,.$$

The series $L_u(x)$ satisfy the equation

$$L_u(x)\cdot L_u(y) = \frac{L_u(x)-L_u(y)}{x-y} + B(x-y)L_u(x) + B(y-x)L_u(y)$$

with $B(T)=\sum_{k=1}^{\infty}\frac{B_k}{k!}T^{k-1}$.

MZV: Recall that the generating series of the harmonic multiple zeta values

$$\mathfrak{T}^*(x_1,\ldots,x_r) = \sum_{k_1,\ldots,k_r\geq 1} \zeta^*(k_1,\ldots,k_r) x_1^{k_1-1}\ldots x_r^{k_r-1}$$

satisfy $$\mathfrak{T}^*(x)\cdot\mathfrak{T}^*(y) = \mathfrak{T}^*(x,y) + \mathfrak{T}^*(y,x) + \frac{\mathfrak{T}^*(x)-\mathfrak{T}^*(y)}{x-y}\,.$$

qMZV: The generating series $\mathfrak{g}(x_1,\ldots,x_r)$ satisfies similar equations, e.g.

$$\begin{aligned}\mathfrak{g}(x)\cdot\mathfrak{g}(y) = \mathfrak{g}(x,y) + \mathfrak{g}(y,x) &+ \frac{\mathfrak{g}(x)-\mathfrak{g}(y)}{x-y}\\ &+ B(x-y)\cdot\mathfrak{g}(x) + B(y-x)\cdot\mathfrak{g}(y)\,,\end{aligned}$$

with $B(T) = \sum_{k=1}^{\infty}\frac{B_k}{k!}T^{k-1}$.

q-analogues of multiple zeta values : harmonic & shuffle product

Let $\Bbbk_1$ and $\Bbbk_2$ be two arbitrary index sets.

We have just seen the following rough picture:

Harmonic product $*$

$$\zeta^*(\Bbbk_1) \cdot \zeta^*(\Bbbk_2) = \zeta^*(\Bbbk_1 * \Bbbk_2)$$
$$g_{\Bbbk_1}(q) \cdot g_{\Bbbk_2}(q) \ = g_{\Bbbk_1 * \Bbbk_2}(q) + \text{ lower weight terms .}$$

Now we want to explain that something similar is true for the shuffle product:

Shuffle product $⧢$

$$\zeta^{⧢}(\Bbbk_1) \cdot \zeta^{⧢}(\Bbbk_2) = \zeta^{⧢}(\Bbbk_1 ⧢ \Bbbk_2)$$
$$g_{\Bbbk_1}(q) \cdot g_{\Bbbk_2}(q) \ = g_{\Bbbk_1 ⧢ \Bbbk_2}(q) + \text{ lower weight terms } + \text{ "derivatives" .}$$

q-analogues of multiple zeta values : harmonic & shuffle product

Let $\mathbb{k}_1$ and $\mathbb{k}_2$ be two arbitrary index sets.

We have just seen the following rough picture:

Harmonic product $*$

$$\zeta^*(\mathbb{k}_1)\cdot\zeta^*(\mathbb{k}_2) = \zeta^*(\mathbb{k}_1 * \mathbb{k}_2)$$
$$g_{\mathbb{k}_1}(q)\cdot g_{\mathbb{k}_2}(q) \;= g_{\mathbb{k}_1 * \mathbb{k}_2}(q) + \text{ lower weight terms .}$$

Now we want to explain that something similar is true for the shuffle product:

Shuffle product $\sqcup\!\sqcup$

$$\zeta^{\sqcup\!\sqcup}(\mathbb{k}_1)\cdot\zeta^{\sqcup\!\sqcup}(\mathbb{k}_2) = \zeta^{\sqcup\!\sqcup}(\mathbb{k}_1 \sqcup\!\sqcup \mathbb{k}_2)$$
$$g_{\mathbb{k}_1}(q)\cdot g_{\mathbb{k}_2}(q) \;= g_{\mathbb{k}_1 \sqcup\!\sqcup \mathbb{k}_2}(q) + \text{ lower weight terms } + \text{ "derivatives" .}$$

$$\text{"derivatives"} = g_{\mathbb{k}_1}(q)\cdot g_{\mathbb{k}_2}(q) - g_{\mathbb{k}_1 \sqcup\!\sqcup \mathbb{k}_2}(q) - \text{ lower weight terms .}$$

