1 2 3 π e $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Numbers & infinite sums

and their appearances in daily life

Studium Generale - 6th December 2018

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About me

- Born in Hamburg (Germany)
- Studied mathematics at Hamburg University
- Since 2017 Assistant Professor at Nagoya University
- Interested in **Number theory**

Goal of this talk

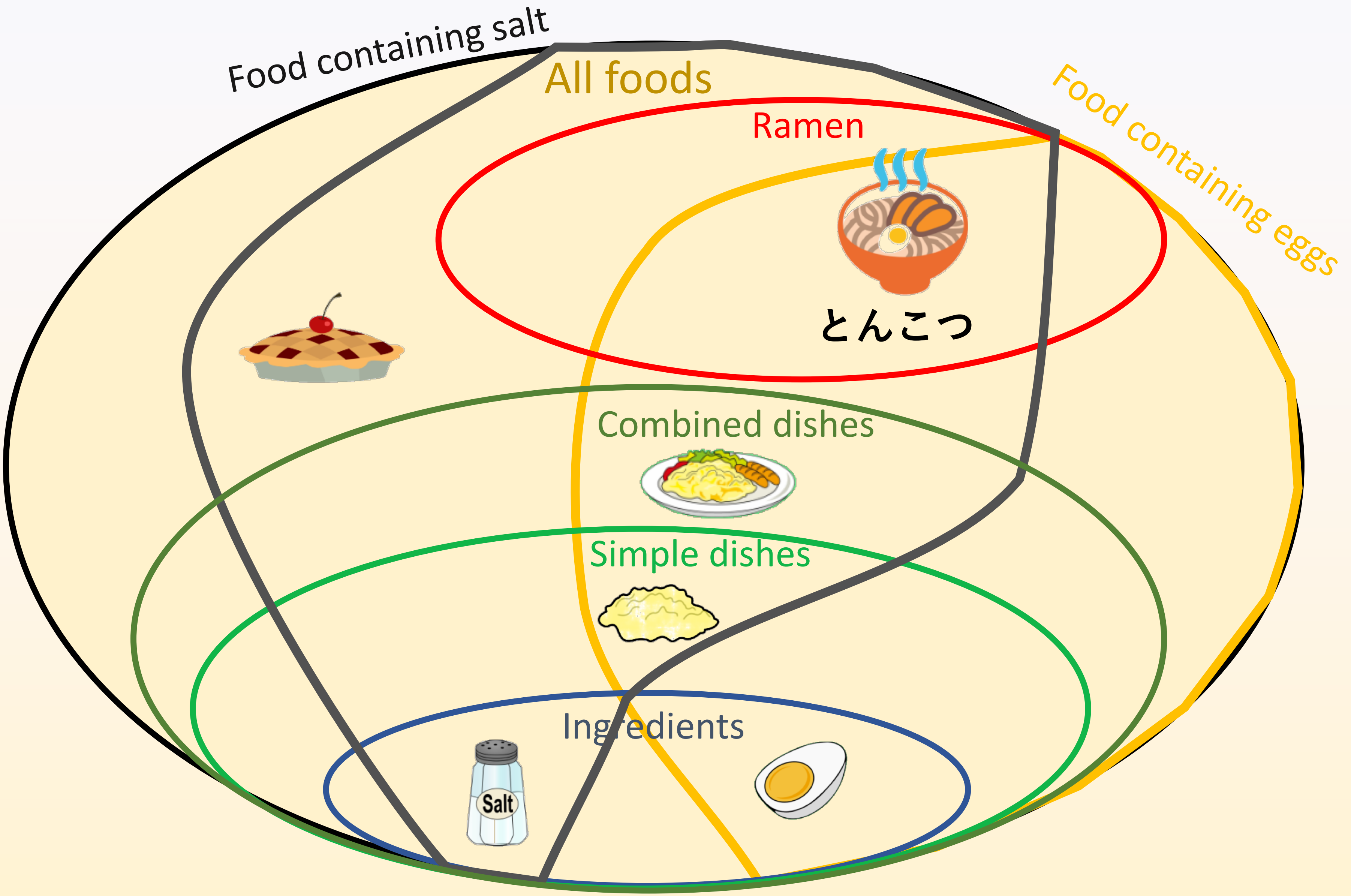
- Talk about the classification of numbers

What does that mean?

- Present some open problems on numbers

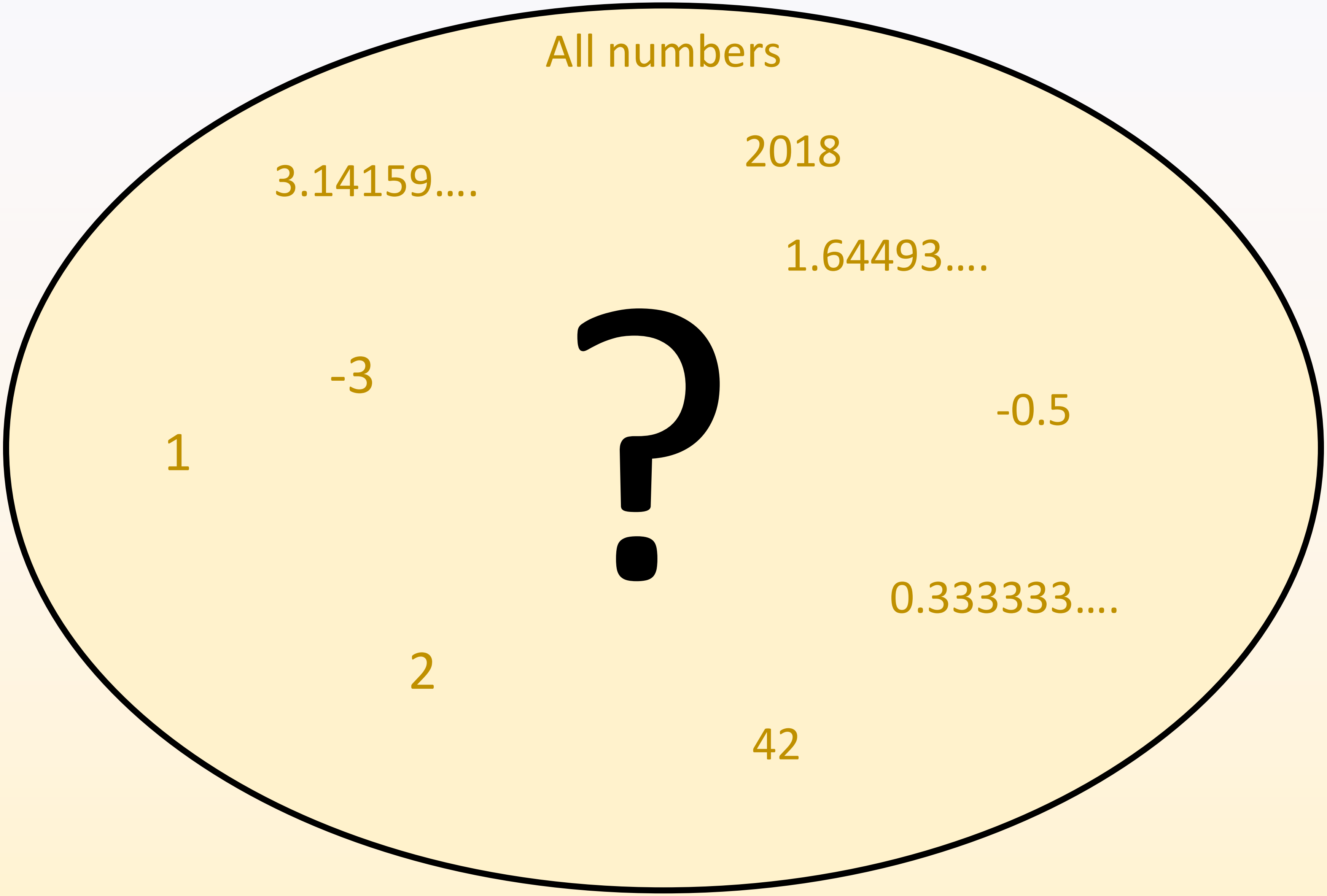
- Give examples of numbers as infinite sums

Classification of food



Pictures: clipartmax.com

Classification of numbers



Natural numbers the simple dishes…

The numbers 1,2,3,4,…. are called natural numbers

We can add (+) and multiply (∗) natural numbers

$$6 = 2 * 3$$

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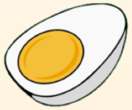
 * 

