The double shuffle structure of certain q-analogues of multiple zeta values and their connections to modular forms

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Various Aspects of Multiple Zeta Values
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	$\mathbb{R}$ Numbers	$\mathcal{O}(\mathbb{H})$ hol. functions	$\mathbb{Q}[[q]]$ q-series
Classical	$\zeta(k)$	$G_k(\tau)$	$\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$
Multiple	$\zeta(k_1, \ldots, k_r)$ $\zeta^{ш}(k_1, \ldots, k_r)$	$G_{k_1,\ldots,k_r}(\tau)$ $G^{ш}_{k_1,\ldots,k_r}(\tau)$	$g^{(d_1,\ldots,d_r)}_{k_1,\ldots,k_r}(q)$ $g^{ш}_{k_1,\ldots,k_r}(q)$
alg. Setup	$\mathfrak{H}^0$ $\mathfrak{H}^1$ ш $\quad *$	$\mathfrak{H}^0$ $\mathfrak{H}^1$ ш $\quad *$	$\mathfrak{H}^0$ $\mathfrak{H}^1$ $\mathfrak{H}^2$ ш $\quad *$ $\quad \boxdot$ $\quad \boxast$

Multiple zeta values

Definition

For $k_1, \ldots, k_{r-1} \geq 1, k_r \geq 2$ define the multiple zeta value by

$$\zeta(k_1, \ldots, k_r) = \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}.$$

By r we denote its depth and $k_1 + \cdots + k_r$ will be called its weight. For the $\mathbb{Q}$-vector space spanned by all multiple zeta values we write $\mathcal{Z}$.

- The product of two MZV can be expressed as a linear combination of MZV with the same weight (harmonic product). e.g:

$$\zeta(k_1) \cdot \zeta(k_2) = \zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1 + k_2)\,.$$

- MZV can be expressed as iterated integrals. This gives another way (shuffle product) to express the product of two MZV as a linear combination of MZV.
- These two products give a number of $\mathbb{Q}$-relations (double shuffle relations) between MZV.

Multiple zeta values - Example for the double shuffle relations

Example:

$$\zeta(3,2) + 3\zeta(2,3) + 6\zeta(1,4) \overset{\text{shuffle}}{=} \zeta(2)\cdot\zeta(3) \overset{\text{harmonic}}{=} \zeta(2,3) + \zeta(3,2) + \zeta(5)\,.$$

$$\Longrightarrow 2\zeta(2,3) + 6\zeta(1,4) \overset{\text{double shuffle}}{=} \zeta(5)\,.$$

But there are more relations between MZV. e.g.:

$$\zeta(1,2) = \zeta(3).$$

These follow from the extended double shuffle relations.

Multiple Eisenstein series

- There are several connections of multiple zeta values to modular forms.
- One of them is given by multiple Eisenstein series $G^{ш}_{k_1,\dots,k_r}(\tau)$. In depth 1 these are the classical Eisenstein series

$$G^{ш}_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \qquad (q = e^{2\pi i \tau}),$$

which are modular forms for even $k > 2$. $\left(\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}\right)$.

- These functions satisfy **some** of the double shuffle relations. For example it is

$$2G^{ш}_{2,3}(\tau) + 6G^{ш}_{1,4}(\tau) = G^{ш}_5(\tau),$$

but

$$G^{ш}_3(\tau) - G^{ш}_{1,2}(\tau) = -\pi i \frac{d}{d\tau} G^{ш}_1(\tau) \neq 0.$$

- There are a lot of open questions regarding multiple Eisenstein series.

Plan

Question

Is the space spanned by all multiple Eisenstein series closed under $\frac{d}{d\tau} = (2\pi i)q\frac{d}{dq}$?

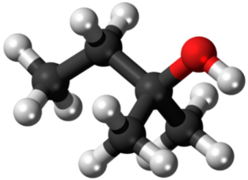
- The multiple Eisenstein series $G^{\shuffle}_{k_1,\ldots,k_r}(\tau)$ can be written as a $\mathcal{Z}[2\pi i]$-linear combination of certain q-series $g^{\shuffle}_{k_1,\ldots,k_r}(q) \in \mathbb{Q}[[q]]$. ($\lambda = 2\pi i$)

$$G^{\shuffle}_{2,3}(\tau) = \zeta(2,3) + 3\zeta(3)\lambda^2 g^{\shuffle}_2(q) + 2\zeta(2)\lambda^3 g^{\shuffle}_3(q) + \lambda^5 g^{\shuffle}_{2,3}(q)\,.$$

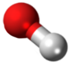
- The q-series $g^{\shuffle}_{k_1,\ldots,k_r}(q)$ can be written in terms of other q-series $g^{(d_1,\ldots,d_r)}_{k_1,\ldots,k_r}(q)$.

$$g^{\shuffle}_{1,2}(q) = g^{(0,0)}_{1,2}(q) + \frac{1}{2}g^{(1)}_2(q) - \frac{1}{2}g^{(0)}_2(q)\,.$$

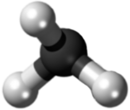
- Most of the algebraic structure and the behavior of $g^{(d_1,\ldots,d_r)}_{k_1,\ldots,k_r}$ under the operator $q\frac{d}{dq}$ is well-understood.



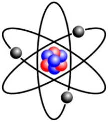
$$G^{\sqcup\!\sqcup}_{k_1,\ldots,k_r}$$



$$\zeta(k_1,\ldots,k_r)$$



$$g^{\sqcup\!\sqcup}_{k_1,\ldots,k_r}$$



$$g^{(d_1,\ldots,d_r)}_{k_1,\ldots,k_r}$$

Algebraic setup - Classical case

- Denote by $\mathfrak{H} = \mathbb{Q}\langle e_0, e_1 \rangle$ the noncommutative polynomial algebra of indeterminates e_0 and e_1 over $\mathbb{Q}$.
- Define its subalgebras $\mathfrak{H}^0$ and $\mathfrak{H}^1$ by

$$\mathfrak{H}^0 = \mathbb{Q} \cdot 1 + e_1 \mathfrak{H} e_0 \subset \mathfrak{H}^1 = \mathbb{Q} \cdot 1 + e_1 \mathfrak{H} \subset \mathfrak{H} \,.$$

- Set $e_k = e_1 e_0^{k-1}$ for $k \geq 1$.
- The monomials $e_{k_1} \ldots e_{k_r}$ form a basis of $\mathfrak{H}^1$.
- The monomials $e_{k_1} \ldots e_{k_r}$ with $k_r \geq 2$ form a basis of $\mathfrak{H}^0$.