The operator D

Today we will be interested in the operator D on $\mathbb{Q}[[q]]$ defined by

$$D := q\frac{d}{dq}\,.$$

- For a q-analogue the operator D increases the **weight by** 2 and the **depth by** 1.
- Given numbers $k_1, \dots, k_r \geq 1$ with $k = k_1 + \dots + k_r$ it is

$$\lim_{q\to 1}(1-q)^{k+2} D g_{k_1,\dots,k_r}(q) = 0\,,$$

  i.e. **formulas for the derivative of q-analogues give relations between MZV**.
- The sub algebra $\mathbb{Q}[\widetilde{G}_2, \widetilde{G}_4, \widetilde{G}_6] \subset \mathcal{G}$ of **quasi-modular forms** is closed under D.

$$\widetilde{G}_{2n}(q) := \frac{1}{2}\frac{B_{2n}}{(2n)!} + g_{2n}(q) \in \mathcal{G}\,.$$

q-analogues of multiple zeta values : shuffle product

Definition

Define for $n_1, \ldots, n_r \geq 1$ the series

$$H\binom{n_1, \ldots, n_r}{x_1, \ldots, x_r} = \sum_{0<d_1<\cdots<d_r} e^{d_1 x_1} \left(\frac{q^{d_1}}{1-q^{d_1}}\right)^{n_1} \ldots e^{d_r x_r} \left(\frac{q^{d_r}}{1-q^{d_r}}\right)^{n_r} .$$

Notice that this series "satisfies" the harmonic product formula. For example:

$$H\binom{n_1}{x_1} \cdot H\binom{n_2}{x_2} = H\binom{n_1, n_2}{x_1, x_2} + H\binom{n_2, n_1}{x_2, x_1} + H\binom{n_1 + n_2}{x_1 + x_2}.$$

q-analogues of multiple zeta values : shuffle product

Definition

Define for $n_1, \dots, n_r \geq 1$ the series

$$H\binom{n_1, \dots, n_r}{x_1, \dots, x_r} = \sum_{0<d_1<\cdots<d_r} e^{d_1 x_1} \left(\frac{q^{d_1}}{1-q^{d_1}}\right)^{n_1} \dots e^{d_r x_r} \left(\frac{q^{d_r}}{1-q^{d_r}}\right)^{n_r} .$$

Notice that this series "satisfies" the harmonic product formula. For example:

$$H\binom{n_1}{x_1} \cdot H\binom{n_2}{x_2} = H\binom{n_1, n_2}{x_1, x_2} + H\binom{n_2, n_1}{x_2, x_1} + H\binom{n_1 + n_2}{x_1 + x_2}.$$

The connection to the series $\mathfrak{g}$ is given by

$$\mathfrak{g}(x_1, \dots, x_r) = H\binom{1, \dots, 1, 1}{x_r - x_{r-1}, \dots, x_2 - x_1, x_1},$$

or equivalently

$$H\binom{1, \dots, 1}{y_1, \dots, y_r} = \mathfrak{g}(y_r, y_{r-1} + y_r, \dots, y_1 + \cdots + y_r).$$

So if we multiply the generating series in depth one we get

$$\begin{aligned}\mathfrak{g}(x)\cdot\mathfrak{g}(y) &= H\binom{1}{x}\cdot H\binom{1}{y}\\ &= H\binom{1,1}{x,y} + H\binom{1,1}{y,x} + H\binom{2}{x_1+x_2}\\ &= \mathfrak{g}(x,x+y) + \mathfrak{g}(y,x+y) + H\binom{2}{x+y}.\end{aligned}$$

q-analogues of multiple zeta values : shuffle product

MZV: Recall that the generating series of the shuffle regularized multiple zeta values

$$\mathfrak{T}^{\sqcup\!\sqcup}(x_1,\ldots,x_r) = \sum_{k_1,\ldots,k_r\geq 1} \zeta^{\sqcup\!\sqcup}(k_1,\ldots,k_r) x_1^{k_1-1}\ldots x_r^{k_r-1}$$

satisfy

$$\mathfrak{T}^{\sqcup\!\sqcup}(x)\cdot\mathfrak{T}^{\sqcup\!\sqcup}(y) = \mathfrak{T}^{\sqcup\!\sqcup}(x,x+y) + \mathfrak{T}^{\sqcup\!\sqcup}(y,x+y)\,.$$

qMZV: The generating series $\mathfrak{g}(x_1,\ldots,x_r)$ satisfies similar equations, e.g.

