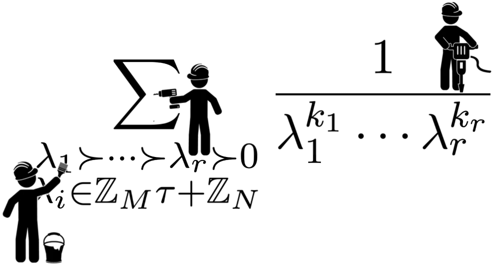
Recent developments and 12 open problems on multiple Eisenstein series

Henrik Bachmann
Nagoya University



$$\sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}_M \tau + \mathbb{Z}_N}} \frac{1}{\lambda_1^{k_1} \cdots \lambda_r^{k_r}}$$

8th December 2024

www.henrikbachmann.com

These slides can be found on my homepage

Multiple zeta values

Definition

For an index $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$ with $k_1 \geq 2, k_2, \ldots, k_r \geq 1$ define the **multiple zeta value** (MZV)

$$\zeta(\mathbf{k}) = \zeta(k_1, \ldots, k_r) = \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}\,.$$

By r we denote its **depth** and $k_1 + \cdots + k_r$ will be called its **weight**.

- $\mathcal{Z}$: $\mathbb{Q}$-algebra of MZVs
- $\mathcal{Z}_k$: $\mathbb{Q}$-vector space of MZVs of weight k.

Algebra setup

Set

$$\mathfrak{H}^0 = \mathbb{Q} + x\mathfrak{H}y \quad \subset \quad \mathfrak{H}^1 = \mathbb{Q} + \mathfrak{H}y \subset \quad \mathfrak{H} = \mathbb{Q}\langle x, y\rangle$$

and write $z_k = x^{k-1}y$ for $k \geq 1$. Note that $\mathfrak{H}^1 = \mathbb{Q}\langle z_1, z_2, \dots\rangle$.

Definition

- $* : \mathfrak{H}^1 \otimes \mathfrak{H}^1 \to \mathfrak{H}^1$: **harmonic product**. For words $w, v \in \mathfrak{H}^1$ and $r, s \geq 1$,

$$\mathbf{1} * w = w = w * \mathbf{1},$$
$$z_r w * z_s v = z_r(w * z_s v) + z_s(z_r w * v) + z_{r+s}(w * v),$$

- $ш : \mathfrak{H} \otimes \mathfrak{H} \to \mathfrak{H}$: **shuffle product**. For words $w, v \in \mathfrak{H}$ and $a, b \in \{x, y\}$,

$$\mathbf{1} ш w = w = w ш \mathbf{1},$$
$$aw ш bv = a(w ш bv) + b(aw ш v),$$

- For $\bullet \in \{*, ш\}$, we denote by $\mathfrak{H}^1_\bullet$ and $\mathfrak{H}^0_\bullet$ the algebras $(\mathfrak{H}^1, \bullet)$ and $(\mathfrak{H}^0, \bullet)$.

Finite double shuffle relations: $\zeta(w)\zeta(v) = \zeta(w * v) = \zeta(w ш v)$ for all $w, v \in \mathfrak{H}^0$.

Algebra setup

We also consider the following subspace of $\mathfrak{H}^0$

$$\mathfrak{H}^{2} = \mathbb{Q} + \langle z_{k_1} \cdots z_{k_r} \mid r \geq 1, k_1, \ldots, k_r \geq 2 \rangle_{\mathbb{Q}}$$

Both $\mathfrak{H}^0$ and $\mathfrak{H}^2$ are closed under $*$ but only $\mathfrak{H}^0$ is closed under $\sqcup\!\sqcup$. We obtain the following inclusion of $\mathbb{Q}$-algebras

$$\begin{aligned} \mathfrak{H}^2_* \subset \mathfrak{H}^0_* \subset \mathfrak{H}^1_*, \\ \mathfrak{H}^0_{\sqcup\!\sqcup} \subset \mathfrak{H}^1_{\sqcup\!\sqcup} \subset \mathfrak{H}_{\sqcup\!\sqcup}. \end{aligned}$$

Theorem (Hoffman, Goncharov)

1. *The algebras $\mathfrak{H}^2_*$ and $\mathfrak{H}^1_*$ are Hopf algebras when equipped with the deconcatination coproduct*

$$\Delta(w) = \sum_{uv=w} u \otimes v.$$

2. *The algebra $\mathfrak{H}^1_{\sqcup\!\sqcup}$ is a Hopf algebra when equipped with the Goncharov coproduct Δ_G.*

Regularization

The multiple zeta values can be viewed as $\mathbb{Q}$-algebra homomorphism $\zeta : \mathfrak{H}^0_\bullet \to \mathcal{Z}$ for $\bullet \in \{*, ш\}$.
Since $\mathfrak{H}^1_\bullet = \mathfrak{H}^0_\bullet[z_1]$ there exist algebra homomorphisms

$$\zeta^\bullet : \mathfrak{H}^1_\bullet \to \mathcal{Z}[T]\,,$$

uniquely determined by $\zeta^\bullet_{|\mathfrak{H}^0} = \zeta$ and $\zeta^*(z_1) = \zeta^ш(y) = T$.