Shuffle product

Define the $\mathbb{Q}$-bilinear commutative product $ш$ on $\mathfrak{H}$ for $a, b \in \{e_0, e_1\}$ and $v, w \in \mathfrak{H}$ by

$$1 \,ш\, w = w \,ш\, 1 = w\,,$$
$$av \,ш\, bw = a(v \,ш\, bw) + b(av \,ш\, w)\,.$$

- The space $\mathfrak{H}$ equipped with this product becomes a commutative $\mathbb{Q}$-algebra which we denote by $\mathfrak{H}_ш$.
- Both $\mathfrak{H}^1$ and $\mathfrak{H}^0$ are closed under $ш$ and by $\mathfrak{H}^1_ш$ and $\mathfrak{H}^0_ш$ we denote the corresponding subalgebras.

Algebraic setup - Classical case - harmonic product

Harmonic (stuffle) product

Define the $\mathbb{Q}$-bilinear commutative product $*$ on $\mathfrak{H}^1$ for $k_1, k_2 \geq 1$ and $v, w \in \mathfrak{H}^1$ by

$$1 * w = w * 1 = w\,,$$
$$e_{k_1} v * e_{k_2} w = e_{k_1}(v * e_{k_2} w) + e_{k_2}(e_{k_1} v * w) + e_{k_1+k_2}(v * w)\,.$$

- The space $\mathfrak{H}^1$ equipped with this product becomes a commutative $\mathbb{Q}$-algebra which we denote by $\mathfrak{H}_*$.
- The subspace $\mathfrak{H}^0$ is also closed under $*$ and by $\mathfrak{H}^0_*$ we denote the corresponding subalgebra.

- View ζ as a $\mathbb{Q}$-linear map from $\mathfrak{H}^0$ to $\mathcal{Z} \subset \mathbb{R}$ which sends the monomials $e_{k_1} \dots e_{k_r}$ to $\zeta(k_1, \dots, k_r)$..
- ζ is an algebra homomorphism from both $\mathfrak{H}^0_{ш}$ and $\mathfrak{H}^0_*$ to $\mathcal{Z}$, i.e. for $w, v \in \mathfrak{H}^0$

$$\zeta(w \text{ ш } v) = \zeta(w) \cdot \zeta(v) = \zeta(w * v) \,.$$

- The map ζ can be extended to algebra homomorphisms

$$\zeta^{ш} : \mathfrak{H}^1_{ш} \to \mathcal{Z}$$

and

$$\zeta^* : \mathfrak{H}^1_* \to \mathcal{Z} \,,$$

which are uniquely determined by $\zeta^{ш}(e_1) = \zeta^*(e_1) = 0$ and $\zeta^{ш}(w) = \zeta^*(w) = \zeta(w)$ for $w \in \mathfrak{H}^0$.

Define for words $u, v \in \mathfrak{H}^1$ the element $\mathrm{ds}(u,v) \in \mathfrak{H}^1$ by

$$\mathrm{ds}(u,v) = u * v - u ⧢ v\,.$$

If both $u, v \in \mathfrak{H}^0$ we have $\zeta(\mathrm{ds}(u,v)) = 0$.

But more generally we have the following Theorem, which conjecturally gives all linear relations between multiple zeta values.

Theorem (Extended double shuffle relations)

For $u \in \mathfrak{H}^0$ and $v \in \mathfrak{H}^1$ it is

$$\zeta^{⧢}(\mathrm{ds}(u,v)) = \zeta^{*}(\mathrm{ds}(u,v)) = 0\,.$$

Algebraic setup - q-analogue case

- Now we want to introduce a similar algebraic setup for our q-series $g^{(d_1,\dots,d_r)}_{k_1,\dots,k_r}(q)$.
- For this consider the space $\mathfrak{H}^2$ spanned by words in the double-indiced letters $e^{(d)}_k$ with $k \geq 1$ and $d \geq 0$, i.e. let

$$\mathfrak{H}^2 = \mathbb{Q}\langle A \rangle$$

  be the noncommutative polynomial algebra of indeterminates $A = \{e^{(d)}_k \mid k \geq 1\,, d \geq 0\}$ over $\mathbb{Q}$
- In the following we will define two products $\circledast$ and $\boxdot$ on $\mathfrak{H}^2$.

Algebraic setup - q-analogue case - "harmonic product analog" $\circledast$

Definition - The product $\circledast$ on $\mathfrak{H}^2$

For $w, v \in \mathfrak{H}^2$, $d_1, d_2 \geq 0$ and $k_1, k_2 \geq 1$ define $1 \circledast w = w \circledast 1 = w$ and

$$
\begin{aligned}
e_{k_1}^{(d_1)} v \circledast e_{k_2}^{(d_2)} w = {} & e_{k_1}^{(d_1)} (v \circledast e_{k_2}^{(d_2)} w) + e_{k_2}^{(d_2)} (e_{k_1}^{(d_1)} v \circledast w) \\
& + \binom{d_1 + d_2}{d_1} e_{k_1+k_2}^{(d_1+d_2)} (v \circledast w) \\
& + \binom{d_1 + d_2}{d_1} \sum_{j=1}^{k_1} \lambda_{k_1,k_2}^{j} e_j^{(d_1+d_2)} (v \circledast w) \\
& + \binom{d_1 + d_2}{d_1} \sum_{j=1}^{k_2} \lambda_{k_2,k_1}^{j} e_j^{(d_1+d_2)} (v \circledast w) \,,
\end{aligned}
$$

where the numbers $\lambda_{a,b}^{j} \in \mathbb{Q}$ for $1 \leq j \leq a$ are defined by

$$
\lambda_{a,b}^{j} := (-1)^{b-1} \binom{a+b-j-1}{a-j} \frac{B_{a+b-j}}{(a+b-j)!} \,.
$$

Theorem

The space $\mathfrak{H}^2$ equipped with the product $\circledast$ becomes a commutative $\mathbb{Q}$-algebra $\mathfrak{H}^2_{\circledast}$.