$$\mathfrak{g}(x)\cdot\mathfrak{g}(y) = \mathfrak{g}(x,x+y) + \mathfrak{g}(y,x+y) + H\binom{2}{x+y}\,.$$

q-analogues of multiple zeta values : shuffle product

MZV: Recall that the generating series of the shuffle regularized multiple zeta values

$$\mathfrak{T}^{\sqcup\!\sqcup}(x_1,\dots,x_r) = \sum_{k_1,\dots,k_r\geq 1} \zeta^{\sqcup\!\sqcup}(k_1,\dots,k_r)x_1^{k_1-1}\dots x_r^{k_r-1}$$

satisfy

$$\mathfrak{T}^{\sqcup\!\sqcup}(x)\cdot\mathfrak{T}^{\sqcup\!\sqcup}(y) = \mathfrak{T}^{\sqcup\!\sqcup}(x,x+y) + \mathfrak{T}^{\sqcup\!\sqcup}(y,x+y)\,.$$

qMZV: The generating series $\mathfrak{g}(x_1,\dots,x_r)$ satisfies similar equations, e.g.

$$\mathfrak{g}(x)\cdot\mathfrak{g}(y) = \mathfrak{g}(x,x+y) + \mathfrak{g}(y,x+y) + H\binom{2}{x+y}\,.$$

- The properties of the function H can be used to define series $g^{\sqcup\!\sqcup}$ satisfying the shuffle product formula of MZV.
- We now want to explain the connection of $H\binom{2}{x+y}$ and the operator $D = q\frac{d}{dq}$.

First notice that

$$D\,\frac{q^d}{1-q^d} = q\frac{d}{dq}\,\frac{q^d}{1-q^d} = d\left(\frac{q^d}{1-q^d}\right)^2 + d\frac{q^d}{1-q^d}\,,$$

which leads to

$$\begin{aligned}\sum_{k>0} Dg_k(q)x^{k-1} = D\mathfrak{g}(x) = DH\binom{1}{x} &= D\sum_{0<d} e^{dx}\,\frac{q^d}{1-q^d}\\ &= \sum_{0<d} de^{dx}\left(\frac{q^d}{1-q^d}\right)^2 + \sum_{0<d} de^{dx}\,\frac{q^d}{1-q^d}\\ &= \frac{d}{dy}\left(H\binom{2}{x+y} + \underbrace{H\binom{1}{x+y}}_{=\mathfrak{g}(x+y)}\right)\Big|_{y=0}\,.\end{aligned}$$

With $H\binom{2}{x+y} = \mathfrak{g}(x)\cdot\mathfrak{g}(y) - \mathfrak{g}(x,x+y) - \mathfrak{g}(y,x+y)$ we get

$$\sum_{k>0} Dg_k(q)x^{k-1} = \frac{d}{dy}\Big(\mathfrak{g}(x)\cdot\mathfrak{g}(y) - \mathfrak{g}(x,x+y) - \mathfrak{g}(y,x+y) + \mathfrak{g}(x+y)\Big)\Big|_{y=0}.$$

In particular this proves that $Dg_k(q) \in \mathcal{G}$.

q-analogues of multiple zeta values : shuffle product & derivative

With $H\binom{2}{x+y} = \mathfrak{g}(x)\cdot\mathfrak{g}(y) - \mathfrak{g}(x, x+y) - \mathfrak{g}(y, x+y)$ we get

$$\sum_{k>0} Dg_k(q)x^{k-1} = \frac{d}{dy}\Big(\mathfrak{g}(x)\cdot\mathfrak{g}(y) - \mathfrak{g}(x, x+y) - \mathfrak{g}(y, x+y) + \mathfrak{g}(x+y)\Big)\Big|_{y=0}.$$

In particular this proves that $Dg_k(q) \in \mathcal{G}$.

Since $\frac{d}{dy}\mathfrak{g}(y)\big|_{y=0}$ equals $g_2(q)$ this can be interpreted as

Derivative of $g_k(q)$ = Failure of the shuffle product formula for $g_k(q)\cdot g_2(q)$ + Lower weight terms

Derivative of $g_k(q)$ $=$ Failure of the shuffle product formula for $g_k(q) \cdot g_2(q)$ $+$ Lower weight terms

Example

The shuffle product formula of $\zeta(3) \cdot \zeta(2)$ reads

$$\zeta(3) \cdot \zeta(2) = \zeta(3,2) + 3\zeta(2,3) + 6\zeta(1,4)\,. \tag{1}$$

The derivative of $g_3(q)$ is given by

$$Dg_3(q) = g_3(q) \cdot g_2(q) - g_{3,2}(q) - 3g_{2,3}(q) - 6g_{1,4}(q) + 3g_4(q)\,. \tag{2}$$

