# Modular forms and multiple zeta values

Henrik Bachmann

Fachbereich Mathematik, Bereich Algebra und Zahlentheorie

Advisor: Prof. Dr. Ulf Kühn

**Notation**: $s_1,\dots,s_l \in \mathbb{N}$, $\tau \in \mathbb{H} = \{x+yi \in \mathbb{C} \mid y>0\}$ and $q = e^{2\pi i \tau}$.

## Multiple zeta values and multiple Eisenstein series

**Definition:** The **multiple zeta value** (**MZV**) of weight $s_1+\dots+s_l$ and length $l$ is defined by

$$\zeta(s_1,\dots,s_l) := \sum_{n_1>n_2>\dots>n_l>0} \frac{1}{n_1^{s_1}\dots n_l^{s_l}} \in \mathbb{R} \qquad (\text{ for } s_1 \geq 2 \text{ and } s_2,\dots,s_l \geq 1)\,.$$

These numbers fulfil a lot of $\mathbb{Q}$-linear relations, which conjecturally all come from the so called extended double shuffle realtions, for example

$$\zeta(3) = \zeta(2,1)\,, \qquad \frac{5197}{691}\zeta(12) = 168\zeta(5,7) + 150\zeta(7,5) + 28\zeta(9,3)\,. \qquad (1)$$

MZV are of particular interest in quantum field theory and number theory [4]. We study their connection to **modular forms** by using multiple Eisenstein series:

**Definition:** The **multiple Eisenstein series** are defined by

$$G_{s_1,\dots,s_l}(\tau) = \sum_{\substack{0\succ\lambda_1\succ\dots\succ\lambda_l\\ \lambda_i\in\mathbb{Z}\tau+\mathbb{Z}}} \frac{1}{\lambda_1^{n_1}\cdots\lambda_l^{s_l}} \qquad (\text{ for } s_1 \geq \mathbf{3} \text{ and } s_2,\dots,s_l \geq \mathbf{2})\,.$$

The $\lambda_i \in \mathbb{Z}\tau+\mathbb{Z}$ are lattices points and $\succ$ is an order on $\mathbb{Z}+\mathbb{Z}\tau$ given by $m_1\tau+n_1 \succ m_2\tau+n_2 :\Leftrightarrow (m_1>m_2 \vee (m_1=m_2 \wedge n_1>n_2))$ .

For $l=1$ and even $s_1>2$ these are the classical Eisenstein series which are **modular forms** for $\mathrm{SL}_2(\mathbb{Z})$.

The case $l=2$ of double Eisenstein series was considered in [4], where the authors also showed a connection of relations between **MZV** and **cusp forms** via period polynomials (second relation in (1)).

**Theorem:** **[1]** The **multiple Eisenstein series** have a Fourier expansion given by an explicit but complicated $\mathbb{Q}[-2\pi i]$-linear combinations of products of **multiple zeta value** and **generating functions of multiple divisor sums** $[s_1,\dots,s_l] \in \mathbb{Q}[[q]]$.

**Examples:** $G_{3,2}(\tau) = \zeta(3,2) + 3\zeta(3)(-2\pi i)^2[2] + 2\zeta(2)(-2\pi i)^3[3] + (-2\pi i)^5[3,2]$.

$$\begin{aligned} G_{3,2,2}(\tau) =& \zeta(3,2,2) + \left(\frac{54}{5}\zeta(2,3) + \frac{51}{5}\zeta(3,2)\right)(2\pi i)^2[2] + \frac{16}{3}\zeta(2,2)(2\pi i)^3[3] \\ &+ 3\zeta(3)(2\pi i)^4[2,2] + 4\zeta(2)(2\pi i)^5[3,2] + (2\pi i)^7[3,2,2]\,, \end{aligned}$$

The **brackets** $[s_1,\dots,s_l]$ in the Fourierexpansion are kind of a combinatorial object which are for $s_1,\dots,s_l>0$ given by

$$[s_1,\dots,s_l] := \frac{1}{(s_1-1)!\dots(s_l-1)!} \sum_{\substack{u_1>\dots>u_l>0\\ v_1,\dots,v_l>0}} v_1^{s_1-1}\dots v_l^{s_l-1} q^{u_1v_1+\dots+u_lv_l} \in \mathbb{Q}[[q]]\,.$$

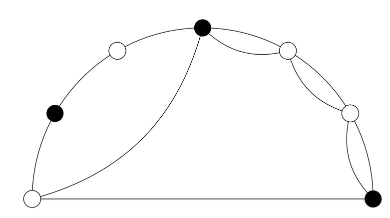
In [2] it is shown that the space spanned by these brackets has the structure of a bi-filtered algebra with a derivation given by $q\frac{d}{dq}$. It has the space of **modular forms** as a subalgebra.