For example we have

$$e_2^{(0)} \circledast e_3^{(0)} = e_2^{(0)} e_3^{(0)} + e_3^{(0)} e_2^{(0)} + e_5^{(0)} - \frac{1}{12} e_3^{(0)} \,,$$
$$e_1^{(1)} \circledast e_1^{(2)} = e_1^{(1)} e_1^{(2)} + e_1^{(2)} e_1^{(1)} + 3e_2^{(3)} - 3e_1^{(3)} \,.$$

Notice that up to the term $-\frac{1}{12} e_3^{(0)}$ the first line looks exactly like the harmonic product

$$e_2 * e_3 = e_2 e_3 + e_3 e_2 + e_5$$

in $\mathfrak{H}^1_*$.

Algebraic setup - q-analogue case - "shuffle product"?

- Recall: The product $\shuffle$ on $\mathfrak{H}^1$ was defined by writing $e_k = e_1 e_0^{k-1}$ and using the shuffle product on $\mathbb{Q}\langle e_0, e_1 \rangle$.
- For the second product $\boxdot$ on $\mathfrak{H}^2$ we will use a different approach.
- We will define an involution $P : \mathfrak{H}^2 \to \mathfrak{H}^2$ and then set for $u, v \in \mathfrak{H}^2$

$$u \boxdot v = P(P(u) \circledast P(v))\,.$$

Algebraic setup - q-analogue case - The map P

Define the following element in $\mathfrak{H}^2[[X_1,\dots,X_r,Y_1,\dots,Y_r]]$

$$M\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} := \sum_{\substack{k_1,\dots,k_r\geq 1\\ d_1,\dots,d_r\geq 0}} e_{k_1}^{(d_1)}\dots e_{k_r}^{(d_r)} X_1^{k_1-1}\dots X_r^{k_r-1}\cdot Y_1^{d_1}\dots Y_r^{d_r}.$$

Definition

For $k_1,\dots,k_r \geq 1, d_1,\dots,d_l \geq 0$ and $w = e_{k_1}^{(d_1)},\dots,e_{k_r}^{(d_r)}$ define $P(w)$ as the coefficients of $X_1^{k_1-1}\dots X_r^{k_r-1}\cdot Y_1^{d_1}\dots Y_r^{d_r}$ in

$$M\binom{Y_r, Y_{r-1}+Y_r,\dots,Y_1+\dots+Y_r}{X_r-X_{r-1}, X_{r-1}-X_{r-2},\dots,X_1}.$$

Define the $\mathbb{Q}$-linear map $P:\mathfrak{H}^2\to\mathfrak{H}^2$ by setting $P(1)=1$ and extending the above definition on monomials linearly to $\mathfrak{H}^2$.

Notice that the map P is an involution on $\mathfrak{H}^2$, i.e. $P(P(w))=w$ for all $w\in\mathfrak{H}^2$.

Algebraic setup - q-analogue case - The map P

For $r = 1$ the definition reads

$$\sum_{\substack{k_1 \geq 1 \\ d_1 \geq 0}} P(e_{k_1}^{(d_1)}) X_1^{k_1-1} Y_1^{d_1} := M\begin{pmatrix} Y_1 \\ X_1 \end{pmatrix} = \sum_{\substack{k_1 \geq 1 \\ d_1 \geq 0}} e_{k_1}^{(d_1)} Y_1^{k_1-1} X_1^{d_1}$$

and therefore $P(e_{k_1}^{(d_1)}) = e_{d_1+1}^{(k_1-1)}$.

Other examples are

$$\begin{aligned} P(e_1^{(2)} e_1^{(1)}) &= e_2^{(0)} e_3^{(0)} + 3e_1^{(0)} e_4^{(0)} \,, \\ P(e_1^{(1)} e_1^{(2)}) &= e_3^{(0)} e_2^{(0)} + 2e_2^{(0)} e_3^{(0)} + 3e_1^{(0)} e_4^{(0)} \end{aligned}$$

which can be obtained by calculation the coefficient of $X_1^0 X_2^0 Y_1^2 Y_2^1$ (resp. $X_1^0 X_2^0 Y_1^1 Y_2^2$) in $M\binom{Y_2, Y_1+Y_2}{X_2-X_1, X_1}$.

Definition - The product $\boxdot$ on $\mathfrak{H}^2$

Define on $\mathfrak{H}^2$ the product $\boxdot$ for $u, v \in \mathfrak{H}^2$ by

$$u \boxdot v = P(P(u) \circledast P(v))\,.$$

Theorem

The space $\mathfrak{H}^2$ equipped with the product $\boxdot$ becomes a commutative $\mathbb{Q}$-algebra $\mathfrak{H}^2_{\boxdot}$.

That this product is commutative and associative which follows from the fact that P is an involution together with the properties of $\circledast$.

Algebraic setup - q-analogue case - "shuffle product"

We have seen before that

$$\begin{aligned}
e_1^{(1)} \circledast e_1^{(2)} &= e_1^{(1)} e_1^{(2)} + e_1^{(2)} e_1^{(1)} + 3e_2^{(3)} - 3e_1^{(3)}\,, \\
P(e_1^{(2)} e_1^{(1)}) &= e_2^{(0)} e_3^{(0)} + 3e_1^{(0)} e_4^{(0)}\,, \\
P(e_1^{(1)} e_1^{(2)}) &= e_3^{(0)} e_2^{(0)} + 2e_2^{(0)} e_3^{(0)} + 3e_1^{(0)} e_4^{(0)}
\end{aligned}$$

and $P(e_{k_1}^{(d_1)}) = e_{d_1+1}^{(k_1-1)}$.