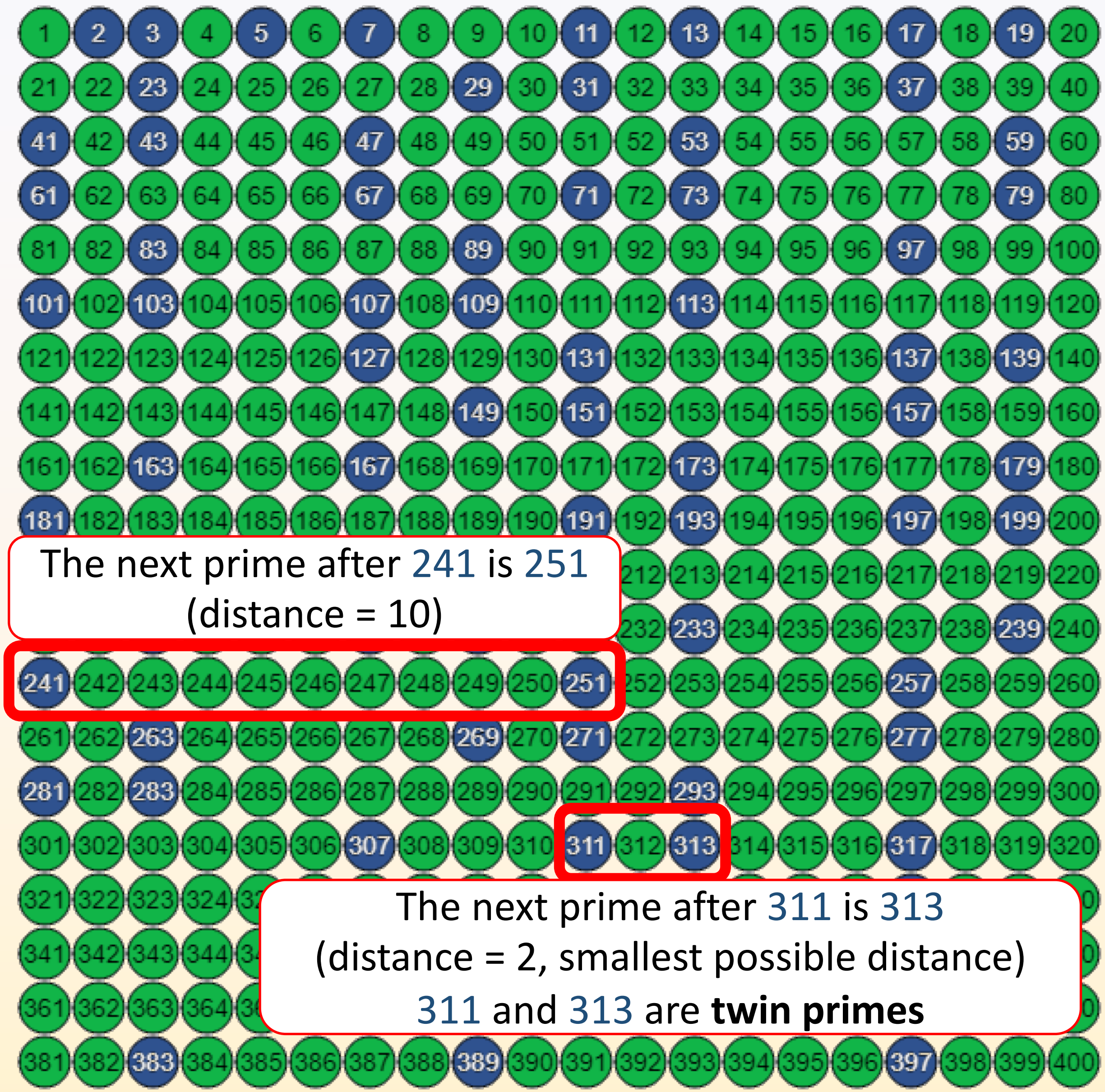
What are the ingredients for natural numbers?

Prime numbers the ingredients...

A natural number greater than 1 is called a **prime number**, if it can not be written as a product of two smaller numbers.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Prime numbers the ingredients...



Open problem 1: Twin prime conjecture

"There are infinitely many primes p such that p + 2 is also prime"

The first few twin primes:

(3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), (59, 61), (71, 73),
(101, 103), (107, 109), (137, 139), …

The best we know (2014):

"There are infinitely many primes p such that the next prime is smaller than p + 246."

Prime numbers the ingredients…

Prime numbers are the "ingredients" of natural numbers with respect to multiplication ($*$).

What about the addition (+) of prime numbers?

Open problem 2: Goldbach's conjecture

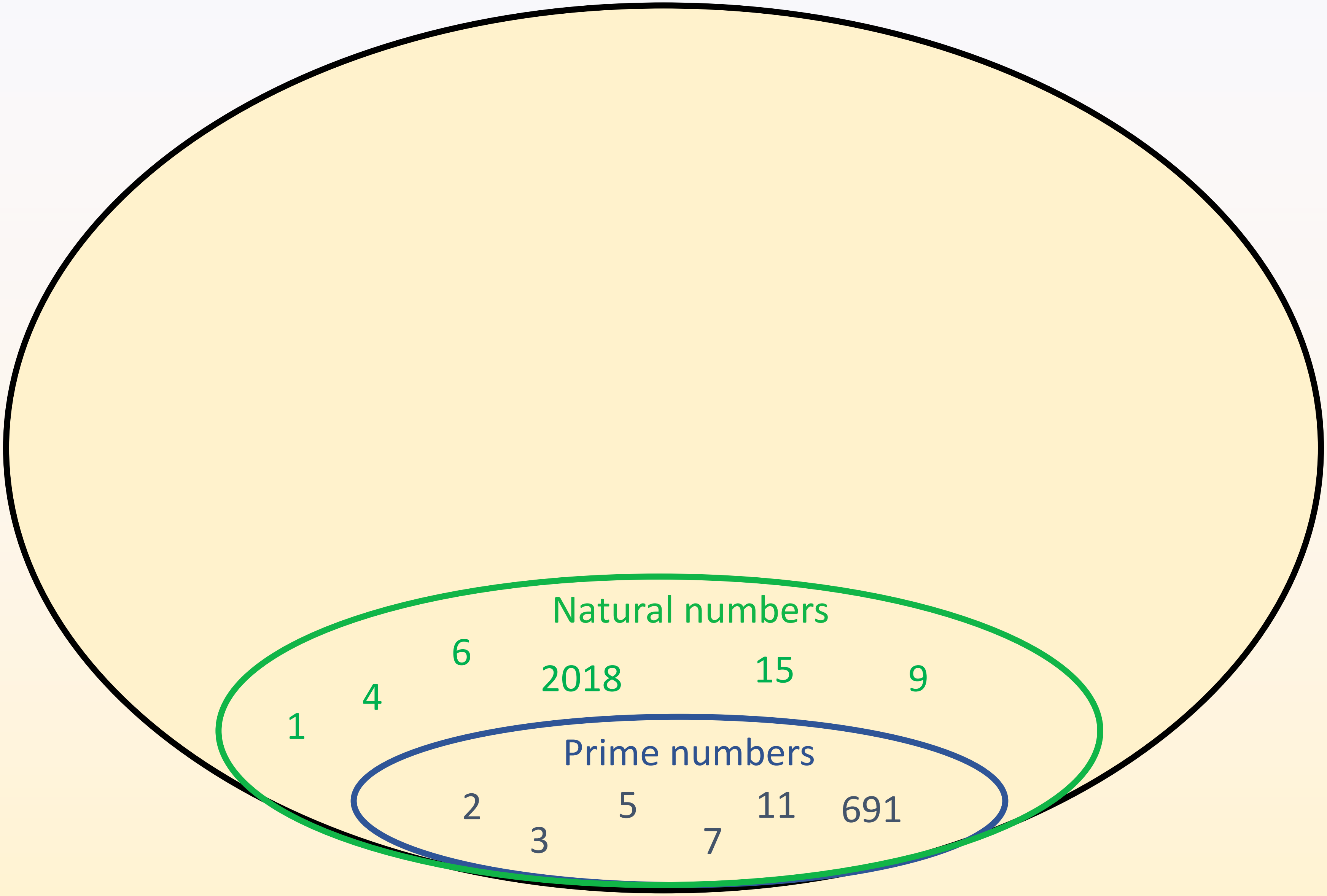
"Every **even** natural number greater than 2 can be expressed as the sum of **two** primes."

$4 = 2 + 2$ $6 = 3 + 3$ $8 = 3 + 5$ $10 = 3 + 7 = 5 + 5$

$12 = 5 + 7$ $14 = 3 + 11 = 7 + 7$ …

$100 = 3 + 97 = 11 + 89 = 17 + 83 = 29 + 71 = 41 + 59 = 47 + 53$

Classification of numbers … so far



Integers

We also have zero 0 and negative numbers -1, -2, -3, -4,...

The natural numbers together with 0 and their negatives are called **integers**.

Mathematicians point of view

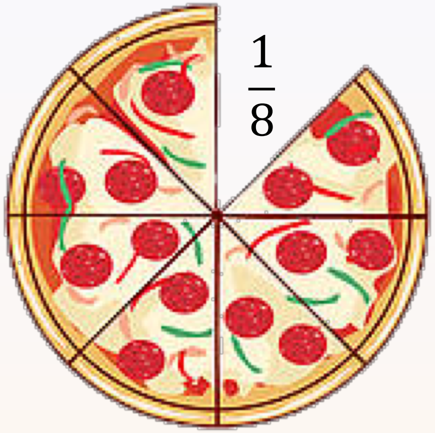
A, B : natural numbers

Integers allow us to solve the following equation for X:

$$X + B = A$$

For example $X = -5$ is the solution of $X + 7 = 2$.

Rational numbers combined dishes...



$\frac{1}{8}$

$\frac{7}{8}$

Numbers given by fractions are called **rational numbers**.