Notice: Multiplying (2) by $(1-q)^5$ and taking the limit $q \to 1$ one obtains (1).

q-analogues of multiple zeta values : derivative

This can be done for arbitrary depth:

Theorem

The derivative of the generating series $\mathfrak{g}$ can be written as

$$
\begin{aligned}
D\mathfrak{g}(x_1,\dots,x_r) &= \mathfrak{g}(x_1,\dots,x_r)\cdot g_2(q) \\
&\quad - \frac{d}{dy}\left(\sum_{j=0}^{r}\mathfrak{g}(x_1,x_2,\dots,x_{r-j},x_{r-j}+y,x_{r-j+1}+y,\dots,x_r+y)\right)\Big|_{y=0} \\
&\quad - \frac{d}{dy}\left(\sum_{j=1}^{r}\mathfrak{g}(x_1,\dots,x_{j-1},x_j+y,\dots,x_r+y)\right)\Big|_{y=0}
\end{aligned}
$$

In particular the space $\mathcal{G}$ is closed under D.

q-analogues of multiple zeta values : derivative

This can be done for arbitrary depth:

Theorem

The derivative of the generating series $\mathfrak{g}$ can be written as

$$D\mathfrak{g}(x_1,\ldots,x_r) = \mathfrak{g}(x_1,\ldots,x_r)\cdot g_2(q) - \frac{d}{dy}\left(\sum_{j=0}^{r}\mathfrak{g}(x_1,x_2,\ldots,x_{r-j},x_{r-j}+y,x_{r-j+1}+y,\ldots,x_r+y)\right)\Big|_{y=0} - \frac{d}{dy}\left(\sum_{j=1}^{r}\mathfrak{g}(x_1,\ldots,x_{j-1},x_j+y,\ldots,x_r+y)\right)\Big|_{y=0}$$

In particular the space $\mathcal{G}$ is closed under D.

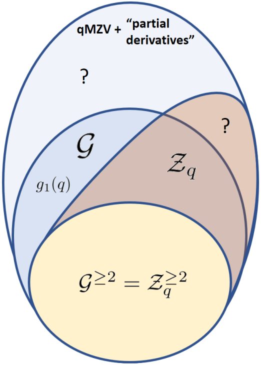
Derivative of $g_{k_1,\ldots,k_r}(q)$ $=$ Failure of the shuffle product formula for $g_{k_1,\ldots,k_r}(q)\cdot g_2(q)$ $+$ Lower weight & depth terms

q-analogues of multiple zeta values : A certain subspace

We now want to study certain subspaces of our q-analogues which we denote by

$$\mathcal{G}^{\geq 2} := \left\langle g_{k_1,\dots,k_r}(q) \mid r \geq 0,\, k_1,\dots,k_r \geq 2 \right\rangle_{\mathbb{Q}} \subset \mathcal{G}\,,$$
$$\mathcal{Z}_q^{\geq 2} := \left\langle \overline{\zeta}_q(k_1,\dots,k_r) \mid r \geq 0,\, k_1,\dots,k_r \geq 2 \right\rangle_{\mathbb{Q}} \subset \mathcal{Z}_q\,.$$

Even though $\mathcal{Z}_q \neq \mathcal{G}$ it is easy to prove that $\mathcal{G}^{\geq 2} = \mathcal{Z}_q^{\geq 2}$.



q-analogues of multiple zeta values : A certain subspace

We now want to study certain subspaces of our q-analogues which we denote by

$$\mathcal{G}^{\geq 2} := \left\langle g_{k_1,\dots,k_r}(q) \mid r \geq 0,\ k_1,\dots,k_r \geq 2 \right\rangle_{\mathbb{Q}} \subset \mathcal{G}\,,$$
$$\mathcal{Z}_q^{\geq 2} := \left\langle \overline{\zeta}_q(k_1,\dots,k_r) \mid r \geq 0,\ k_1,\dots,k_r \geq 2 \right\rangle_{\mathbb{Q}} \subset \mathcal{Z}_q\,.$$

Even though $\mathcal{Z}_q \neq \mathcal{G}$ it is easy to prove that $\mathcal{G}^{\geq 2} = \mathcal{Z}_q^{\geq 2}$.