Example

The product $e_2^{(0)} \boxdot e_3^{(0)}$ in $\mathfrak{H}^2$ is therefore given by

$$\begin{aligned}
e_2^{(0)} \boxdot e_3^{(0)} &= P(P(e_2^{(0)}) \circledast P(e_3^{(0)})) = P(e_1^{(1)} \circledast e_1^{(2)}) \\
&= P(e_1^{(1)} e_1^{(2)} + e_1^{(2)} e_1^{(1)} + 3e_2^{(3)} - 3e_1^{(3)}) \\
&= e_3^{(0)} e_2^{(0)} + 3e_2^{(0)} e_3^{(0)} + 6e_1^{(0)} e_4^{(0)} + 3e_4^{(1)} - 3e_4^{(0)}\,.
\end{aligned}$$

Compare this to the shuffle product $e_2 \sqcup\!\sqcup\, e_3 = e_3 e_2 + 3e_2 e_3 + 6e_1 e_4$ on $\mathfrak{H}^1_{\sqcup\!\sqcup}$.

In analogy to $\zeta^{ш}$ and ζ^{*}, which are algebra homomorphism from $\mathfrak{H}^1_{ш}$ (resp. $\mathfrak{H}^1_{*}$) to $\mathbb{R}$, we will now define a map

$$\mathfrak{g} : \mathfrak{H}^2 \longrightarrow \mathbb{Q}[[q]]$$

which will be an algebra homomorphism from both $\mathfrak{H}^2_{\circledast}$ and $\mathfrak{H}^2_{\boxdot}$ to $\mathbb{Q}[[q]]$.

q-analogues - the series $g^{(d_1,\dots,d_r)}_{k_1,\dots,k_r}$

Definition

For $k_1,\dots,k_r \geq 1, d_1,\dots,d_l \geq 0$ we define the following q-series in $\mathbb{Q}[[q]]$

$$g^{(d_1,\dots,d_r)}_{k_1,\dots,k_r}(q) := \sum_{\substack{0<u_1<\dots<u_r\\0<v_1,\dots,v_r}} \frac{u_1^{d_1}}{d_1!}\dots\frac{u_r^{d_r}}{d_r!}\cdot\frac{v_1^{k_1-1}\dots v_l^{k_r-1}}{(k_1-1)!\dots(k_r-1)!}\cdot q^{u_1v_1+\dots+u_rv_r}\,.$$

By $k_1+\dots+k_r+d_1+\dots+d_r$ we denote its weight and by r its depth.

Since q will be fixed the whole time we will also write $g^{(d_1,\dots,d_r)}_{k_1,\dots,k_r}$ instead of $g^{(d_1,\dots,d_r)}_{k_1,\dots,k_r}(q)$.

q-analogues - the series $g^{(d_1,\dots,d_r)}_{k_1,\dots,k_r}$

Definition

For $k_1,\dots,k_r \geq 1, d_1,\dots,d_l \geq 0$ we define the following q-series in $\mathbb{Q}[[q]]$

$$g^{(d_1,\dots,d_r)}_{k_1,\dots,k_r}(q) := \sum_{\substack{0<u_1<\dots<u_r \\ 0<v_1,\dots,v_r}} \frac{u_1^{d_1}}{d_1!}\dots\frac{u_r^{d_r}}{d_r!}\cdot\frac{v_1^{k_1-1}\dots v_l^{k_r-1}}{(k_1-1)!\dots(k_r-1)!}\cdot q^{u_1v_1+\dots+u_rv_r}\,.$$

By $k_1+\dots+k_r+d_1+\dots+d_r$ we denote its weight and by r its depth.

Since q will be fixed the whole time we will also write $g^{(d_1,\dots,d_r)}_{k_1,\dots,k_r}$ instead of $g^{(d_1,\dots,d_r)}_{k_1,\dots,k_r}(q)$.

Example: In depth one we have

$$g^{(0)}_k = \sum_{\substack{0<u_1 \\ 0<v_1}} \frac{v_1^{k-1}}{(k-1)!}q^{u_1v_1} = \frac{1}{(k-1)!}\sum_{n>0}\sigma_{k-1}(n)q^n\,,$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

q-analogues - the space $\mathcal{G}$

For the $\mathbb{Q}$-vector space spanned by all of these q-series we write

$$\mathcal{G} := \left\langle g_{k_1,\dots,k_r}^{(d_1,\dots,d_r)} \mid r \geq 0, k_1,\dots,k_r \geq 1\,, d_1,\dots,d_r \geq 0 \right\rangle_{\mathbb{Q}}\,,$$

where we set $g_{k_1,\dots,k_r}^{(d_1,\dots,d_r)} = 1$ for $r = 0$.

In the case $d_1 = \dots = d_r = 0$ we write

$$g_{k_1,\dots,k_r} := g_{k_1,\dots,k_r}^{(0,\dots,0)}$$

and denote the subspace spanned by all of these by

$$\mathcal{G}^{(0)} := \left\langle g_{k_1,\dots,k_r} \mid r \geq 0, k_1,\dots,k_r \geq 1 \right\rangle_{\mathbb{Q}} \subset \mathcal{G}\,.$$

q-analogues - the map $\mathfrak{g}$

We now consider the following $\mathbb{Q}$-linear map from $\mathfrak{H}^2$ to $\mathcal{G}$ which we define on the monomials by

$$\mathfrak{g} : \mathfrak{H}^2 \longrightarrow \mathcal{G}\,,$$
$$w = e_{k_1}^{(d_1)} \dots e_{k_r}^{(d_r)} \longmapsto \mathfrak{g}(w) := g_{k_1,\dots,k_r}^{(d_1,\dots,d_r)}\,.$$

Set $\mathfrak{g}(1) = 1$ and extend it linearly to $\mathfrak{H}^2$.

q-analogues - the map $\mathfrak{g}$

Theorem

We have the following statements for the map $\mathfrak{g}$.