Theorem (Ihara-Kaneko-Zagier)

Define the $\mathbb{R}$-linear map $\rho : \mathbb{R}[T] \to \mathbb{R}[T]$ by

$$\cdots + \frac{1}{2}\rho(T^2)u^2 + \cdots = \rho(e^{Tu}) := \exp\left(Tu + \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\zeta(n)u^n\right) = \cdots + \frac{1}{2}\left(T^2 + \zeta(2)\right)u^2 + \dots .$$

Then we have

$$\zeta^ш = \rho \circ \zeta^*.$$

($\rightsquigarrow$ **extended double shuffle relations**)

Algebra of formal MZV

Let $\mathrm{reg}_* : \mathfrak{H}^1_* \cong \mathfrak{H}^0_*[z_1] \to \mathfrak{H}^0_*$ be the alg. hom. that is the identity on $\mathfrak{H}^0$ and maps $y = z_1$ to 0.

Definition

The algebra of **formal multiple zeta values** is defined by

$$\mathcal{Z}^f := \mathfrak{H}^0_* \big/ \mathrm{EDS}_*,$$

where EDS_* is the ideal in $\mathfrak{H}^0_*$ generated by $\mathrm{reg}_*(w * v - w \sqcup\!\sqcup\, v)$ for all $w \in \mathfrak{H}^0, v \in \mathfrak{H}^1$.

For $k_1 \geq 2$ and $k_2, ..., k_r \geq 1$, we denote the class of $z_{k_1} \cdots z_{k_r}$ in $\mathcal{Z}^f$ by $\zeta^f(k_1, ..., k_r)$.

Conjecture

The linear map

$$\mathcal{Z}^f \longrightarrow \mathcal{Z}$$

$$\zeta^f(k_1, \ldots, k_r) \longmapsto \zeta(k_1, \ldots, k_r)$$

is an algebra isomorphism.

Definition (Gangl-Kaneko-Zagier 2006 (Level 2: Kaneko-Tasaka 2013, Level 4: Kina 2024))

For $k \geq 1$ the **formal double zeta space** in weight k is the $\mathbb{Q}$-vector space

$$\mathcal{D}_k = \left\langle Z_k, Z_{k_1,k_2}, P_{k_1,k_2} \mid k_1 + k_2 = k, k_1, k_2 \geq 1 \right\rangle_{\mathbb{Q}} \Big/ \text{(1)} \,,$$

spanned by symbols Z_k, Z_{k_1,k_2}, and P_{k_1,k_2}, satisfying the relations

$$P_{k_1,k_2} = Z_{k_1,k_2} + Z_{k_2,k_1} + Z_{k_1+k_2} = \sum_{j=1}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) Z_{j,k_1+k_2-j} \,. \qquad (1)$$

For a $\mathbb{Q}$-vector space A we call an element in $\mathrm{Hom}_{\mathbb{Q}}(\mathcal{D}_k, A)$ a **realization of $\mathcal{D}_k$ in A**.

For $A = \mathbb{R}$ there exists a realization with such that for $k, k_1, k_2 \geq 2$

$$Z_k \longmapsto \zeta(k), \qquad P_{k_1,k_2} \longmapsto \zeta(k_1)\zeta(k_2), \qquad Z_{k_1,k_2} \longmapsto \zeta(k_1, k_2).$$

Eisenstein series realization

Proposition (Gangl-Kaneko-Zagier)

For $k \geq 3$ there exists a realization of $\mathcal{D}_k$ in $\mathbb{C}[\![q]\!]$ such that

$$Z_k \longmapsto \mathbb{G}(k),$$

$$P_{k_1,k_2} \longmapsto \mathbb{G}(k_1)\mathbb{G}(k_2) + \frac{\delta_{k_1,2}}{2k_2}\mathbb{G}'(k_2) + \frac{\delta_{k_2,2}}{2k_1}\mathbb{G}'(k_1).$$

Here the **Eisenstein series** are defined for even $k \geq 4$ by

$$\mathbb{G}(k) = \mathbb{G}(k;\tau) = \frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k} = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\,,$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ is the divisor sum, $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ and $q = e^{2\pi i\tau}$.

Question

What is the image of Z_{k_1,k_2} in above realization? $\longrightarrow$ double Eisenstein series.

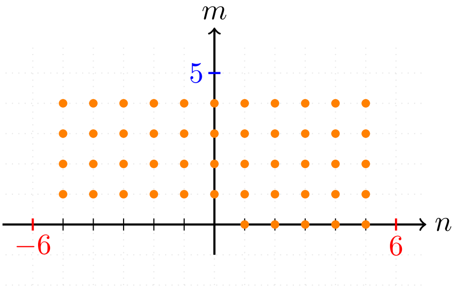
Order on lattices

For $M \geq 1$ set

$$\mathbb{Z}_M = \{m \in \mathbb{Z} \mid |m| < M\}\,.$$

and for $\tau \in \mathbb{H}$ define on $\mathbb{Z}\tau + \mathbb{Z}$ the **order** $\succ$ by

$$m_1\tau + n_1 \succ m_2\tau + n_2 \quad :\Leftrightarrow \quad (m_1 > m_2) \text{ or } (m_1 = m_2 \text{ and } n_1 > n_2)\,.$$



All the points $\lambda \in \mathbb{Z}_5 i + \mathbb{Z}_6$ satisfying $\lambda \succ 0$.

Multiple Eisenstein series

Definition

For integers $k_1, \dots, k_r \geq 1$, and $M, N \geq 1$ we define the **truncated multiple Eisenstein series** by

$$\mathbb{G}_{M,N}(k_1, \dots, k_r; \tau) = \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}_M \tau + \mathbb{Z}_N}} \frac{1}{\lambda_1^{k_1} \cdots \lambda_r^{k_r}} .$$

For $k_1, \dots, k_r \geq 2$ the **multiple Eisenstein series** are defined by

$$\mathbb{G}(k_1, \dots, k_r) = \mathbb{G}(k_1, \dots, k_r; \tau) = \lim_{M \to \infty} \lim_{N \to \infty} \mathbb{G}_{M,N}(k_1, \dots, k_r; \tau) .$$

We denote the $\mathbb{Q}$-vector space spanned by all MES by

$$\mathcal{E} = \langle \mathbb{G}(k_1, \dots, k_r) \mid r \geq 0, k_1, \dots, k_r \geq 2 \rangle_{\mathbb{Q}} ,$$

where we set $\mathbb{G}(k_1, \dots, k_r) = 1$ for $r = 0$.