Theorem

Define for $2 \leq s \leq k$ the numbers $\alpha_{s,k} \in \mathbb{Q}$ by

$$\sum_{s=2}^{k} \frac{\alpha_{s,k}}{(s-1)!} X^{s-1} := \binom{X}{k-1} = \frac{X(X-1)\dots(X-k+2)}{(k-1)!}\,.$$

Then we have for $k_1,\dots,k_r \geq 2$

$$\overline{\zeta}_q(k_1,\dots,k_r) = \sum_{\substack{2 \leq s_j \leq k_j \\ 1 \leq j \leq r}} \alpha_{s_1,k_1} \dots \alpha_{s_r,k_r}\, g_{s_1,\dots,s_r}(q)\,.$$

q-analogues of multiple zeta values : A certain subspace

We now want to study certain subspaces of our q-analogues which we denote by

$$\mathcal{G}^{\geq 2} := \left\langle g_{k_1,\dots,k_r}(q) \mid r \geq 0,\, k_1,\dots,k_r \geq 2 \right\rangle_{\mathbb{Q}} \subset \mathcal{G}\,,$$
$$\mathcal{Z}_q^{\geq 2} := \left\langle \overline{\zeta}_q(k_1,\dots,k_r) \mid r \geq 0,\, k_1,\dots,k_r \geq 2 \right\rangle_{\mathbb{Q}} \subset \mathcal{Z}_q\,.$$

Even though $\mathcal{Z}_q \neq \mathcal{G}$ it is easy to prove that $\mathcal{G}^{\geq 2} = \mathcal{Z}_q^{\geq 2}$.

Theorem

Define for $2 \leq s \leq k$ the numbers $\beta_{k,s} \in \mathbb{Q}$ by

$$\sum_{2 \leq s \leq k < \infty} \beta_{k,s}\, T^{k-s} X^k = \frac{XT}{T + 1 - e^{XT}} - X\,.$$

Then we have for $k_1,\dots,k_r \geq 2$

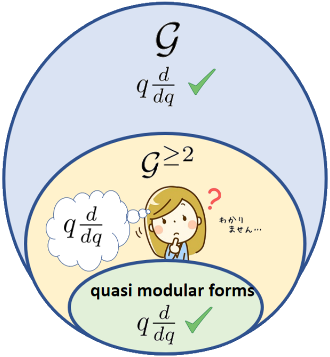
$$g_{k_1,\dots,k_r}(q) = \sum_{\substack{2 \leq s_j \leq k_j \\ 1 \leq j \leq r}} \beta_{s_1,k_1} \dots \beta_{s_r,k_r}\, \overline{\zeta}_q(s_1,\dots,s_r)\,.$$

q-analogues of multiple zeta values : A certain subspace

We now want to study certain subspaces of our q-analogues which we denote by

$$\mathcal{G}^{\geq 2} := \left\langle g_{k_1,\dots,k_r}(q) \mid r \geq 0,\ k_1,\dots,k_r \geq 2 \right\rangle_{\mathbb{Q}} \subset \mathcal{G}\,,$$
$$\mathcal{Z}_q^{\geq 2} := \left\langle \overline{\zeta}_q(k_1,\dots,k_r) \mid r \geq 0,\ k_1,\dots,k_r \geq 2 \right\rangle_{\mathbb{Q}} \subset \mathcal{Z}_q\,.$$

Even though $\mathcal{Z}_q \neq \mathcal{G}$ it is easy to prove that $\mathcal{G}^{\geq 2} = \mathcal{Z}_q^{\geq 2}$.



q-analogues of multiple zeta values : A certain subspace

We now want to study certain subspaces of our q-analogues which we denote by

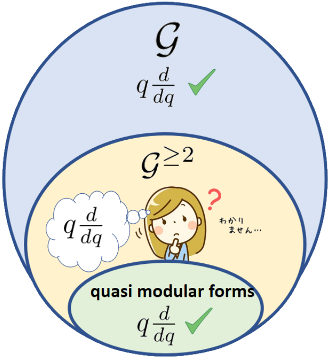
$$\mathcal{G}^{\geq 2} := \left\langle g_{k_1,\dots,k_r}(q) \mid r \geq 0,\ k_1,\dots,k_r \geq 2 \right\rangle_{\mathbb{Q}} \subset \mathcal{G}\,,$$

$$\mathcal{Z}_q^{\geq 2} := \left\langle \overline{\zeta}_q(k_1,\dots,k_r) \mid r \geq 0,\ k_1,\dots,k_r \geq 2 \right\rangle_{\mathbb{Q}} \subset \mathcal{Z}_q\,.$$

Even though $\mathcal{Z}_q \neq \mathcal{G}$ it is easy to prove that $\mathcal{G}^{\geq 2} = \mathcal{Z}_q^{\geq 2}$.