**Idea:** Extend the definition of the **multiple Eisenstein series** $G_{s_1,\dots,s_l}$ for $s_1\geq 2$ and $s_2,\dots,s_l \geq 1$ by studying the space of the **generating functions of multiple divisor sums** and a connection to the **coproduct** on the space of formal iterated integrals.

## Shuffle regularized multiple Eisenstein series

There exists an algebra $\mathcal{I}^1$ of so called formal iterated integrals which was introduced by Goncharov in [5]. It is spanned by formal symbols $I(s_1,\dots,s_l)$, with $s_1,\dots,s_l \geq 1$, which satisfy relations coming from real iterated integrals. The product is given by the shuffle product. This space has the structure of a Hopf algebra, i.e. in particular it has a **coproduct** $\Delta : \mathcal{I}^1 \longrightarrow \mathcal{I}^1 \oplus \mathcal{I}^1$ which can be computed in a nice combinatorial way by summing over all dissections of a half circle.

**Example:** $\Delta(I(2,3)) = I(2,3)\otimes 1 + 3I(3)\otimes I(2) + 2I(2)\otimes I(3) + 1\otimes I(2,3)$.



$\longrightarrow I(2)\otimes I(3)$

**Observation ([4, 3]):** The coefficients in the **coproduct** are the same as in the Fourier expansion of **multiple Eisenstein series**.

**Proposition:** **[3]** There exist algebra homomorphisms $Z^{ш} : \mathcal{I}^1 \to \mathbb{R}$ and $\mathfrak{g}^{ш} : \mathcal{I}^1 \to \mathbb{C}[[q]]$ such that $Z^{ш}(I(s_1,\dots,s_l)) = \zeta(s_l,\dots,s_1)$ and $\mathfrak{g}^{ш}(I(s_1,\dots,s_l)) = [s_l,\dots,s_1]$ for $s_1,\dots,s_l \geq 2$.

**Definition:** For integers $s_1,\dots,s_l \geq 1$ define the **shuffle regularized multiple Eisenstein series**, as the image of the generator $I(s_1,\dots,s_l)$ in $\mathcal{I}^1$ under the algebra homomorphism $(Z^{ш}\otimes\mathfrak{g}^{ш})\circ\Delta$:

$$G^{ш}_{s_1,\dots,s_l}(\tau) := (Z^{ш}\otimes\mathfrak{g}^{ш})\circ\Delta\big(I(s_1,\dots,s_l)\big).$$

**Theorem:** **[3]** For all $s_1,\dots,s_l \geq 1$ the **shuffle regularized multiple Eisenstein series** $G^{ш}_{s_1,\dots,s_l}$ have the following properties:

- They fulfill the shuffle product, i.e. we have an algebra homormorphism $\mathcal{I}^1 \to \mathbb{C}[[q]]$ by sending the generators $I(s_1,\dots,s_l)$ to $G^{ш}_{s_1,\dots,s_l}(q)$.
- For integers $s_1,\dots,s_l \geq 2$ they equal the **multiple Eisenstein series**

$$G^{ш}_{s_1,\dots,s_l}(q) = G_{s_l,\dots,s_1}(q)$$

and therefore they fulfill the stuffle product in these cases.

**References**

[1] **H. Bachmann**, "Multiple Zeta-Werte und die Verbindung zu Modulformen durch Multiple Eisensteinreihen", Master thesis, University of Hamburg, 2012.

[2] **H. Bachmann**, U. Kühn, "The algebra of generating functions for multiple divisor sums and applications to multiple zeta values", arXiv:1309.3920 [math.NT].

[3] **H. Bachmann**, K. Tasaka, "The algebra of multiple Eisenstein series and the Goncharov coproduct", in preparation.

[4] H. Gangl, M. Kaneko, D. Zagier, "Double zeta values and modular forms", in "Automorphic forms and zeta functions" p. 71-106, World Sci. Publ., Hackensack, NJ, 2006.

[5] A.B. Goncharov, "Galois symmetries of fundamental groupoids and noncommutative geometry", Duke Math. J., **128**(2) (2005), 209–284.

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Research Training Group 1670
MATHEMATICS INSPIRED BY STRING THEORY AND QUANTUM FIELD THEORY

Deutsche Forschungsgemeinschaft