Some facts

Some simple facts on multiple Eisenstein series:

- MES are holomorphic functions on the upper-half plane $\mathbb{H}$, but in general they are not modular.
- The product of multiple Eisenstein series can also be expressed by the **harmonic product** formula, e.g.

$$\mathbb{G}(4) \cdot \mathbb{G}(3) = \mathbb{G}(4,3) + \mathbb{G}(3,4) + \mathbb{G}(7)\,.$$

- The space $\mathcal{E}$ is a $\mathbb{Q}$-algebra and we have an algebra homomorphism

$$\begin{aligned} \mathbb{G} : \mathfrak{H}^2_* &\to \mathcal{E} \\ w = z_{k_1} \dots z_{k_r} &\longmapsto \mathbb{G}(w) := \mathbb{G}(k_1, \dots, k_r)\,. \end{aligned}$$

- We have

$$\begin{aligned} \lim_{\tau \to i\infty} \mathbb{G}(k_1, \dots, k_r; \tau) &= \zeta(k_1, \dots, k_r), \\ \mathbb{G}(k_1, \dots, k_r; \tau + 1) &= \mathbb{G}(k_1, \dots, k_r; \tau). \end{aligned}$$

The q-series g

Definition

For $k_1, \ldots, k_r \geq 1$ we define the q-series $g(k_1, \ldots, k_r) \in \mathbb{Q}[\![q]\!]$ by

$$g(k_1, \ldots, k_r; \tau) = g(k_1, \ldots, k_r) = \sum_{\substack{m_1 > \cdots > m_r > 0 \\ n_1, \ldots, n_r > 0}} \frac{n_1^{k_1 - 1}}{(k_1 - 1)!} \cdots \frac{n_r^{k_r - 1}}{(k_r - 1)!} q^{m_1 n_1 + \cdots + m_r n_r} .$$

In the case $r = 1$ these are the generating series of divisor-sums $\sigma_{k-1}(n) = \sum_{d|n} n^{k-1}$

$$g(k) = \sum_{m,n>0} \frac{n^{k-1}}{(k-1)!} q^{mn} = \frac{1}{(k-1)!} \sum_{n>0} \sigma_{k-1}(n) q^n ,$$

and they can be viewed as q**-analogues of multiple zeta values**, since for $k_1 \geq 2, k_2, \ldots, k_r \geq 1$ we have

$$\lim_{q \to 1} (1-q)^{k_1 + \cdots + k_r} g(k_1, \ldots, k_r) = \zeta(k_1, \ldots, k_r) .$$

Fourier expansion

$$\hat{g}(k_1,\ldots,k_r) := (-2\pi i)^{k_1+\cdots+k_r} g(k_1,\ldots,k_r) \in \mathbb{Q}[\pi i][\![q]\!]\,.$$

Theorem (Gangl-Kaneko-Zagier 2006 ($r=2$), B. 2012 ($r\geq 2$))

For $k_1,\ldots,k_r \geq 2$ there exist explicit $\alpha^{k_1,\ldots,k_r}_{l_1,\ldots,l_r,j} \in \mathbb{Z}$, such that for $q=e^{2\pi i\tau}$ we have

$$\mathbb{G}(k_1,\ldots,k_r) = \zeta(k_1,\ldots,k_r) + \sum_{\substack{0<j<r \\ l_1+\cdots+l_r=k_1+\cdots+k_r}} \alpha^{k_1,\ldots,k_r}_{l_1,\ldots,l_r,j}\, \zeta(l_1,\ldots,l_j)\hat{g}(l_{j+1},\ldots,l_r) + \hat{g}(k_1,\ldots,k_r)\,.$$

In particular, $\mathbb{G}(k_1,\ldots,k_r) = \zeta(k_1,\ldots,k_r) + \sum_{n>0} a_{k_1,\ldots,k_r}(n)q^n$ *for some* $a_{k_1,\ldots,k_r}(n) \in \mathcal{Z}[\pi i]$.

Examples

$$\mathbb{G}(k) = \zeta(k) + \hat{g}(k)\,,$$

$$\mathbb{G}(3,2) = \zeta(3,2) + 3\zeta(3)\hat{g}(2) + 2\zeta(2)\hat{g}(3) + \hat{g}(3,2)\,.$$

Definition

For $k_1, \dots, k_r \geq 1, N \geq 1$ and $x \in \mathbb{C}\backslash\mathbb{Z}$ define the **(truncated) multitangent function** by

$$\Psi_N(k_1, \dots, k_r; x) := \sum_{\substack{N > n_1 > \dots > n_r > -N \\ n_i \in \mathbb{Z}}} \frac{1}{(x+n_1)^{k_1} \cdots (x+n_r)^{k_r}}.$$

For $k_1, k_r \geq 2$ the **multitangent function** is given by $\Psi(k_1, \dots, k_r; x) = \lim_{N\to\infty} \Psi_N(k_1, \dots, k_r; x)$.