Conjecture

The space $\mathcal{G}^{\geq 2}$ is close under $D = q\frac{q}{dq}$.



Motivation 1: Multiple Eisenstein series

For $k_1, \ldots, k_r \geq 2$ the multiple Eisenstein series $G_{k_1,\ldots,k_r}(\tau)$ is defined by

$$G_{k_1,\ldots,k_r}(\tau) = \sum_{\substack{0 \prec \lambda_1 \prec \cdots \prec \lambda_r \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \cdots \lambda_r^{k_r}} \,,$$

where $\tau \in \{x + iy \in \mathbb{C} \mid y > 0\}$ is an element in the upper half plane and the order $\prec$ on $\mathbb{Z}\tau + \mathbb{Z}$ is defined by

$$m_1\tau + n_1 \prec m_2\tau + n_2 :\Leftrightarrow (m_1 < m_2) \vee (m_1 = m_2 \wedge n_1 < n_2)\,.$$

Motivation 1: Multiple Eisenstein series

For $k_1, \ldots, k_r \geq 2$ the multiple Eisenstein series $G_{k_1,\ldots,k_r}(\tau)$ is defined by

$$G_{k_1,\ldots,k_r}(\tau) = \sum_{\substack{0 \prec \lambda_1 \prec \cdots \prec \lambda_r \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \cdots \lambda_r^{k_r}},$$

where $\tau \in \{x + iy \in \mathbb{C} \mid y > 0\}$ is an element in the upper half plane and the order $\prec$ on $\mathbb{Z}\tau + \mathbb{Z}$ is defined by

$$m_1\tau + n_1 \prec m_2\tau + n_2 :\Leftrightarrow (m_1 < m_2) \vee (m_1 = m_2 \wedge n_1 < n_2).$$

Theorem

Setting $q = \exp(2\pi i \tau)$ the $\mathbb{C}$-vector space spanned by all multiple Eisenstein series $G_{k_1,\ldots,k_r}(\tau)$ with $k_1, \ldots, k_r \geq 2$ equals $\mathbb{C} \otimes \mathcal{G}^{\geq 2}$.

Conjecture

The $\mathbb{C}$-vector space spanned by all $G_{k_1,\ldots,k_r}(\tau)$ with $k_1, \ldots, k_r \geq 2$ is closed under

$$\frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq} = D.$$

Motivation 2: Hilbert Scheme of surfaces

q-analogues of multiple zeta values also appear in algebraic geometry.

- S : nonsingular quasi-projective surface
- $\mathrm{Hilb}(S,n)$: Hilbert scheme (parametrizes 0-dim. length n subschemes of S)

In a recent work A. Okounkov introduces for a **characteristic class** f on S a q-series

$$\langle f \rangle = \sum_{n>0} \left(\int_{\mathrm{Hilb}(S,n)} \dots \right) q^n \,.$$

Conjecture (Okounkov)

For every characteristic class f on S it is $\langle f \rangle \in \mathcal{G}^{\geq 2}$.

Using geometric arguments one can show that for a certain characteristic class c on S it is

$$D\langle f \rangle = q\frac{d}{dq}\langle f \rangle = \langle f \cdot c \rangle - g_2(q)\,,$$

which also lead Okounkov to the Conjecture that $\mathcal{G}^{\geq 2}$ is closed under D.

Derivatives in $\mathcal{G}^{\geq 2}$

We can not use the formula from the Theorem before, since we have for example

$$Dg_3(q) = g_3(q) \cdot g_2(q) - g_{3,2}(q) - 3g_{2,3}(q) - 6g_{1,4}(q) + 3g_4(q)\,.$$

All elements on the right side are in $\mathcal{G}^{\geq 2}$ except for $g_{1,4}(q)$.

Derivatives in $\mathcal{G}^{\geq 2}$

$$Dg_3(q) = g_3(q)\cdot g_2(q) - g_{3,2}(q) - 3g_{2,3}(q) - 6g_{1,4}(q) + 3g_4(q)\,.$$

All elements on the right side are in $\mathcal{G}^{\geq 2}$ except for $g_{1,4}(q)$.