- The map $\mathfrak{g}$ is invariant under P, i.e. for all $w \in \mathfrak{H}^2$ it is

$$\mathfrak{g}(P(w)) = \mathfrak{g}(w)\,.$$

- $\mathfrak{g}$ is an algebra homorphism from $\mathfrak{H}^2$ to $\mathbb{Q}[[q]]$ with respect to both products $\circledast$ and $\boxdot$, i.e. we have for all $u, v \in \mathfrak{H}^2$

$$\mathfrak{g}(u \boxdot v) = \mathfrak{g}(u) \cdot \mathfrak{g}(v) = \mathfrak{g}(u \circledast v)\,,$$

  where $\cdot$ denotes the usual multiplication of formal q-series in $\mathbb{Q}[[q]]$

- In particular the space $\mathcal{G} = \mathfrak{g}(\mathfrak{H}^2) \subset \mathbb{Q}[[q]]$ is an $\mathbb{Q}$-algebra.

q-analogues - the map $\mathfrak{g}$

- The invariance under the map P can be explained by interpreting the coefficients of the series $g^{(d_1,\dots,d_r)}_{k_1,\dots,k_r}$ as a sum over partitions.

$$\mathfrak{g}(w) = \sum_{n>0} \left(\sum_{|\yng(4,3,2,1,1)|=n} f(\yng(4,3,2,1,1)) \right) q^n = \sum_{n>0} \left(\sum_{|\yng(4,3,2,1,1)|=n} f(\yng(5,3,2,1)) \right) q^n = \mathfrak{g}(P(w))\,.$$

The action of P on the coefficient correspond to conjugating ($\yng(4,3,2,1,1) \longrightarrow \yng(5,3,2,1)$) the partitions.

- For $\mathfrak{g}(u)\cdot\mathfrak{g}(v) = \mathfrak{g}(u \circledast v)$ we use explicit formulas for the generating functions of $g^{(d_1,\dots,d_r)}_{k_1,\dots,k_r}$.
- Finally $\mathfrak{g}(u \boxdot v) = \mathfrak{g}(u)\cdot\mathfrak{g}(v)$ follows from the two statement above, since

$$\mathfrak{g}(u \boxdot v) = \mathfrak{g}(P(P(u) \circledast P(v))) = \mathfrak{g}(P(u) \circledast P(v)) = \mathfrak{g}(P(u))\cdot\mathfrak{g}(P(v)) = \mathfrak{g}(u)\cdot\mathfrak{g}(v)\,.$$

The Theorem provides a large family of linear relations between the q-series $g_{k_1,\dots,k_r}^{(d_1,\dots,d_r)}$.
We have seen before that

$$e_2^{(0)} \circledast e_3^{(0)} = e_2^{(0)} e_3^{(0)} + e_3^{(0)} e_2^{(0)} + e_5^{(0)} - \frac{1}{12} e_3^{(0)},$$
$$e_2^{(0)} \boxdot e_3^{(0)} = e_3^{(0)} e_2^{(0)} + 3e_2^{(0)} e_3^{(0)} + 6e_1^{(0)} e_4^{(0)} + 3e_4^{(1)} - 3e_4^{(0)}$$

and therefore we obtain the relation

$$\begin{aligned} 0 &= \mathfrak{g}(e_2^{(0)} \circledast e_3^{(0)}) - \mathfrak{g}(e_2^{(0)} \boxdot e_3^{(0)}) \\ &= g_5 - 2g_{2,3} - 6g_{1,4} - 3g_4^{(1)} + 3g_4 - \frac{1}{12} g_3 . \end{aligned}$$

as an analogue of $0 = \zeta(5) - 2\zeta(2,3) - 6\zeta(1,4)$.

$\mathfrak{H}^1$ and $\mathfrak{H}^0$ have a natural embedding in $\mathfrak{H}^2$, by sending a monomial $e_{k_1} \dots e_{k_r}$ to $e^{(0)}_{k_1} \dots e^{(0)}_{k_r}$. We view both $\mathfrak{H}^1$ and $\mathfrak{H}^0$ as subspaces of $\mathfrak{H}^2$, i.e.

$$\mathfrak{H}^0 \subset \mathfrak{H}^1 \subset \mathfrak{H}^2 .$$

Proposition

The spaces $\mathfrak{H}^1$ and $\mathfrak{H}^0$ are closed under $\circledast$ and therefore we also have for $u, v \in \mathfrak{H}^1$ (resp. $\mathfrak{H}^0$) that

$$\mathfrak{g}(u) \cdot \mathfrak{g}(v) = \mathfrak{g}(u \circledast v) .$$

In particular the space $\mathcal{G}^{(0)}$ is a subalgebra of $\mathcal{G}$.

Notice that the analogue statement for the product $\boxdot$ is false, since $e_2 \boxdot e_3 \notin \mathfrak{H}^1$.

Define for $k \in \mathbb{N}$ the map $Z_k : \mathbb{Q}[[q]] \to \mathbb{R} \cup \{\infty\}$ by

$$Z_k(f) = \lim_{q \to 1} (1-q)^k f(q)\,.$$

Proposition

If $k_r - d_r \geq 2$ and $k_j - d_j \geq 1$ for $j = 1, \dots, r-1$, then

$$Z_{k_1+\dots+k_r}\left(g^{(d_1,\dots,d_r)}_{k_1,\dots,k_r}\right) = \frac{1}{d_1! \dots d_r!} \zeta(k_1 - d_1, \dots, k_r - d_r)\,.$$

In particular for $k_r \geq 2$ we have

$$Z_k\left(g_{k_1,\dots,k_r}\right) = \begin{cases} \zeta(k_1,\dots,k_r)\,, & k_1 + \dots + k_r = k, \\ 0\,, & k_1 + \dots + k_r < k\,. \end{cases}$$

We will now introduce the second q-series $g^{ш}$, which appear in the Fourier expansion of Multiple Eisenstein series. For this we need the following series:

Definition

Define for $n_1, \ldots, n_r \geq 1$ the series

$$H\binom{n_1, \ldots, n_r}{x_1, \ldots, x_r} = \sum_{0<d_1<\cdots<d_r} e^{d_1 x_1} \left(\frac{q^{d_1}}{1-q^{d_1}}\right)^{n_1} \cdots e^{d_r x_r} \left(\frac{q^{d_r}}{1-q^{d_r}}\right)^{n_r}.$$

Notice that this series "satisfies" the harmonic product formula. For example:

$$H\binom{n_1}{x_1} \cdot H\binom{n_2}{x_2} = H\binom{n_1, n_2}{x_1, x_2} + H\binom{n_2, n_1}{x_2, x_1} + H\binom{n_1+n_2}{x_1+x_2}.$$

q-analogues - $g^{\sqcup\!\sqcup}$

Definition

- For $k_1, \dots, k_r \geq 1$ define the q-series $g^{\sqcup\!\sqcup}_{k_1,\dots,k_r}(q) \in \mathbb{Q}[[q]]$ as the coefficients of the following generating function:

$$g_{\sqcup\!\sqcup}(x_1, \dots, x_r) = \sum_{k_1,\dots,k_r \geq 1} g^{\sqcup\!\sqcup}_{k_1,\dots,k_r}(q) x_1^{k_1-1} \dots x_r^{k_r-1}$$

$$:= \sum_{m=1}^{r} \sum_{\substack{i_1+\dots+i_m=r \\ i_1,\dots,i_m \geq 1}} \frac{1}{i_1! \dots i_m!} H\binom{i_1, i_2, \dots, i_m}{x_r - x_{r-i_1}, x_{r-i_1} - x_{r-i_1-i_2}, \dots, x_{i_m}}.$$

  Again we also write $g^{\sqcup\!\sqcup}_{k_1,\dots,k_r}$ instead of $g^{\sqcup\!\sqcup}_{k_1,\dots,k_r}(q)$.

- Define the $\mathbb{Q}$-linear map $\mathfrak{g}^{\sqcup\!\sqcup}$ from $\mathfrak{H}^1$ to $\mathbb{Q}[[q]]$ on the monomials by

$$\mathfrak{g}^{\sqcup\!\sqcup} : \mathfrak{H}^1 \longrightarrow \mathbb{Q}[[q]],$$
$$w = e_{k_1} \dots e_{k_r} \longmapsto \mathfrak{g}^{\sqcup\!\sqcup}(w) := g^{\sqcup\!\sqcup}_{k_1,\dots,k_r}$$

  and set $\mathfrak{g}^{\sqcup\!\sqcup}(1) = 1$.

q-analogues - $g^{\sqcup\!\sqcup}$

Theorem

- *For all* $k_1, \ldots, k_r \geq 1$ *we have* $g^{\sqcup\!\sqcup}_{k_1,\ldots,k_r} \in \mathcal{G}$.
- *In the cases* $k_1, \ldots, k_{r-1} \geq 2$, $k_r \geq 1$ *it is* $g^{\sqcup\!\sqcup}_{k_1,\ldots,k_r} = g_{k_1,\ldots,k_r} \in \mathcal{G}^{(0)}$.
- *The map* $\mathfrak{g}^{\sqcup\!\sqcup}$ *is an algebra homomorphism from* $\mathfrak{H}^1_{\sqcup\!\sqcup}$ *to* $\mathcal{G}$.

q-analogues - $g^{ш}$

Theorem

- *For all* $k_1,\dots,k_r \geq 1$ *we have* $g^{ш}_{k_1,\dots,k_r} \in \mathcal{G}$.
- *In the cases* $k_1,\dots,k_{r-1} \geq 2$, $k_r \geq 1$ *it is* $g^{ш}_{k_1,\dots,k_r} = g_{k_1,\dots,k_r} \in \mathcal{G}^{(0)}$.
- *The map* $\mathfrak{g}^{ш}$ *is an algebra homomorphism from* $\mathfrak{H}^1_{ш}$ *to* $\mathcal{G}$.

Proof ideas:

- The first two statements follow again by using explicit expressions for the generating functions of the q-series $g^{(d_1,\dots,d_r)}_{k_1,\dots,k_r}$.
- The third statement uses results (Exponentialmap) by Hoffman on quasi-shuffle products to turn the "harmonic" product of H into the "shuffle" product.
  We then use the general case of the following fact

$$\mathfrak{g}^{ш}(e_{k_1}) \cdot \mathfrak{g}^{ш}(e_{k_2}) = \mathfrak{g}^{ш}(e_{k_1} \,ш\, e_{k_2})$$

$$\Longleftrightarrow$$

$$g^{\sharp}_{ш}(x_1) \cdot g^{\sharp}_{ш}(x_2) = g^{\sharp}_{ш}(x_1,x_2) + g^{\sharp}_{ш}(x_2,x_1)\,,$$

where $g^{\sharp}_{ш}(x_1,\dots,x_r) = g_{ш}(x_1, x_1+x_2, \dots, x_1+\dots+x_r)$.

q-analogues - $g^{\sqcup\!\sqcup}$

There are explicit formulas to write $g^{\sqcup\!\sqcup}$ in terms of g.

Proposition

- In depth two it is
$$g^{\sqcup\!\sqcup}_{k_1,k_2} = g_{k_1,k_2} + \delta_{k_1,1} \cdot \frac{1}{2}\left(g^{(1)}_{k_2} - g_{k_2}\right)$$
- And in depth three it is
$$\begin{aligned} g^{\sqcup\!\sqcup}_{k_1,k_2,k_3} = g_{k_1,k_2,k_3} &+ \delta_{k_1,1} \cdot \frac{1}{2}\left(g^{(1,0)}_{k_2,k_3} - g_{k_2,k_3}\right) \\ &+ \delta_{k_2,1} \cdot \frac{1}{2}\left(g^{(0,1)}_{k_1,k_3} - g^{(1,0)}_{k_1,k_3} - g_{k_1,k_3}\right) \\ &+ \delta_{k_1 \cdot k_2,1} \cdot \left(\frac{1}{6} g^{(2)}_{k_3} - \frac{1}{4} g^{(1)}_{k_3} + \frac{1}{6} g_{k_3}\right) . \end{aligned}$$

Derivatives of g

We will now focus on the action of the operator $\mathrm{d} = q\frac{d}{dq}$.