In depth one we have for $k \geq 2$ the **Lipschitz formula** $(q = e^{2\pi i\, \tau})$

$$\Psi(k; \tau) = \sum_{n \in \mathbb{Z}} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{d>0} d^{k-1} q^d .$$

In particular for $k_1, \dots, k_r \geq 2$ we get

$$\hat{g}(k_1, \dots, k_r) = \sum_{m_1 > \dots > m_r > 0} \Psi_{k_1}(m_1 \tau) \cdots \Psi_{k_r}(m_r \tau) .$$

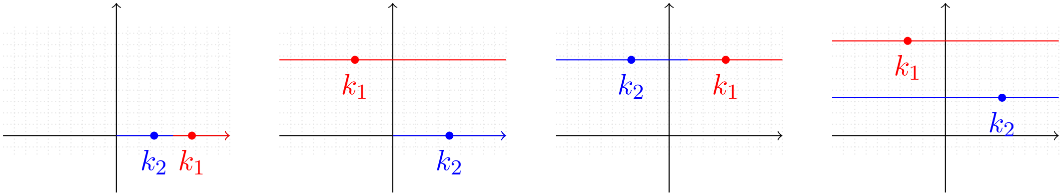
Fourier expansion - Multitangent functions

We can write $\mathbb{G}$ as sums over Ψ. For example, in depth two we have:

$$\mathbb{G}(k_1, k_2; \tau) = \sum_{m_1\tau+n_1 \succ m_2\tau+n_2 \succ 0} \frac{1}{(m_1\tau+n_1)^{k_1}(m_2\tau+n_2)^{k_2}}$$

$$= \left(\sum_{\substack{m_1=m_2=0\\ n_1>n_2>0}} + \sum_{\substack{m_1>m_2=0\\ n_1\in\mathbb{Z}, n_2>0}} + \sum_{\substack{m_1=m_2>0\\ n_1>n_2}} + \sum_{\substack{m_1>m_2>0\\ n_1,n_2\in\mathbb{Z}}} \right) \frac{1}{(m_1\tau+n_1)^{k_1}(m_2\tau+n_2)^{k_2}}$$

$$= \zeta(k_1,k_2) + \sum_{m>0} \Psi(k_1; m\tau)\zeta(k_2) + \sum_{m>0} \Psi(k_1,k_2; m\tau) + \sum_{m_1>m_2>0} \Psi(k_1; m_1\tau)\Psi(k_2; m_2\tau)\,.$$



Fourier expansion - Multitangent functions

Theorem (Bouillot 2011)

For $k_1,\dots,k_r \geq 1$ with $k_1, k_r \geq 2$ and $k = k_1+\dots+k_r$ the multitangent function can be written as

$$\Psi(k_1,\dots,k_r;\tau) = \sum_{\substack{1\leq j\leq r\\ l_1+\dots+l_r=k}} (-1)^{\bullet} \prod_{\substack{1\leq i\leq r\\ i\neq j}} \binom{l_i-1}{k_i-1} \zeta(l_1,\dots,l_{j-1})\,\Psi(l_j;\tau)\,\zeta(l_r,l_{r-1},\dots,l_{j+1})\,.$$

where $\bullet = l_1+\dots+l_{j-1}+k_j+k$. Moreover, the terms with $\Psi(1;\tau)$ vanish.

Proof: Partial fraction decomposition and antipote relation.

Corollary

For $w_1,\dots,w_l \in \mathfrak{H}^2$ we have

$$\sum_{m_1>\dots>m_l>0} \Psi(w_1;m_1\tau)\dots\Psi(w_l;m_l\tau) \subset \mathcal{Z}[\pi i]\otimes\langle g(w)\mid w\in\mathfrak{H}^2\rangle_{\mathbb{Q}}\,.$$

In particular, $\mathbb{C}\otimes\mathcal{E}$ is spanned (as a $\mathbb{C}$-vector space) by $g(w)$ with $w\in\mathfrak{H}^2$.

Some works on multiple Eisenstein series (MES)

Depth \ Level	1	2	4	all
2	Kaneko (2004) Gangl-Kaneko-Zagier (2006) F Tasaka (2020) B.-Kuehn-Matthes (2021) F	Kaneko-Tasaka (2013) F Sun (202?)	Kina (2024) Kina F (202?)	
all	B. (2012) B.-Tasaka (2017) B. (2019) B.-Burmester (2023) B.-van-Ittersum-Matthes F Burmester (2023) F			Yuan-Zhao (2015/2016) Kitada (2023) Kanno (2024)

Other variants: Schur MES (Yu 202?)

F: Formal space

What relations do MES satisfy?

Multiple zeta values satisfy various relations. For example,

$$\zeta(2)^2 = \frac{5}{2}\zeta(4)\,, \qquad \zeta(5) = 2\zeta(3,2) + 6\zeta(4,1)\,.$$

Question

Do multiple Eisenstein series satisfy these relations?

The first relation is not satisfied, since

$$\mathbb{G}(2)^2 = \frac{5}{2}\mathbb{G}(4) - \frac{(2\pi i)^2}{2}\mathbb{G}'(2)\,.$$

The second relation can not be satisfied since $\mathbb{G}_{4,1}$ is not defined.

Question

1. Are there "natural" extensions of the algebra homomorphism $\mathbb{G} : \mathfrak{H}^2_* \to \mathcal{O}(\mathbb{H})$ to $\mathfrak{H}^1_*$ or $\mathfrak{H}^1_{\sqcup\!\sqcup}$?
2. How to include derivatives in our algebraic setup?
3. What are the relations among MES?

Shuffle regularized multiple Eisenstein series

Question

Is there a "natural" construction of an algebra homomorphism $\mathbb{G}^{ш} : \mathfrak{H}^1_{ш} \to \mathcal{O}(\mathbb{H})$?