$$\frac{a}{b}$$

a ← numerator

b ← denominator

<u>Mathematicians point of view</u>

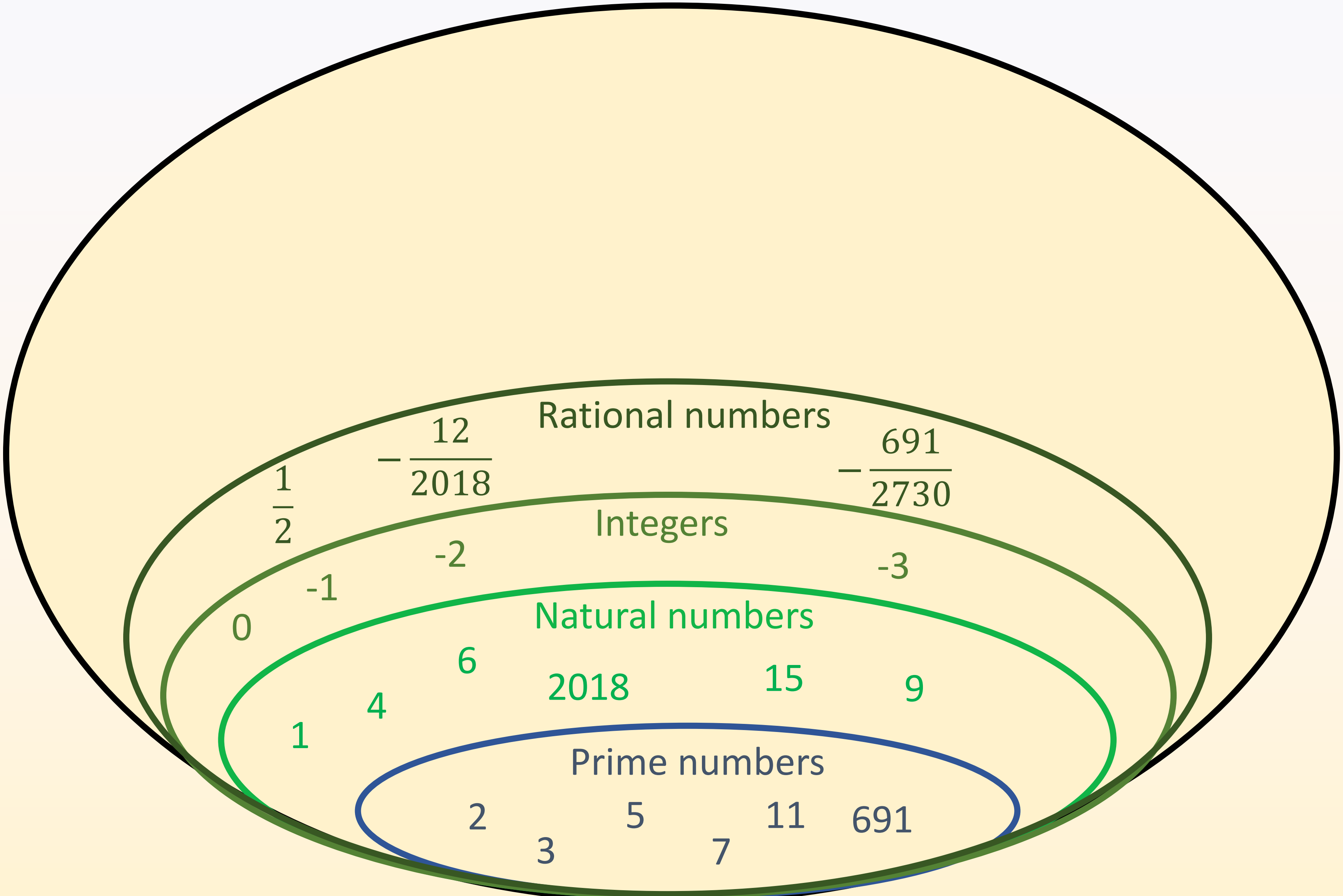
A, B, C : natural numbers

Rational numbers allow us to solve the following equation for X:

$$C\,X + B = A$$

For example $X = \frac{2}{5}$ is the solution of $5\,X + 7 = 9$.

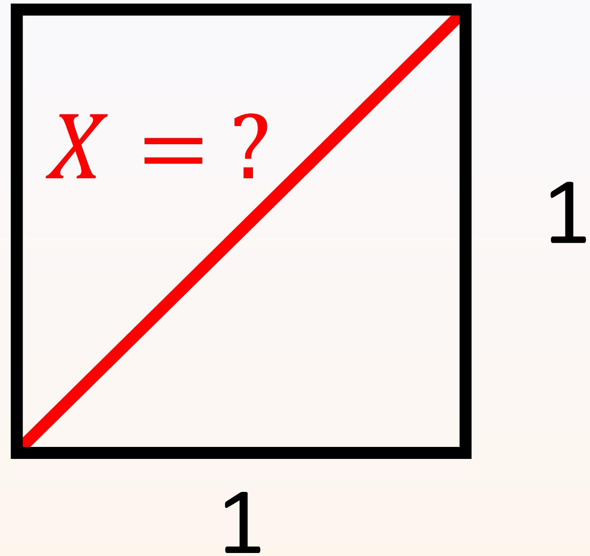
Classification of numbers … so far



Algebraic numbers

$$X^2 = 1^2 + 1^2 = 2$$

$$X = \sqrt{2} \approx 1.414\ldots$$



Mathematicians point of view

$A_0, A_1, \ldots, A_n$: integers

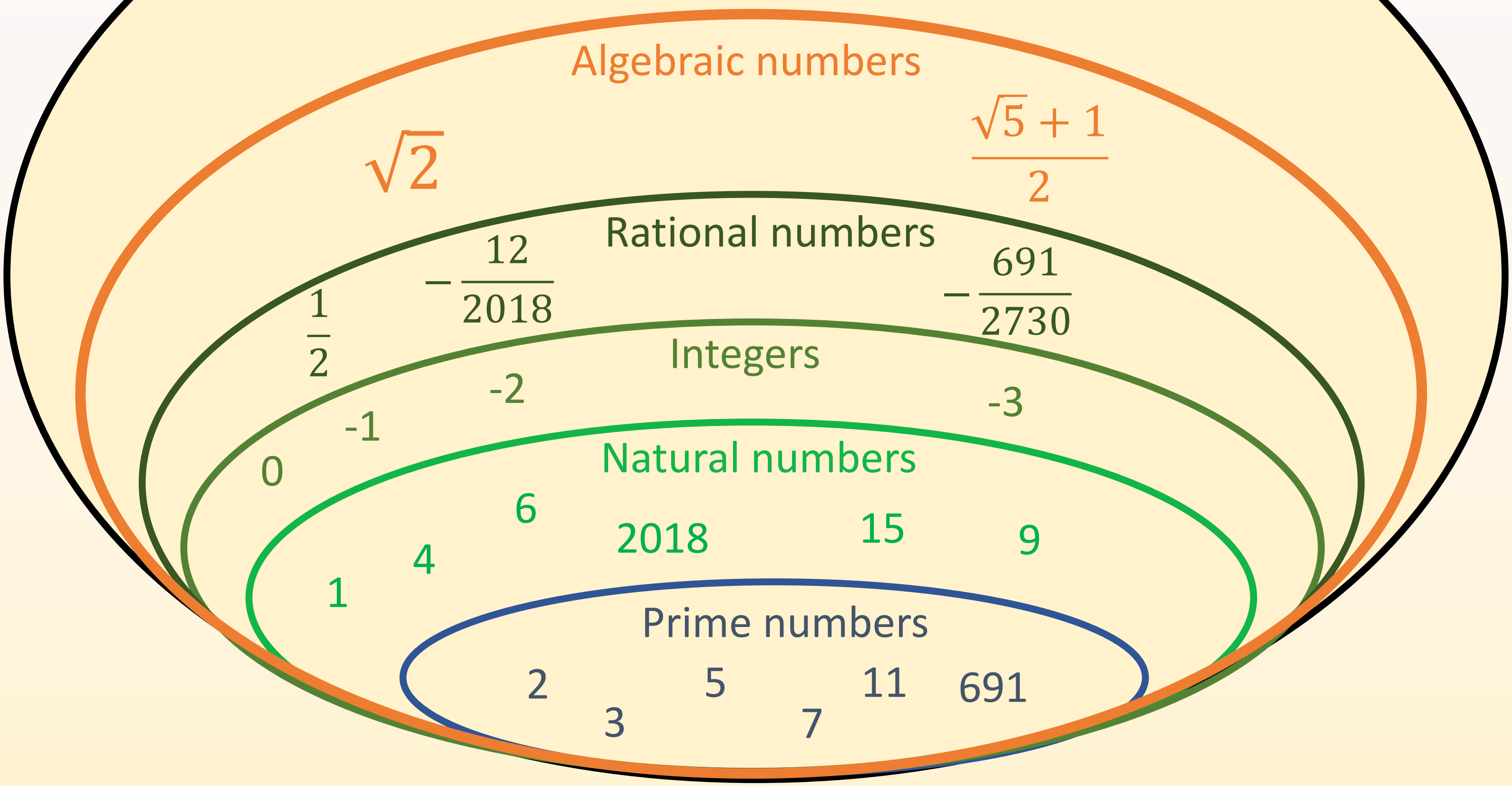
Algebraic numbers are given as solutions for X of polynomial equations:

$$A_n X^n + \ldots + A_2 X^2 + A_1 X + A_0 = 0$$

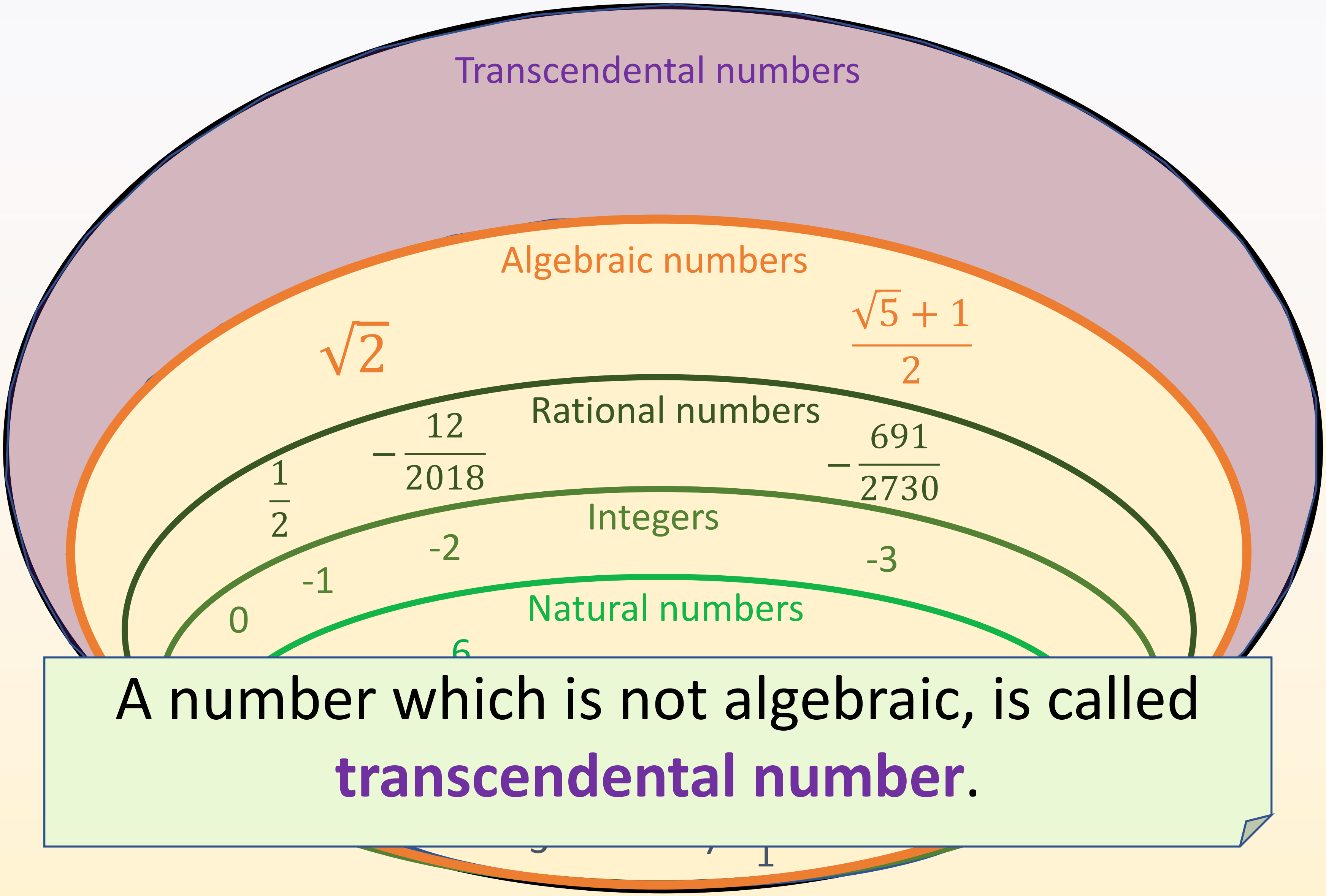
For example $X = \sqrt{2}$ is the solution of $1\,X^2 + 0\,X - 2 = 0$.

Classification of numbers … so far

- Every rational number is also an algebraic number.
- But not all algebraic numbers are rational!
- For example $\sqrt{2}$ can **not** be written as a fraction $\frac{a}{b}$.

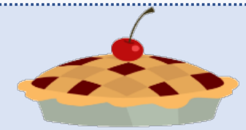


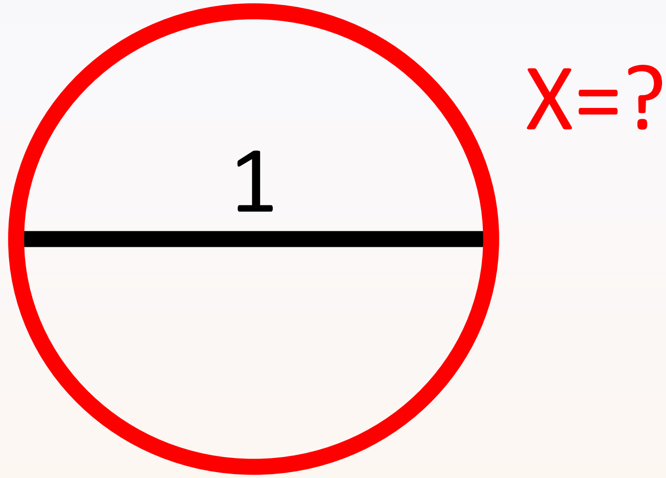
Transcendental numbers



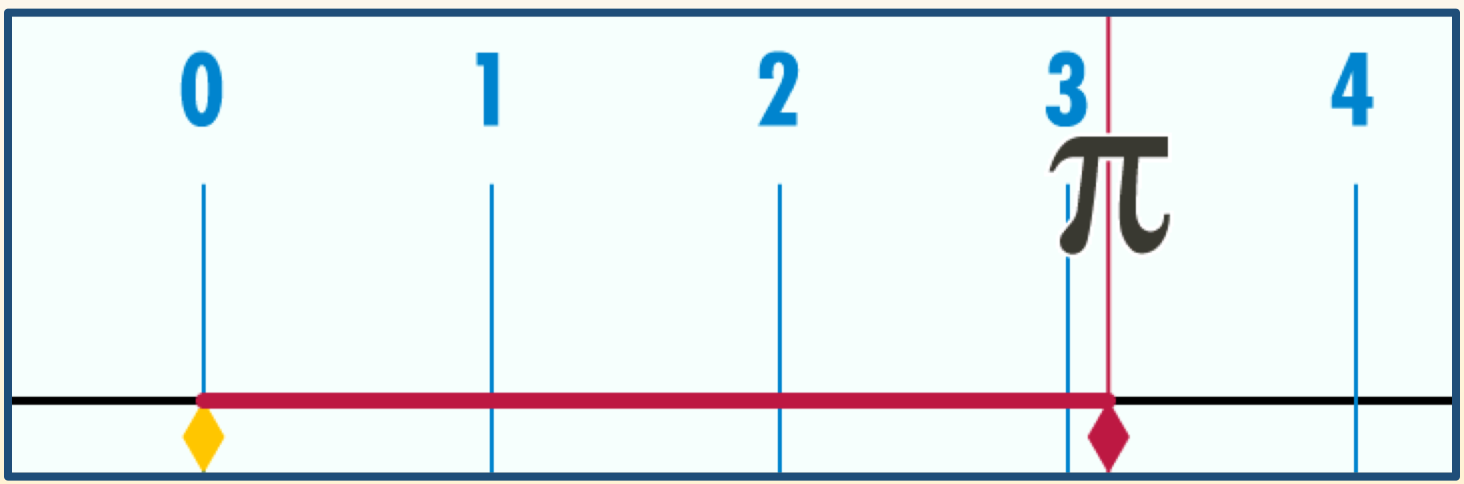
A number which is not algebraic, is called **transcendental number**.

π





$$\pi = 3.141592.....$$



Digits of Pi π

$$\pi = 3.14159265358979323846264338327$$
$$950288419716939937510\ldots\ldots\ldots$$
$$\ldots 595806461454780858841206201855$$
$$3557722052941993520\ldots\ldots.$$

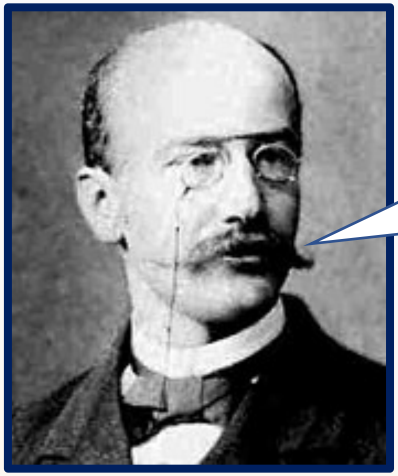
32688872-th position

Open problem 3: π **is normal**

“The decimal representation of π contains every finite combination of digits “

Pi-search page: http://www.angio.net/pi/

Pi π



π is transcendental!

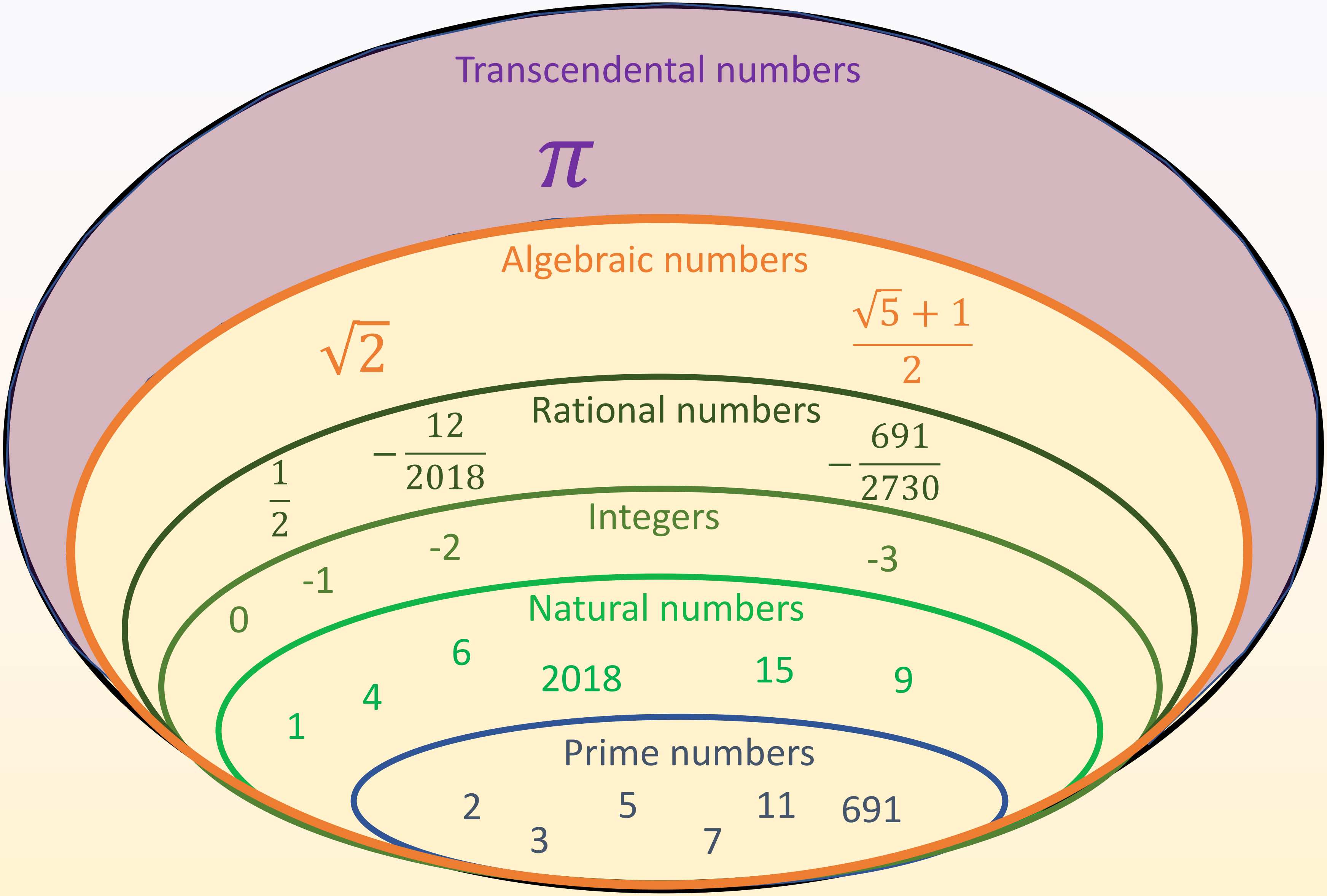
Ferdinand von Lindemann
(1852 – 1939)