Proposition

For $k_1, \dots, k_r \geq 1, d_1, \dots, d_r \geq 0$ we have

$$\mathrm{d}\, g^{(d_1,\dots,d_r)}_{k_1,\dots,k_r} = \sum_{j=1}^{r} (d_j + 1) \cdot k_j \cdot g^{(d_1,\dots,d_j+1,\dots,d_r)}_{k_1,\dots,k_j+1,\dots,k_r} .$$

In particular the space $\mathcal{G}$ is closed under d.

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In particular the space $\mathcal{G}$ is closed under d.

Proof: This is an easy consequence of the definition

$$g^{(d_1,\ldots,d_r)}_{k_1,\ldots,k_r} := \sum_{\substack{0<u_1<\cdots<u_r \\ 0<v_1,\ldots,v_r}} \frac{u_1^{d_1}}{d_1!} \cdots \frac{u_r^{d_r}}{d_r!} \cdot \frac{v_1^{k_1-1} \ldots v_l^{k_r-1}}{(k_1-1)! \ldots (k_r-1)!} \cdot q^{u_1 v_1 + \cdots + u_r v_r}$$

together with

$$\mathrm{d} \sum_{n \geq 0} a_n q^n = \sum_{n>0} n \cdot a_n q^n\,.$$

Derivatives of g

Even though it is

$$\mathrm{d}\, g_{k_1,k_2} = \mathrm{d}\, g^{(0,0)}_{k_1,k_2} = k_1 g^{(1,0)}_{k_1+1,k_2} + k_2 g^{(0,1)}_{k_1,k_2+1}\,,$$

we have the following result:

Theorem

The subalgebra $\mathcal{G}^{(0)} \subset \mathcal{G}$ is closed under the operator $\mathrm{d} = q\frac{d}{dq}$.

The proof uses double shuffle relations for the functions g to show that (for example)

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More generally we expect the following strong statement:

Conjecture

It is $\mathcal{G}^{(0)} = \mathcal{G}$.

Derivatives of $g^{\sqcup\!\sqcup}$

Define for $k \geq 1$

$$\mathcal{G}^{\sqcup\!\sqcup}_{\leq k} := \left\langle g^{\sqcup\!\sqcup}_{k_1,\dots,k_r} \in \mathcal{G} \mid k_1 + \dots + k_r \leq k \right\rangle_{\mathbb{Q}} ,$$

We expect that $\mathrm{d}\,\mathcal{G}^{\sqcup\!\sqcup}_{\leq k} \subset \mathcal{G}^{\sqcup\!\sqcup}_{\leq k+2}$, but so far we only know the following:

Theorem

For $k \geq 1$ and $\mathrm{d} = q\frac{d}{dq}$ we have

$$\frac{1}{k}\,\mathrm{d}\, g^{\sqcup\!\sqcup}_{k} = (k+1) g^{\sqcup\!\sqcup}_{k+2} - \sum_{n=2}^{k+1} (2^n - 2) g^{\sqcup\!\sqcup}_{k+2-n,n} \,.$$

Multiple Eisenstein series <-> $g^{ш}$

For the space spanned by all multiple Eisenstein series we write

$$\mathcal{E}_k = \langle G^{ш}_{k_1,\ldots,k_r} \mid k = k_1 + \cdots + k_r, k_1, \ldots, k_r \geq 1 \rangle_{\mathbb{Q}} \,.$$

Expectation (Rough statement)

We expect that the map F, given by

$$F : \mathcal{E}_k \longrightarrow \mathcal{G}^{ш}_{\leq k} \Big/ \mathcal{G}^{ш}_{\leq k-1}$$
$$G^{ш}_{k_1,\ldots,k_r} \longmapsto g^{ш}_{k_1,\ldots,k_r}$$

is an isomorphism of $\mathbb{Q}$-algebras, which respects the action of d.

Good thing: It is easy to obtain results in the space $\mathcal{G}^{ш}_{\leq k} \Big/ \mathcal{G}^{ш}_{\leq k-1}$!

Derivatives of $g^{ш}$ modulo lower weight - Depth 1

Proposition

For $k \geq 1$ we have

$$\mathrm{d}\, g_k^{ш} = k \cdot g_{k+1}^{(1)} \equiv 2k \cdot \mathfrak{g}^{ш}(\mathrm{ds}(e_1, e_{k+1})) \qquad \mod \mathcal{G}_{\leq k+1}^{ш}$$

Proof: Notice that

$$g_{k_1,k_2}^{ш} = g_{k_1,k_2} + \delta_{k_1,1} \cdot \frac{1}{2} g_{k_2}^{(1)} \mod \mathcal{G}_{\leq k_1+k_2-1}^{ш}$$

and

$$g_{k_1}^{ш} \cdot g_{k_2}^{ш} = g_{k_1} \cdot g_{k_2} = g_{k_1,k_2} + g_{k_2,k_1} + g_{k_1+k_2} \mod \mathcal{G}_{\leq k_1+k_2-1}^{ш}.$$

With this we can "measure" the failure of the double shuffle relations for the series $g^{ш}$:

$$\begin{aligned}
\mathfrak{g}^{ш}(\mathrm{ds}(e_{k_1}, e_{k_2})) &= \mathfrak{g}^{ш}(e_{k_1} * e_{k_2}) - \mathfrak{g}^{ш}(e_{k_1} ш e_{k_2}) \\
&= g_{k_1,k_2}^{ш} + g_{k_2,k_1}^{ш} + g_{k_1+k_2}^{ш} - g_{k_1}^{ш} \cdot g_{k_2}^{ш} \\
&\equiv \frac{1}{2}\delta_{k_1,1} g_{k_2}^{(1)} + \frac{1}{2}\delta_{k_2,1} g_{k_1}^{(1)} \qquad \mod \mathcal{G}_{\leq k_1+k_2-1}^{ш}.
\end{aligned}$$