Equipped with the **Goncharov coproduct** Δ_G the algebra $\mathfrak{H}^1_{ш}$ becomes a Hopf algebra.

There exist explicit formulas for Δ_G, e.g.

$$\Delta_G(z_3 z_2) = z_3 z_2 \otimes 1 + 3 z_2 \otimes z_3 + 2 z_3 \otimes z_2 + 1 \otimes z_3 z_2 \,.$$

Compare this to the Fourier expansion of $\mathbb{G}_{3,2}$:

$$\mathbb{G}(3,2;\tau) = \zeta(3,2) + 3\hat{g}(2)\zeta(3) + 2\hat{g}(3)\zeta(2) + \hat{g}(3,2)\,.$$

Theorem (B.-Tasaka 2017 (Level N: Kanno 2024))

We have

$$\mathbb{G} = (m \circ (\hat{g} \otimes \zeta) \circ \Delta_G)_{|\mathfrak{H}^2} \,,$$

where m denotes the multiplication.

Shuffle regularized multiple Eisenstein series

Proposition (B.-Tasaka 2017, Level N: Kitada 2023)

There exists an algebra homomorphism $\hat{g}^{\sqcup\mathchoice{}{}{}{}\!\sqcup} : \mathfrak{H}^1_{\sqcup\!\sqcup} \to \mathcal{O}(\mathbb{H})$ with $\hat{g}^{\sqcup\!\sqcup}_{|\mathfrak{H}^2} = \hat{g}$.

Definition

Define the **shuffle regularized multiple Eisenstein series** as the algebra homomorphism $\mathbb{G}^{\sqcup\!\sqcup} : \mathfrak{H}^1_{\sqcup\!\sqcup} \to \mathcal{O}(\mathbb{H})$

$$\mathbb{G}^{\sqcup\!\sqcup} = m \circ (\hat{g}^{\sqcup\!\sqcup} \otimes \zeta^{\sqcup\!\sqcup}) \circ \Delta_G$$

By the previous mentioned results we have $\mathbb{G}^{\sqcup\!\sqcup}_{|\mathfrak{H}^2} = \mathbb{G}$.

Corollary

The shuffle regularized multiple Eisenstein series satisfy the **restricted double shuffle relations**, i.e.

$$\mathbb{G}^{\sqcup\!\sqcup}(w \sqcup\!\sqcup v - w * v) = 0 \qquad (w, v \in \mathfrak{H}^2)\,.$$

Stuffle regularized multiple Eisenstein series

With a similar idea one can also obtain a regularization for the harmonic product:

Proposition/Definition (B. 2019)

There exists an algebra homomorphism $\hat{g}^* : \mathfrak{H}^1_* \to \mathcal{O}(\mathbb{H})$ such that the **stuffle regularized multiple Eisenstein series** defined as the algebra homomorphism $\mathbb{G}^* : \mathfrak{H}^1_* \to \mathcal{O}(\mathbb{H})$

$$\mathbb{G}^* = \left(m \circ (\hat{g}^* \otimes \zeta^*) \circ \Delta\right).$$

satisfies $\mathbb{G}^*_{|\mathfrak{H}^2} = \mathbb{G}$.

These two regularizations are...

1. not uniquely determined by the image of z_1.
2. not the same even on admissible indices, e.g. $\mathbb{G}^*(2,1,1) \neq \mathbb{G}^{\sqcup\!\sqcup}(2,1,1)$.

Open problem 1

1. Find/Understand "all" shuffle and stuffle regularized multiple Eisenstein series.
2. Describe their relationship (Is there an analogue to $\zeta^{\sqcup\!\sqcup} = \rho \circ \zeta^*$?)

Relations among $\mathbb{G}^\bullet$

One can check that $\mathbb{G}^{ш}$ satisfy more relations than the restricted double shuffle relations, e.g.

$$\mathbb{G}^{ш}(z_2 ш z_2 z_1 - z_2 * z_2 z_1) = 0\,.$$

But they satisfy less relations than MZV, e.g. we have $\zeta(3) - \zeta(2,1) = 0$, but for $\bullet \in \{*, ш\}$

$$\mathbb{G}^\bullet(3) - \mathbb{G}^\bullet(2,1) = \frac{(2\pi i)^2}{2} q \frac{d}{dq} \mathbb{G}^\bullet(1)\,.$$

Question

How to describe all relations among multiple Eisenstein series algebraically?

Observations/Conjecture

- The q-series g modulo lower weight satisfy the same relations as MES.
- All relations among g are a consequence of a certain involution (swap) and an analogue of the stuffle product.

Algebraic setup (Burmester 2023)

As an extension of $\mathfrak{H}^1 = \mathbb{Q}\langle z_1, z_2, \ldots \rangle$ and $\mathfrak{H}$ define

$$\widehat{\mathfrak{H}}^1 = \mathbb{Q} + \sum_{k \geq 1} z_k \widehat{\mathfrak{H}} \qquad \subset \qquad \widehat{\mathfrak{H}} = \mathbb{Q}\langle z_0, z_1, z_2, \ldots \rangle.$$

($\widehat{\mathfrak{H}}^1$= words in $z_0, z_1, z_2 \ldots$ not starting in z_0)

Extend the **harmonic product** to $\widehat{\mathfrak{H}}^1 \otimes \widehat{\mathfrak{H}}^1 \to \widehat{\mathfrak{H}}^1$ for words $w, v \in \widehat{\mathfrak{H}}^1$ and $r, s \geq 0$ by

$$\mathbf{1} * w = w = w * \mathbf{1},$$
$$z_r w * z_s v = z_r(w * z_s v) + z_s(z_r w * v) + \delta_{r \cdot s \neq 0}\, z_{r+s}(w * v),$$

and obtain $\mathbb{Q}$-algebras $\mathfrak{H}^1_* \subset \widehat{\mathfrak{H}}^1_*$.