Above formula was obtained by considering the coefficient of x^2 in

$$\begin{aligned} D\mathfrak{g}(x) &= \sum_{0<d} de^{dx}\left(\frac{q^d}{1-q^d}\right)^2 + \sum_{0<d} de^{dx}\frac{q^d}{1-q^d} \\ &= \frac{d}{dy}\Big(H\binom{2}{x+y} + H\binom{1}{x+y}\Big)\Big|_{y=0} \\ &= \frac{d}{dy}\Big(\mathfrak{g}(x)\cdot\mathfrak{g}(y) - \mathfrak{g}(x,x+y) - \mathfrak{g}(y,x+y) + \mathfrak{g}(x+y)\Big)\Big|_{y=0}\,. \end{aligned}$$

Derivatives in $\mathcal{G}^{\geq 2}$

$$Dg_3(q) = 3g_1(q) \cdot g_4(q) - 6g_{1,4}(q) - 3g_{2,3}(q) - 3g_{3,2}(q) - 3g_{4,1}(q) + 3g_4(q)\,.$$

All elements on the right side are in $\mathcal{G}^{\geq 2}$ except for $g_{1,4}(q)$, $g_{4,1}(q)$ and $g_1(q) \cdot g_4(q)$.

Above formula was obtained by considering the coefficient of x^2 in

$$\begin{aligned} D\mathfrak{g}(x) &= \sum_{0<d} de^{dx} \left(\frac{q^d}{1-q^d}\right)^2 + \sum_{0<d} de^{dx} \frac{q^d}{1-q^d} \\ &= \frac{d}{dx}\left(H\binom{2}{x+y} + H\binom{1}{x+y}\right)\Big|_{y=0} \\ &= \frac{d}{dx}\Big(\mathfrak{g}(x) \cdot \mathfrak{g}(y) - \mathfrak{g}(x, x+y) - \mathfrak{g}(y, x+y) + \mathfrak{g}(x+y)\Big)\Big|_{y=0}\,. \end{aligned}$$

Clearly the $\frac{d}{dy}$ can also be replaced by $\frac{d}{dx}$.

Derivatives in $\mathcal{G}^{\geq 2}$

$$Dg_3(q) = 5g_5(q) - 4g_{2,3}(q) - 6g_{3,2}(q) + \frac{7}{12} g_3(q)\,.$$

All the elements on the right side are in $\mathcal{G}^{\geq 2}$.

Above formula was obtained by considering the coefficient of x^2 in

$$\begin{aligned} D\mathfrak{g}(x) &= \sum_{0<d} d e^{dx} \left(\frac{q^d}{1-q^d} \right)^2 + \sum_{0<d} d e^{dx} \frac{q^d}{1-q^d} \\ &= \left(2\frac{d}{dx} - \frac{d}{dy} \right) \left(H\binom{2}{x+y} + H\binom{1}{x+y} \right) \Big|_{y=0} \\ &= \left(2\frac{d}{dx} - \frac{d}{dy} \right) \Big(\mathfrak{g}(x) \cdot \mathfrak{g}(y) - \mathfrak{g}(x, x+y) - \mathfrak{g}(y, x+y) + \mathfrak{g}(x+y) \Big) \Big|_{y=0}\,. \end{aligned}$$

Instead of $\frac{d}{dy}$ and $\frac{d}{dx}$ we can also use $2\frac{d}{dx} - \frac{d}{dy}$.

(and evaluate the product by using the harmonic product).

Derivatives in $\mathcal{G}^{\geq 2}$: Depth one

Theorem

For $k \geq 1$ the derivative of $g_k(q)$ is given by

$$Dg_k(q) = q\frac{d}{dq}g_k(q) = (2k-1)g_{k+2}(q) - \sum_{j=2}^{k}(k+j-1)g_{j,k+2-j}(q) - g_{k,2}(q)$$

$$+ \sum_{j=2}^{k} \frac{B_{k+2-j}}{(k+2-j)!}(3k-j+1)g_j(q) + (-1)^k \frac{B_k}{k!} g_2(q)\,.$$

In particular $Dg_k(q) \in \mathcal{G}^{\geq 2}$ for $k \geq 2$.

Example:

$$q\frac{d}{dq}g_2(q) = 3g_4(q) - 4g_{2,2}(q) + \frac{1}{2}g_2(q)\,.$$

Notice that by multiplying both sides with $(1-q)^{k+2}$ and taking the limit $q \to 1$ we obtain

$$(2k-1)\zeta(k+2) = \sum_{j=2}^{k}(k+j-1)\zeta(j,k+2-j) + \zeta(k,2)\,.$$

Derivatives of $\mathcal{G}^{\geq 2}$: Higher depths

From the Theorem we obtain inductively the following corollary

Corollary

For every $k \geq 2$ we have $Dg_{k,\dots,k}(q) \in \mathcal{G}^{\geq 2}$.