This means you will **never** find integers $A_0, A_1, \ldots, A_n$ such that

$$A_n \pi^n + \ldots + A_2\pi^2 + A_1\pi + A_0 = 0$$

$$7\pi - 22 \neq 0$$
$$2\pi^2 - 6\pi - 1 \neq 0$$
$$\pi^3 - 22\pi^2 + 4\pi - 12 \neq 0$$
$$\ldots.$$

Classification of numbers … so far



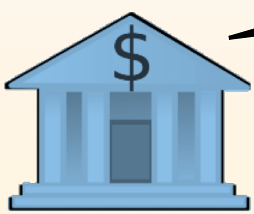
Nice banks

We offer you an interest rate of 100% each year!

Nice bank

After one year: $(1+1)=2$

We also offer you an interest rate of 100% each year.
But at our bank you get 50% after every 6 month!



Nicer bank

After 6 month: $\left(1+\frac{1}{2}\right)=1.5$

After one year: $\left(1+\frac{1}{2}\right)*\left(1+\frac{1}{2}\right)=2.25$

Pictures: clipartmax.com

Nice banks

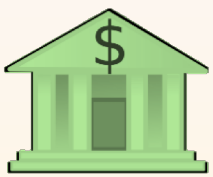
Nice bank

After one year: $(1+1)=2$

Nicer bank

After one year: $\left(1+\frac{1}{2}\right)^{2}=2.25$

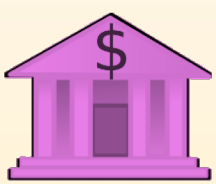
We apply the interest after every month!



Super nice bank

After one year: $\left(1+\frac{1}{12}\right)^{12}=2.613\ldots$

We apply the interest after every day!



Super super nice bank

After one year: $\left(1+\frac{1}{365}\right)^{365}=2.714\ldots$

"compounding frequency"

The nicest bank & Euler's number

The nicest bank applies the interest rate continuously ("infinitely often")

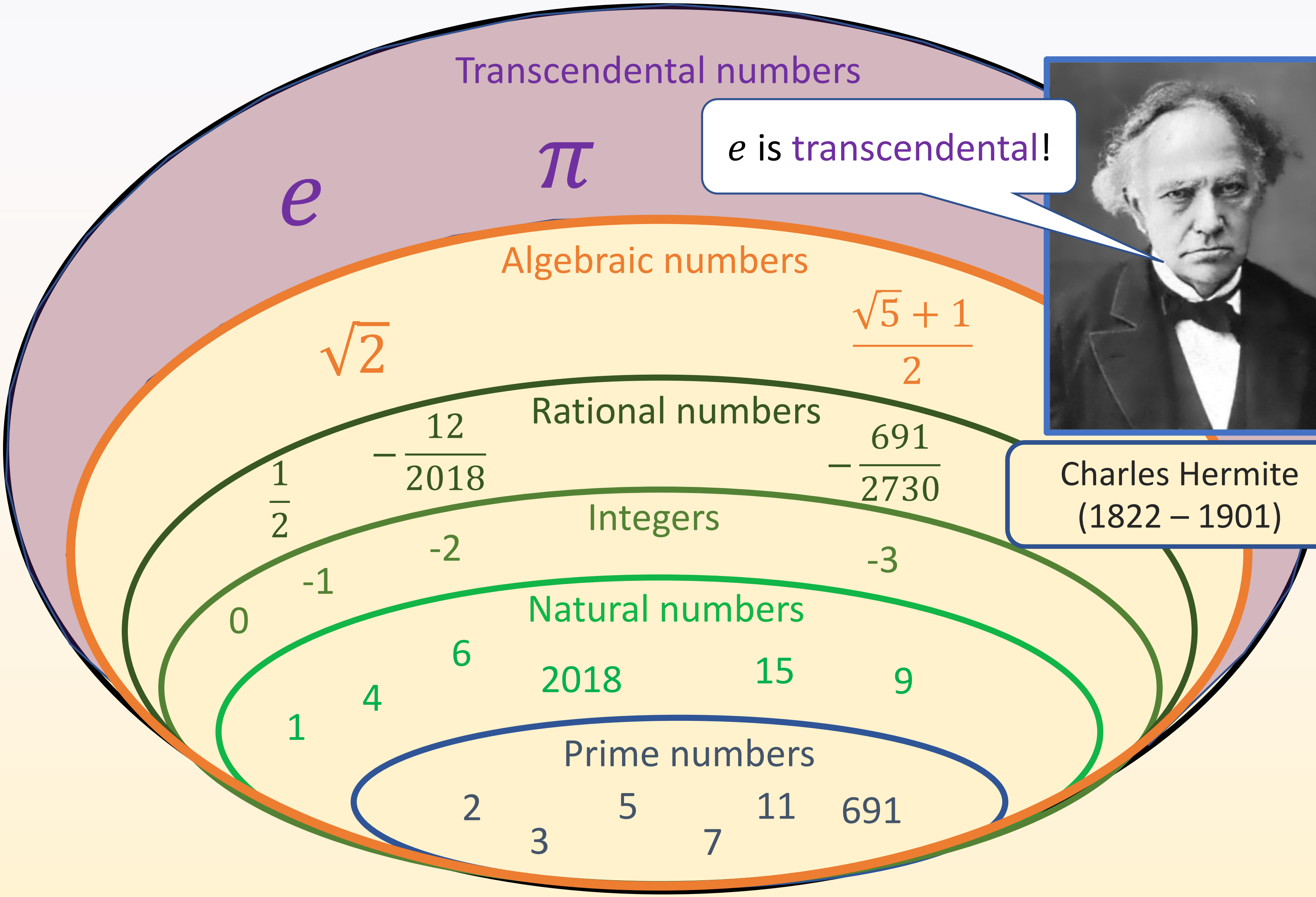
$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.71828182845\ldots$$

The number e is called **Euler's number**.

Leonhard Euler
(1707 – 1783)

Pictures: wikipedia.org

Classification of numbers … so far



Charles Hermite
(1822 – 1901)

Pictures: wikipedia.org

Infinite sums

Finite sums:

$1 = 1$

$1 + 2 = 3$

$1 + 2 + 3 = 6$

$1 + 2 + 3 + 4 = 10$

…

$1 + 2 + 3 + \ldots + 100 = 5050$

This sum gets bigger and bigger and therefore the infinite sum

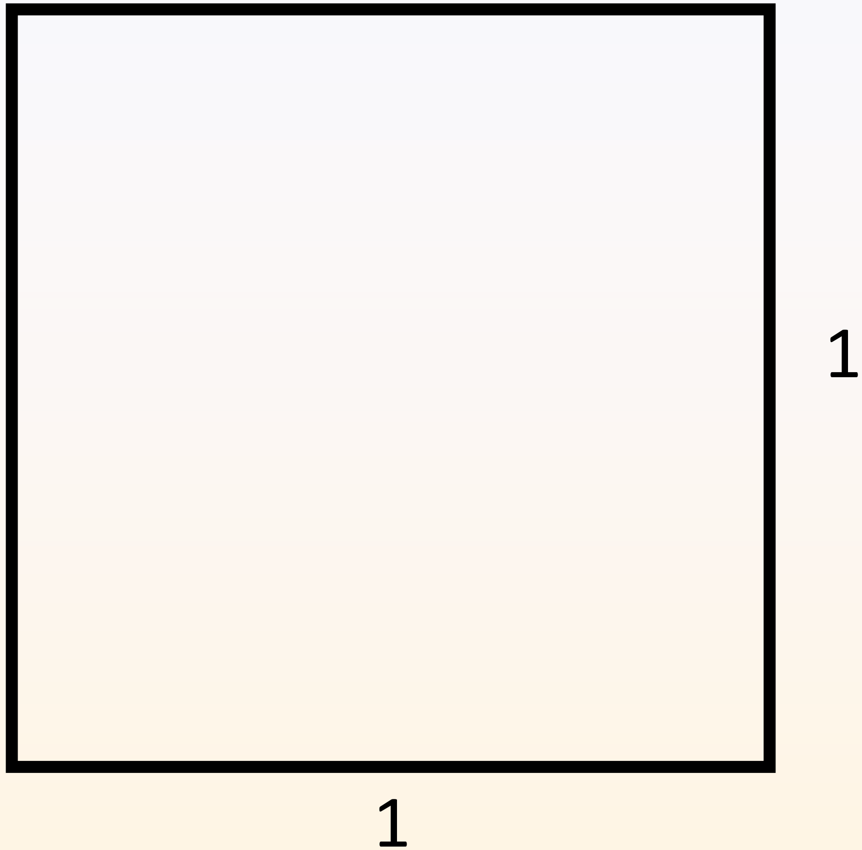
$$\cancel{1 + 2 + 3 + 4 + \ldots + 100 + \ldots + 43432423 + \ldots\ldots\ldots}$$

does not make sense.

But there are infinite sums which make sense!