Define the **swap** σ as the linear map given for $k_1, \ldots, k_r, m_1, \ldots, m_r \geq 1$ by

$$\sigma : \widehat{\mathfrak{H}}^1 \longrightarrow \widehat{\mathfrak{H}}^1,$$
$$z_{k_1} z_0^{m_1 - 1} z_{k_2} z_0^{m_2 - 1} \ldots z_{k_r} z_0^{m_r - 1} \longmapsto z_{m_r} z_0^{k_r - 1} \ldots z_{m_1} z_0^{k_1 - 1}.$$

Formal MES

Definition

The algebra of **formal multiple Eisenstein series** $\mathcal{G}^f$ is the $\mathbb{Q}$-algebra defined by

$$\mathcal{G}^f = \widehat{\mathfrak{H}}^1_* \big/ \mathfrak{S}$$

where $\mathfrak{S}$ is the $*$-ideal generated by $\{w - \sigma(w) \mid w \in \widehat{\mathfrak{H}}^1\}$.

By $G^f(k_1, \ldots, k_r)$ we denote the class of $z_{k_1} \cdots z_{k_r}$ for $k_1 \geq 1, k_2, \ldots, k_r \geq 0$.
Claim: The $G^f(k_1, \ldots, k_r)$ satisfy the same relations as (regularized) MES. In particular, define

$$\mathcal{E}^f = \langle G^f(k_1, \ldots, k_r) \mid r \geq 0, k_1, \ldots, k_r \geq 2 \rangle_{\mathbb{Q}} .$$

Conjecture

We have an isomorphism of $\mathbb{Q}$-algebras

$$\mathcal{E}^f \longrightarrow \mathcal{E}$$
$$G^f(k_1, \ldots, k_r) \longmapsto \mathbb{G}(k_1, \ldots, k_r).$$

Formal MES

Conjecturally we actually do not need to introduce this big space $\widehat{\mathfrak{H}}^1$.

Conjecture

We have

$$\mathcal{G}^f \cong \left. \mathfrak{H}^1_* \middle/ \mathfrak{S} \cap \mathfrak{H}^1 \right.$$

Open problem 2

1. Describe $\mathfrak{S} \cap \mathfrak{H}^1$ explicitly.
   (This means we could give a definition of formal MES as a quotient of $\mathfrak{H}^1$)
2. Describe $\mathfrak{S} \cap \mathfrak{H}^2$ explicitly.
   (This means describing (conjecturally all) the relations satisfied by $\mathbb{G}$)
3. Prove above conjectures.

$\mathfrak{sl}_2$-algebras

Definition

An algebra A is called **$\mathfrak{sl}_2$-algebra** if there exists a Lie-algebra homomorphism $\mathfrak{sl}_2 \to \mathrm{Der}(A)$.

In other words, there exist derivations $D, W, \delta \in \mathrm{Der}(A)$ satisfying the commutator relations

$$[W, D] = 2D, \quad [W, \delta] = -2\delta, \quad [\delta, D] = W.$$

Example The algebra of **quasimodular forms** $\widetilde{\mathcal{M}} = \mathbb{Q}[G(2), G(4), G(6)]$ with

$$D = q\frac{d}{dq}, \qquad W(G(k)) = kG(k), \qquad \delta(G(2)) = -\frac{1}{2}, \quad \delta(G(4)) = \delta(G(6)) = 0$$

is an $\mathfrak{sl}_2$ algebra. Here the Eisenstein series $G(k)$ are given for $k \geq 2$ by

$$G(k) = (-2\pi i)^{-k}\mathbb{G}(k) = -\frac{B_k}{2k!} + \frac{1}{(k-1)!}\sum_{m,n\geq 1} m^{k-1}\, q^{mn}, \qquad (B_k = k\text{th Bernoulli number}).$$

Results

Theorem (B.-van-Ittersum, 2023)

There exist explicit derivations W, D, δ *on* $\mathcal{G}^f$ *such that*

- $\mathcal{G}^f$ *is an* $\mathfrak{sl}_2$*-algebra;*
- *the subalgebra* $\mathbb{Q}[G^f(2), G^f(4), G^f(6)] \subset \mathcal{G}^f$ *is isomorphic to* $\widetilde{\mathcal{M}}$ *as an* $\mathfrak{sl}_2$*-algebra.*

Theorem (B.-van-Ittersum, 2023)

There exists a surjective algebra homomorphism (The "formal projection to the constant term")

$$\pi : \mathcal{G}^f \to \mathcal{Z}^f,$$

with $\pi(G^f(k_1, \ldots, k_r)) = \zeta^f(k_1, \ldots, k_r)$*. The kernel of* π *can be described explicitly.*

Theorem (B.-Burmester, 2023)

There exists an algebra homomorphism $\mathcal{G}^f \to \mathbb{Q}[\![q]\!]$ *with* $G^f(k) \mapsto G(k)$*.*

Open problems: Closed under derivation

Conjectures

- The subspace spanned by all $G^f(k_1,\dots,k_r)$ (resp. $\mathbb{G}(k_1,\dots,k_r)$) is closed under D (resp. $\frac{d}{d\tau}$).
- These subspaces are $\mathfrak{sl}_2$-algebras.