Derivatives of $g^{\sqcup\!\sqcup}$ modulo lower weight - Depth 1

The analogue statement of

$$\mathrm{d}\, g_k^{\sqcup\!\sqcup} = k \cdot g_{k+1}^{(1)} \equiv 2k \cdot \mathfrak{g}^{\sqcup\!\sqcup}(\mathrm{ds}(e_1, e_{k+1})) \qquad \mathrm{mod}\ \mathcal{G}_{\leq k+1}^{\sqcup\!\sqcup}$$

is also known for Eisenstein series:

Theorem (Kaneko)

For $k \geq 1$, the derivative of the Eisenstein series $G_k^{\sqcup\!\sqcup}$ can be written as

$$\begin{aligned}(2\pi i)^2\, \mathrm{d}\, G_k^{\sqcup\!\sqcup} &= G_{1,k+1}^{\sqcup\!\sqcup} + G_{k+1,1}^{\sqcup\!\sqcup} + G_{k+2}^{\sqcup\!\sqcup} - G_{k+1}^{\sqcup\!\sqcup} \cdot G_1^{\sqcup\!\sqcup} \\ &= 2k \cdot G^{\sqcup\!\sqcup}(\mathrm{ds}(e_1, e_{k+1}))\,,\end{aligned}$$

where in the last line we (by abuse of notation) interpret the multiple Eisenstein series as a map $G^{\sqcup\!\sqcup} : \mathfrak{H}^1 \to \mathbb{C}[[q]]$.

Derivatives of $g^{ш}$ modulo lower weight - Depth 2

Lemma

For $k_1, k_2, k_3 \geq 1$ and $k = k_1 + k_2 + k_3$ we have

$$\mathfrak{g}^{ш}(\mathrm{ds}(e_{k_1}, e_{k_2}e_{k_3})) \equiv \delta_{k_1,1}\frac{1}{2}g^{(0,1)}_{k_2,k_3} + \delta_{k_3,1}\frac{1}{2}\left(g^{(0,1)}_{k_2,k_1} - g^{(1,0)}_{k_2,k_1}\right) + \delta_{k_1 \cdot k_2,1}\frac{1}{3}g^{(2)}_{k_3} + \delta_{k_2 \cdot k_3,1}\frac{1}{6}g^{(2)}_{k_1} \mod \mathcal{G}^{ш}_{\leq k-1}\,.$$

Proposition

For $k_1, k_2 \geq 2$ we have

$$\mathrm{d}\, g^{ш}_{k_1,k_2} \equiv 2k_1\left(\mathfrak{g}^{ш}(\mathrm{ds}(e_1, e_{k_1+1}e_{k_2})) - \mathfrak{g}^{ш}(\mathrm{ds}(e_{k_2}, e_{k_1+1}e_1))\right) + 2k_2 \cdot \mathfrak{g}^{ш}(\mathrm{ds}(e_1, e_{k_1}e_{k_2+1})) \mod \mathcal{G}^{ш}_{\leq k_1+k_2+1}$$

Proof: Use the lemma together with

$$\mathrm{d}\, g^{ш}_{k_1,k_2} = \mathrm{d}\, g_{k_1,k_2} = k_1 g^{(1,0)}_{k_1+1,k_2} + k_2 g^{(0,1)}_{k_1,k_2+1}\,.$$

Derivatives of q-analogues

We also have a similar result for $\mathrm{d}\, g^{\sqcup\!\sqcup}_{k_1,k_2,k_3}$, which leads to the following:

Conjecture

For $k_1, k_2, k_3 \geq 2$ the derivative of the Double and Triple Eisenstein series are given by

$$(-2\pi i)^2 \,\mathrm{d}\, G_{k_1,k_2} = 2k_1 \left(G^{\sqcup\!\sqcup}(\mathrm{ds}(e_1, e_{k_1+1}e_{k_2})) - G^{\sqcup\!\sqcup}(\mathrm{ds}(e_{k_2}, e_{k_1+1}e_1))\right) + 2k_2 \cdot G^{\sqcup\!\sqcup}(\mathrm{ds}(e_1, e_{k_1}e_{k_2+1}))\,,$$

$$\begin{aligned}(-2\pi i)^2 \,\mathrm{d}\, G_{k_1,k_2,k_3} = {} & 2k_1 \cdot G^{\sqcup\!\sqcup}(\mathrm{ds}(e_1, e_{k_1+1}e_{k_2}e_{k_3}) + \mathrm{ds}(e_{k_3}, e_{k_2}e_{k_1+1}e_1)) \\ & +2k_1 \cdot G^{\sqcup\!\sqcup}(\mathrm{ds}(e_{k_3}, e_{k_1+1+k_2}e_1) - \mathrm{ds}(e_{k_1+1}e_1, e_{k_2}e_{k_3})) \\ & +2k_2 \cdot G^{\sqcup\!\sqcup}(\mathrm{ds}(e_1, e_{k_1}e_{k_2+1}e_{k_3}) - \mathrm{ds}(e_{k_3}, e_{k_1}e_{k_2+1}e_1)) \\ & +2k_3 \cdot G^{\sqcup\!\sqcup}(\mathrm{ds}(e_1, e_{k_1}e_{k_2}e_{k_3+1}))\end{aligned}$$

Example:

$$\begin{aligned}\mathrm{d}\, G^{\sqcup\!\sqcup}_{2,2} \overset{?}{=} {} & 8G^{\sqcup\!\sqcup}_{2,3,1} - 4G^{\sqcup\!\sqcup}_{1,2,3} + 4G^{\sqcup\!\sqcup}_{1,3,2} + 24G^{\sqcup\!\sqcup}_{1,4,1} - 4G^{\sqcup\!\sqcup}_{2,1,3} - 4G^{\sqcup\!\sqcup}_{2,2,2} \\ & + 4G^{\sqcup\!\sqcup}_{2,4} + 4G^{\sqcup\!\sqcup}_{3,3} + 4G^{\sqcup\!\sqcup}_{4,2} - 4G^{\sqcup\!\sqcup}_{5,1}\,.\end{aligned}$$