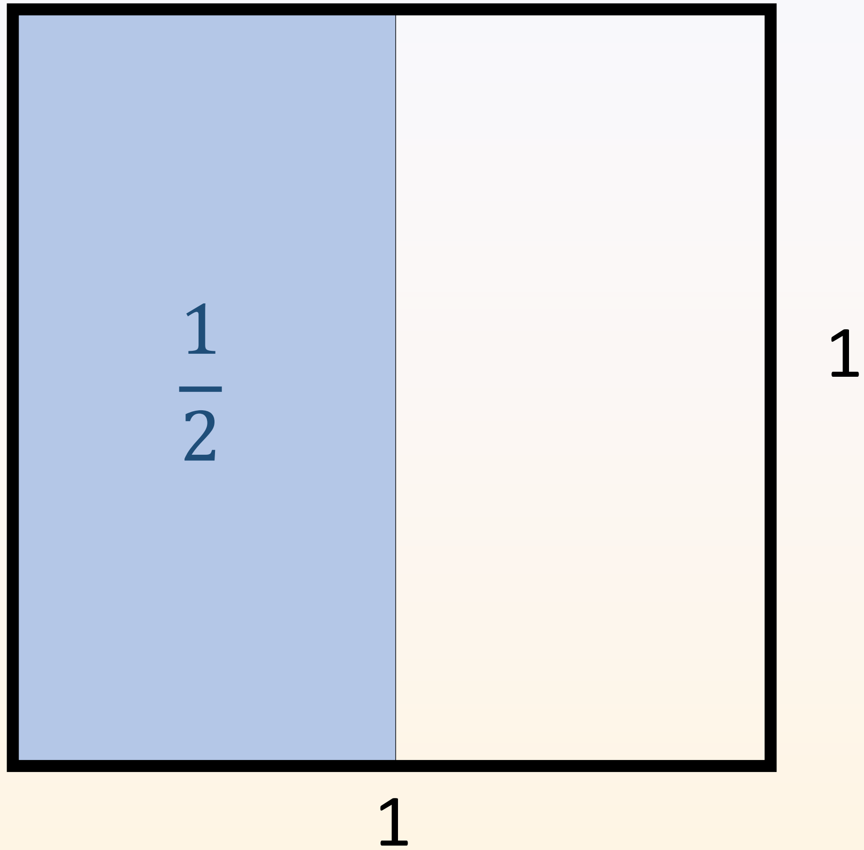
$$1 + 0.1 + 0.01 + 0.001 + 0.0001 + \ldots = 1.1111111\ldots$$

Infinite sums - Example



Area of □ = 1

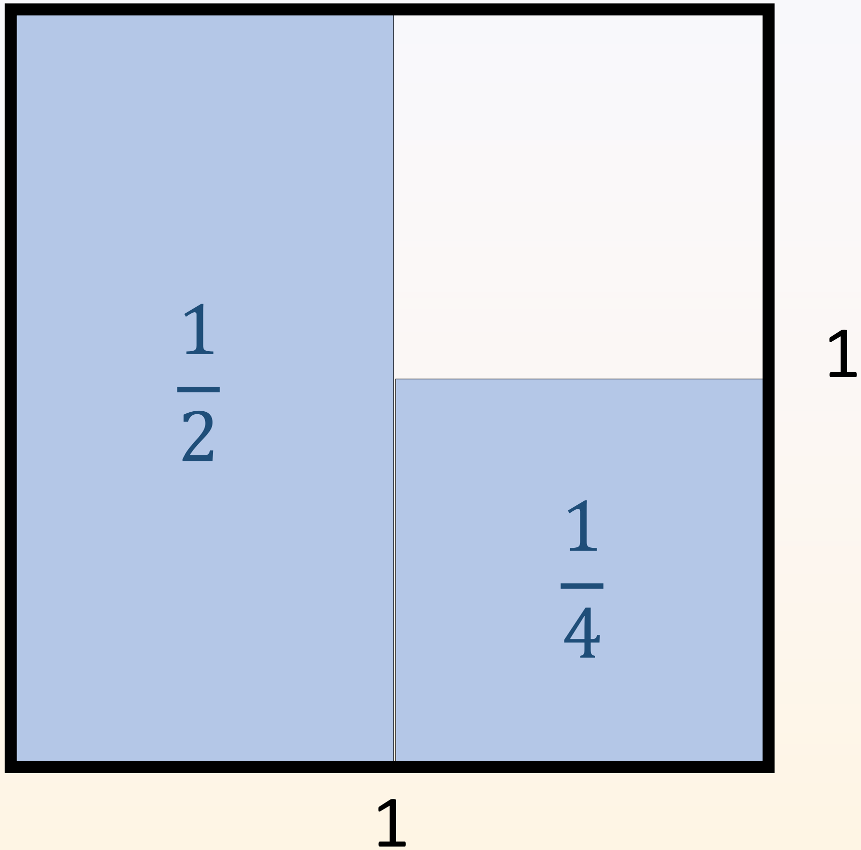
Infinite sums - Example



Area of $\square$ = 1

Area of blue part = $\frac{1}{2} = 0.5$

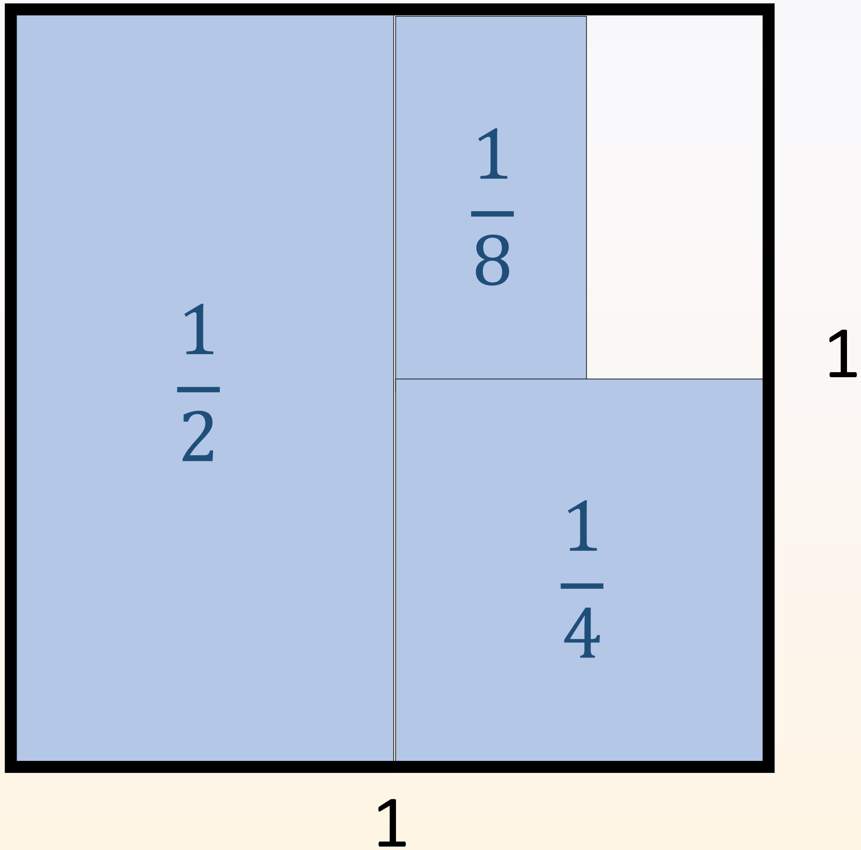
Infinite sums - Example



Area of $\square$ = 1

Area of blue part = $\frac{1}{2}+\frac{1}{4}=0.75$

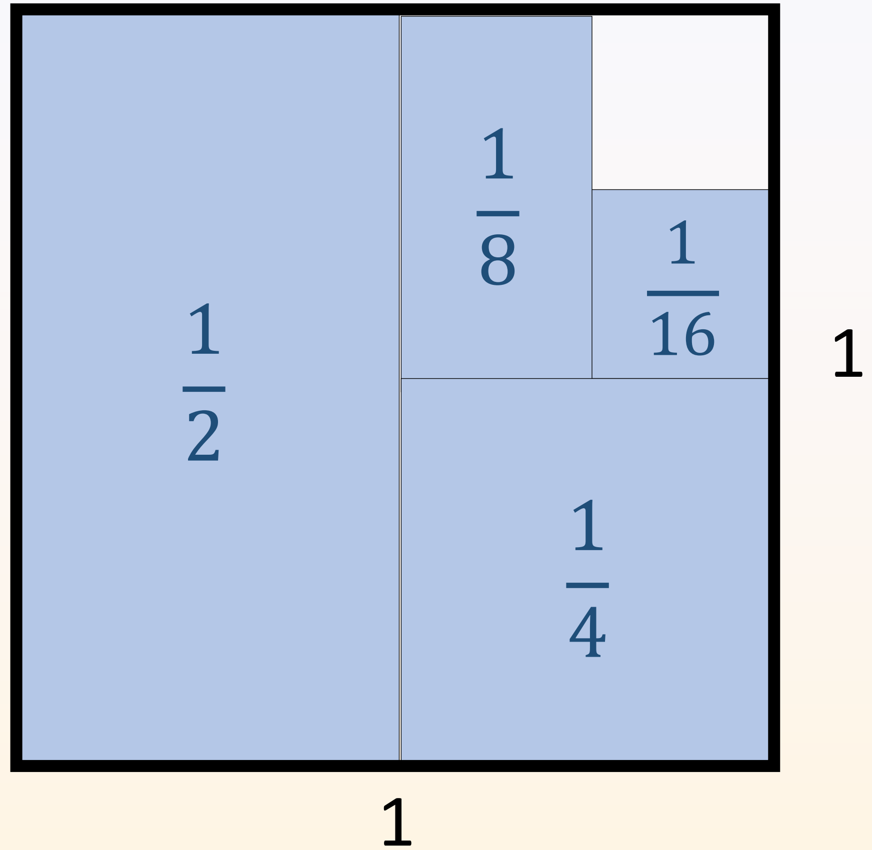
Infinite sums - Example



Area of $\square$ = 1

Area of blue part = $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=0.875$