Derivatives of $\mathcal{G}^{\geq 2}$: Higher depths

From the Theorem we obtain inductively the following corollary

Corollary

For every $k \geq 2$ we have $Dg_{k,\dots,k}(q) \in \mathcal{G}^{\geq 2}$.

For **even** k this can also proven without the Theorem by showing that

$$g_{k,\dots,k}(q) \in \mathbb{Q}[\widetilde{G}_2(q), \widetilde{G}_4(q), \widetilde{G}_6(q)]\,.$$

Theorem

The series $g_{\{2\}^r}(q) = g_{2,\dots,2}(q)$ is the coefficient of X^{2r+1} in

$$2\arcsin\left(\frac{X}{2}\right) \exp\left(\sum_{j\geq 1} \frac{(-1)^{j-1}}{j} \widetilde{G}_{2j}(q) \left(2\arcsin\left(\frac{X}{2}\right)\right)^{2j}\right)\,.$$

Proof idea: Use an explicit formula for the Fourier expansion of Multiple Eisenstein series.

Derivatives of $\mathcal{G}^{\geq 2}$: Depth one for $\overline{\zeta}_q$

We can also obtain the $\overline{\zeta}_q$ version of our Theorem:

Theorem

For $k \geq 3$ the derivative of $\overline{\zeta}_q(k)$ is given by

$$\begin{aligned} D\overline{\zeta}_q(k) = & (2k-1)\overline{\zeta}_q(k+2) + 3(k-1)\overline{\zeta}_q(k+1) + (k-1)\overline{\zeta}_q(k) \\ & - \sum_{j=2}^{k-2}(k+j-1)\left(\overline{\zeta}_q(j,k+2-j) + \overline{\zeta}_q(j+1,k-j)\right) \\ & - 2k\overline{\zeta}_q(k,2) - (2k-2)\overline{\zeta}_q(k-1,3) - k\overline{\zeta}_q(2,k-1) \end{aligned}$$

and for $k = 2$ it is

$$D\overline{\zeta}_q(2) = 3\overline{\zeta}_q(4) + 3\overline{\zeta}_q(3) + \overline{\zeta}_q(2) - 4\overline{\zeta}_q(2,2)\,.$$

In particular $D\overline{\zeta}_q(k) \in \mathcal{Z}_q^{\geq 2} = \mathcal{G}^{\geq 2}$ for $k \geq 2$.

Notice that the depth one part is simpler but the depth two part is in weight $k+2$ and $k+1$.

Derivatives of $\mathcal{G}^{\geq 2}$: Open questions

- Except for the example before and numerical experiments, there are no results (that I am aware of) in higher depths.

Questions

- Can we use the a similar idea for higher depth by using our formula for $D\mathfrak{g}(x_1, \dots, x_r)$?
- Are there results on the derivatives in $\mathcal{Z}_q$ or $\mathcal{Z}_q^{\geq 2}$ for higher depths? (日本語で?)
- Is there another (better?) model to study the operator $D = q\frac{d}{dq}$?

Dimensions of $\mathcal{G}^{\geq 2}$

In his work Okounkov also proposes a conjecture for the dimension of the associated graded algebra of $\mathcal{G}^{\geq 2}$. For this let

$$\mathcal{G}^{\geq 2}_k = \left\langle g_{k_1,\dots,k_r}(q) \in \mathcal{G}^{\geq 2} \mid r \geq 0,\ k_1 + \dots + k_r = k \right\rangle_{\mathbb{Q}}$$

and set $\mathrm{gr}_0\, \mathcal{G}^{\geq 2} = \mathbb{Q}$ and for $k \geq 1$

$$\mathrm{gr}_k\, \mathcal{G}^{\geq 2} = \mathcal{G}^{\geq 2}_k \Big/ \mathcal{G}^{\geq 2}_{k-1}\,.$$

Conjecture (Okounkov)

The dimension $d_k = \dim_{\mathbb{Q}} \mathrm{gr}_k\, \mathcal{G}^{\geq 2}$ is given by

$$\sum_{k\geq 0} d_k x^k = \frac{1}{1 - x^2 - x^3 - x^4 - x^5 + x^8 + x^9 + x^{10} + x^{11} + x^{12}}.$$

Slides are available here: **www.henrikbachmann.com**