Open problem 3

1. Show that the space $\mathbb{C}\otimes\mathcal{E}$ is closed under $\frac{d}{d\tau}$.
2. Show that the $\mathbb{Q}$-vector spaced spanned by all $g(k_1,\dots,k_r)$ with $k_1,\dots,k_r \geq 2$ is closed under $q\frac{d}{dq}$.
   (This is also conjectured by Okounkov in a different context)

Open problems: Linear independence

Using that a modular forms f of weight k satisfies $f\left(-\frac{1}{\tau}\right) = \tau^k f(\tau)$ one can show the following:

Proposition

Modular forms with different weights are linearly independent over $\mathbb{C}$.

Multiple Eisenstein series do not satisfy this transformation property, but we still expect that there are no linear relations among MES of different weight.

Open problem 4

Show that multiple Eisenstein series of different weights are linearly independent.

Open problems: Dimension & Basis

Conjectures (B.-Kühn, Okounkov)

- The dimension of $\mathcal{G}^f$ in weight k are given by

$$\sum_{k\geq 0} \dim_{\mathbb{Q}} \mathcal{G}_k^f X^k = \frac{1}{1-X-X^2-X^3+X^6+X^7+X^8+X^9}$$
$$= 1+X+2X^2+4X^3+7X^4+13X^5+23X^6+41X^7+\ldots.$$

- The dimension of $\mathcal{E}$ (resp. $\mathcal{E}^f$) in weight k are by

$$\sum_{k\geq 0} \dim_{\mathbb{Q}} \mathcal{E}_k X^k = \frac{1}{1-X^2-X^3-X^4-X^5+X^8+X^9+X^{10}+X^{11}+X^{12}}$$
$$= 1+X^2+X^3+2X^4+3X^5+7X^7+\ldots.$$

Open problem 5

Come up with a conjectural basis for these spaces.

Open problems: Modularity

Most of multiple Eisenstein series $\mathbb{G}(k_1, \ldots, k_r; \tau)$ are not modular since in general

$$\mathbb{G}(k_1, \ldots, k_r; -\frac{1}{\tau}) \neq \tau^{k_1+\cdots+k_r}\mathbb{G}(k_1, \ldots, k_r; \tau).$$

Question

For which $\alpha(w) \in \mathbb{Q}$ is the sum

$$\sum_{w \in \mathfrak{H}^2} \alpha(w)\mathbb{G}(w)$$

a (quasi)modular form?

Simple examples are $\mathbb{G}(2k, \ldots, 2k)$ for $k \geq 1$.

Open problem 6

- Find non-trivial examples of linear combinations of MES which are modular.
- Find a criteria to decide when a linear combination of MES is modular.

Open problems: L-series / Mellin transform

Whenever one has a q-series $f = \sum_{n\geq 0} a_n q^n$ one can consider its L-series $L(f,s) = \sum_{n\geq 1} \frac{a_n}{n^s}$.

Theorem (Heckes converse theorem - rough version)

f is modular $\iff$ $L(f,s)$ satisfies a functional equation & analytic. cont. & growth condition.

Question

How does the L-series of $\mathbb{G}(k_1,\dots,k_r)$ (or equivalently) $g(k_1,\dots,k_r)$ look like?

In the case $r=1$ we get the classical L-series of the Eisenstein series

$$L(g(k),s) \doteq \sum_{n\geq 1} \frac{1}{n^s} \sum_{a|n} a^{k-1} = \sum_{a,b\geq 1} \frac{a^{k-1}}{(ab)^s} = \zeta(s)\zeta(s-k+1).$$

Open problems: L-series / Mellin transform

Recall that in general depth we had

$$g(k_1,\dots,k_r)=\sum_{\substack{m_1>\dots>m_r>0\\ n_1,\dots,n_r>0}}\frac{n_1^{k_1-1}}{(k_1-1)!}\cdots\frac{n_r^{k_r-1}}{(k_r-1)!}q^{m_1n_1+\dots+m_rn_r}.$$

For $r=2$ we get

$$L(g(k_1,k_2),s)\doteq\sum_{n>0}\frac{1}{n^s}\sum_{\substack{m_1>m_2>0\\ n_1,n_2>0\\ m_1n_1+m_2n_2=n}}n_1^{k_1-1}n_2^{k_2-1}=\sum_{\substack{m_1>m_2>0\\ n_1,n_2>0}}\frac{n_1^{k_1}n_2^{k_2}}{(m_1n_1+m_2n_2)^s}=???.$$

Question

1. Are the values $L(g(k_1,\dots,k_r),s)$ for large enough $s\in\mathbb{N}$ again MZV?
2. Do these L-functions satisfy some functional equations?

Open problem 7

Study the L-series of MES and/or of q-analogues of MZV.

Open problems: Hecke operators on MES

We can view MES as functions on lattices

$$\mathbb{G}_{k_1,\dots,k_r}(\Lambda) := \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_r \succ 0 \\ \lambda_i \in \Lambda}} \frac{1}{\lambda_1^{k_1} \dots \lambda_r^{k_r}}$$

For a lattice function F of weight k the n-th **Hecke operator** acts by

$$(T_n \cdot F)(\Lambda) := n^{k-1} \sum_{\substack{\Lambda' \in \mathcal{L} \\ [\Lambda:\Lambda']=n}} F(\Lambda'),$$

where the sum is over all sublattices of rank n.

Question

1. Can one write $T_N \mathbb{G}$ again as MES of level N?
2. Are there "eigenforms"?

Open problem 8

Study and describe the action of T_n on $\mathbb{G}$.