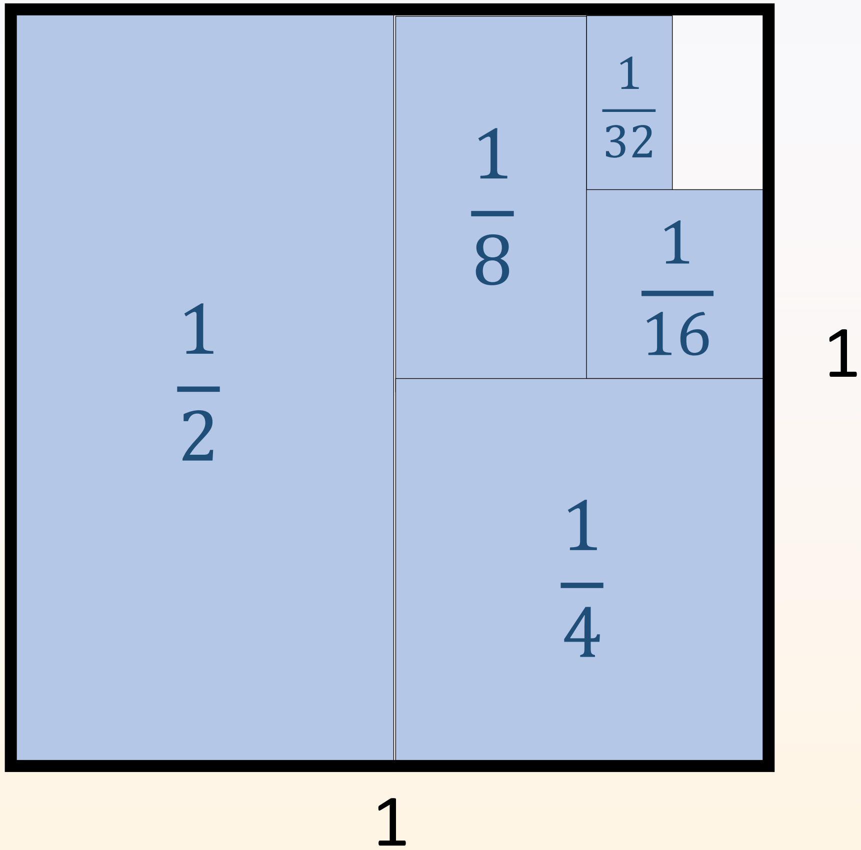
Infinite sums - Example



Area of □ = 1

Area of blue part = $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=0.93\ldots$

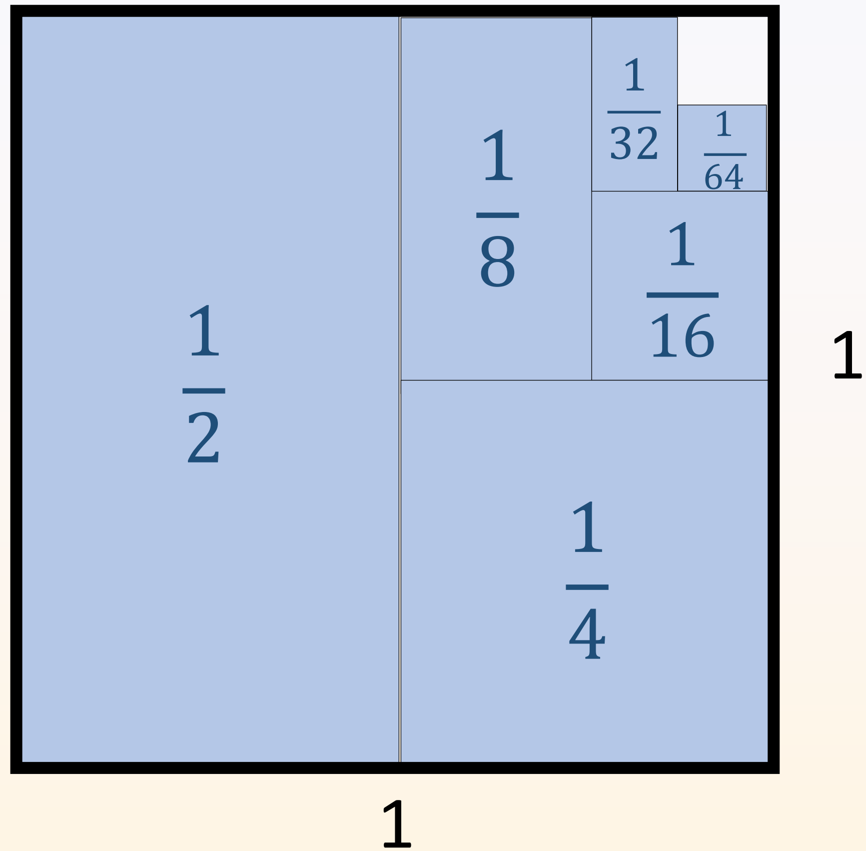
Infinite sums - Example



Area of $\square$ = 1

Area of blue part = $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}=0.96\ldots$

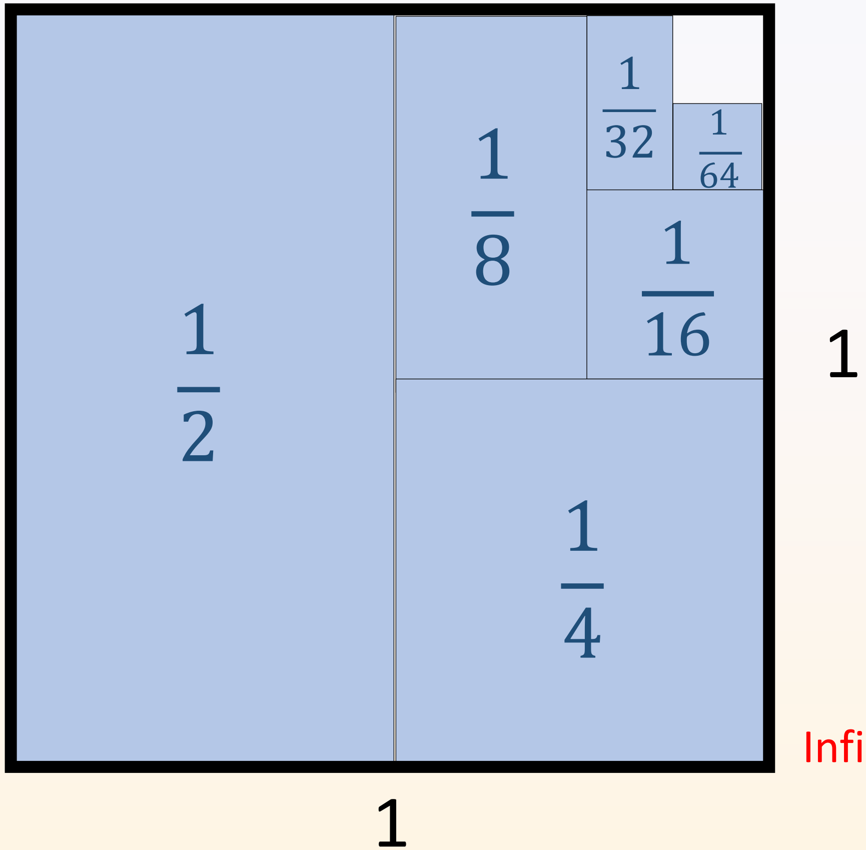
Infinite sums - Example



Area of $\square$ = 1

Area of blue part = $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}=0.98\ldots$

Infinite sums - Example



Infinite number of terms (no end)

$$\text{Area of } \square = 1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots$$

$$\text{Area of blue part} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} = 0.98\ldots$$

Geometric series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots=1$$

Mathematical notation:

$$\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots=1$$

For any natural number A greater than 1 we have

$$\sum_{n=1}^{\infty}\frac{1}{A^n}=\frac{1}{A-1}$$

Another infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots = 1$$

What happens if we change this sum a little bit?

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

$$= \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = ??$$

Another infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = ???$$

$$\frac{1}{1^2} = 1$$

$$\frac{1}{1^2} + \frac{1}{2^2} = 1 + \frac{1}{4} = 1.25$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} = 1 + \frac{1}{4} + \frac{1}{9} = 1.3611 \ldots$$

$$\frac{1}{1^2} + \cdots + \frac{1}{100^2} = 1 + \cdots + \frac{1}{10000} = 1.6349 \ldots$$

???

Another infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

But where is the circle??