Open problems: Symmetric & Finite MES

Define for $k_1, \ldots, k_r \geq 2$ the **symmetric MES**

$$\mathbb{G}_{\mathcal{S}}(k_1, \ldots, k_r; \tau) = \mathbb{G}_{\mathcal{S}}(k_1, \ldots, k_r) = \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_r \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z} \setminus \{0\}}} \frac{1}{\lambda_1^{k_1} \ldots \lambda_r^{k_r}}.$$

It is easy to see that we have

$$\mathbb{G}_{\mathcal{S}}(k_1, \ldots, k_r) = \sum_{j=0}^{r} (-1)^{k_{j+1} + \cdots + k_r} \mathbb{G}(k_1, \ldots, k_j) \mathbb{G}(k_r, \ldots, k_{j+1}).$$

Open problem 9

- Find a good definition of $\mathbb{G}_{\mathcal{S}}(k_1, \ldots, k_r)$ for $k_1, \ldots, k_r \geq 1$.
- Find a definition of "finite MES" and consider an analogue of the Kaneko-Zagier conjecture.

Open problems: MSW type formula

Maesaka, Seki and Watanabe introduced a discretization of iterated integrals.

Theorem (MSW-Formula)

For $N > 0$ define

$$\zeta^{\flat}_{<N}(k_1,\ldots,k_r) = \sum_{\substack{0<n_{i1}\le\cdots\le n_{ik_i}<N \quad (1\le i\le r) \\ n_{ik_i}<n_{(i+1)1} \quad (1\le i<r)}} \prod_{i=1}^{r} \frac{1}{(N-n_{i1})n_{i2}\cdots n_{ik_i}}.$$

Then we have

$$\zeta_{<N}(k_1,\ldots,k_r) = \sum_{0<m_1<\cdots<m_r<N} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} = \zeta^{\flat}_{<N}(k_1,\ldots,k_r).$$

This gives a way of proving the duality formula and the double shuffle relations without using iterated integrals.

Open problem 10

Find a good definition of $\mathbb{G}^{\flat}_{M,N}$.

Open problems: Fourier coefficients

Define the **multiple Eisenstein coefficients** $\gamma_n(k_1,\dots,k_r)$ by

$$G^{\sqcup\!\sqcup}(k_1,\dots,k_r)=\sum_{n\geq 0}\gamma_n(k_1,\dots,k_r)q^n\,.$$

These can also be seen as maps $\gamma_n:\mathfrak{H}^1\to\mathcal{Z}[\pi i]=\mathcal{Z}+\pi i\mathcal{Z}$.

Basic facts

- $\gamma_0(k_1,\dots,k_r)=\zeta^{\sqcup\!\sqcup}(k_1,\dots,k_r)$.
- For all $n\geq 0$ and $w,v\in\mathfrak{H}^2$ we have

$$\gamma_n(w\sqcup\!\sqcup\, v-w*v)=0\,.$$

Relations among γ_n $\rightsquigarrow$ Relations among elements in $\mathcal{Z}[\pi i]$.

Open Problems: MES coefficients - Relation Examples

Examples One can show $G^{ш}(2,2,1)+6G^{ш}(3,1,1)-G^{ш}(2,3)-G^{ш}(4,1)=0$. This gives

$$\gamma_1(2,2,1)+6\gamma_1(3,1,1)-\gamma_1(2,3)-\gamma_1(4,1)=3\zeta(3)(2\pi i)^2-3\zeta(2,1)(2\pi i)^2=0$$

from which we deduce $\zeta(2,1)=\zeta(3)$.

This example shows that relations among the γ can give (some) extended double shuffle relations.

Questions

Can we obtain all extended double shuffle relations of MZV?

Open problem 11

- Study the Fourier coefficients of MES.
- Study the relations among MZV which follow from the relations among MES.

Definition

A **modular form** of weight k is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ on the upper half-plane $\mathbb{H}$, satisfying:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \quad \text{for all} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

and is holomorphic at ∞.

Proposition

The space of modular forms is given by $\mathbb{C}[\mathbb{G}(4), \mathbb{G}(6)]$.

Questions

Can we find a notion of "multiple modular forms" such that the space of them will be $\mathcal{E}$?

Open problem 12

Find a framework in which multiple Eisenstein series arise naturally, analogous to classical modular forms.

12 open problems on multiple Eisenstein series

1. **Regularization of MES:** Describe the regularization of multiple Eisenstein series and their relationships.
2. **Relations among MES:** Explicitly describe $\mathfrak{S} \cap \mathfrak{H}^1$ and $\mathfrak{S} \cap \mathfrak{H}^2$.
3. **Closed Under Derivation:** Prove closure of $\mathcal{E}$ under $\frac{d}{d\tau}$ and $g(k_1, \ldots, k_r)$ under $q\frac{d}{dq}$.
4. **Linear Independence of MES:** Show that MES of different weights are linearly independent.
5. **Dimension and Basis of MES:** Determine a conjectural basis for the spaces of MES.
6. **Modularity of MES:** Find criteria or examples of modular linear combinations of MES.
7. **L-Series of MES:** Study L-series and Mellin transforms of MES.
8. **Hecke Operators on MES:** Describe the action of Hecke operators T_n on MES.
9. **Symmetric and Finite MES:** Define symmetric and finite MES, and consider finite analogues.
10. **MSW-Type Formula for MES:** Find a MSW-type formula for MES.
11. **Fourier Coefficients of MES:** Study relations among MES coefficients and implications for MZV.
12. **Intrinsic Definition of MES:** Develop a framework where MES arise naturally, analogous to modular forms.