Leonhard Euler
(1707 – 1783)

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^8} = \frac{\pi^8}{9450}$$

Pictures: wikipedia.org

Riemann zeta values

For any natural number k greater than 1 the numbers

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} = \frac{1}{1^k} + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots$$

are called **Riemann zeta values.**

Euler's formulas imply:
If k is even then $\zeta(k)$ is transcendental

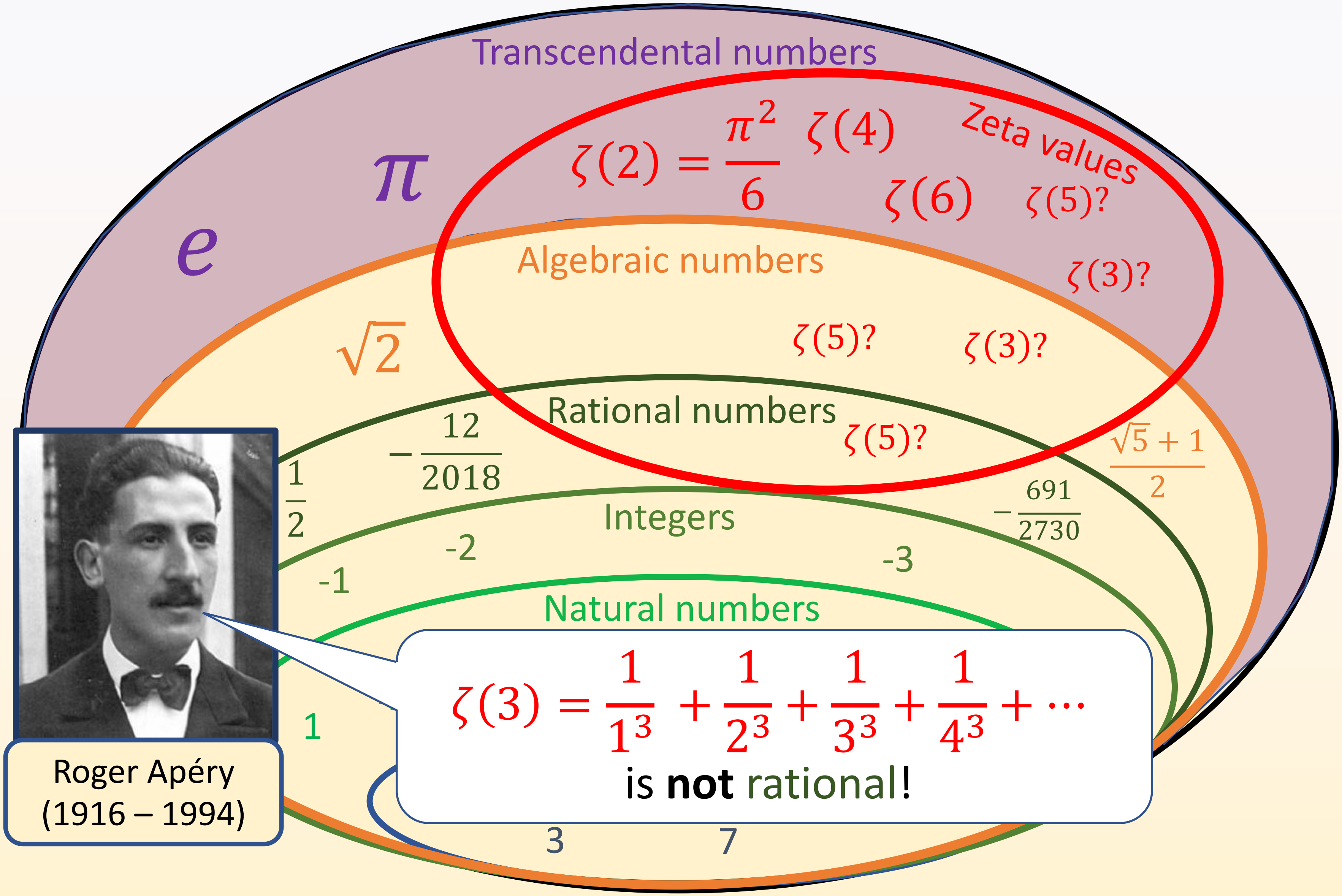
Bernhard Riemann
(1826 – 1866)

Open problem 4:

$\zeta(k)$ is transcendental for all natural numbers k greater than 1

Pictures: wikipedia.org

Classification of numbers



“All numbers”

So far by “all numbers” we talked about numbers which have a decimal representation, e.g. 5.323123…. .

These numbers are called **real numbers**.

Recall: Algebraic numbers

$$X^2 - 2 = 0 \qquad X = \sqrt{2}$$

?

?

$$X^2 + 2 = 0 \qquad X = \sqrt{-2}$$

?

?

Complex numbers

To be able to take the square root of negative numbers, one introduces an additional imaginary number i, which satisfies

$$i^2 = -1 \qquad i = \sqrt{-1} \quad \sqrt{-2} = \sqrt{2}i$$

The numbers of the form $a + b\,i$ are called **complex numbers**.

a, b : real numbers

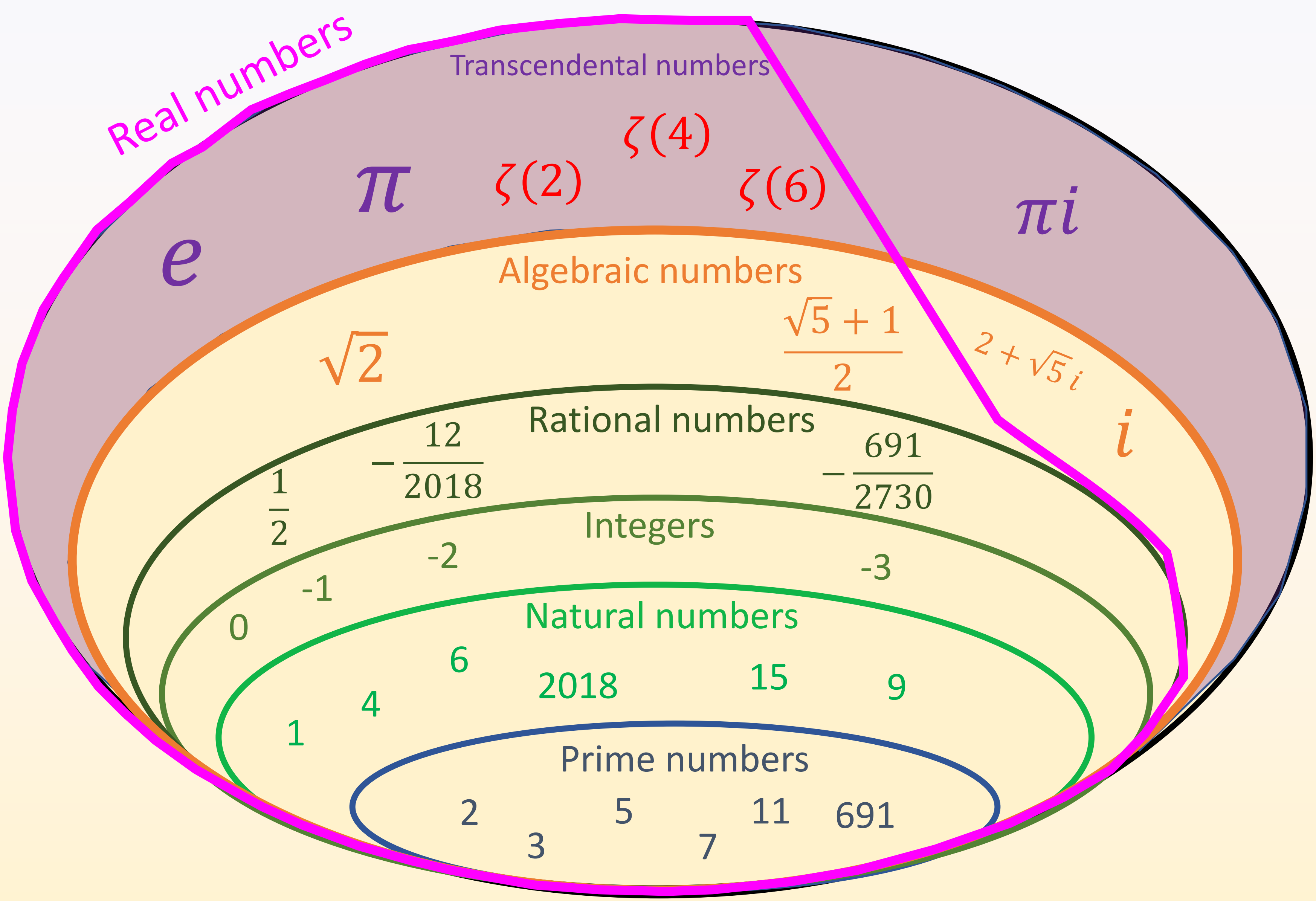
Complex numbers can be added and multiplied like real numbers

$$(1 + 2i) + (-2 + 3i) = (1 - 2) + (2 + 3)i = -1 + 5i$$

$$(1 + 2i) * (-2 + 3i) = -2 + 3i - 4i + 6 * \overbrace{i * i}^{-1}$$

$$= -8 - 1\,i$$

Classification of complex numbers



One more identity

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.71828182845\ldots$$

This can also be written as a nice infinite sum:

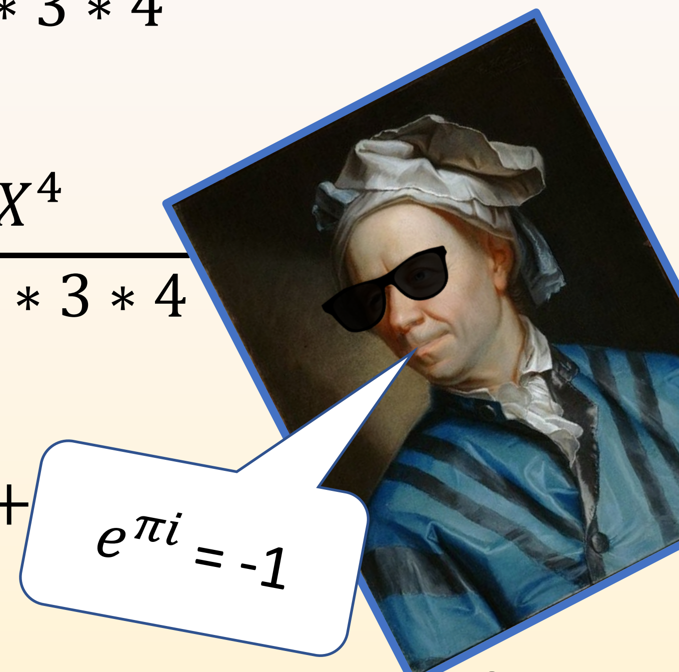
$$e = 1 + \frac{1}{1} + \frac{1}{1*2} + \frac{1}{1*2*3} + \frac{1}{1*2*3*4} + \cdots$$

More general for a complex number X:

$$e^X = 1 + \frac{X}{1} + \frac{X^2}{1*2} + \frac{X^3}{1*2*3} + \frac{X^4}{1*2*3*4}$$

$X = \pi i$:

$$e^{\pi i} = 1 + \frac{\pi i}{1} + \frac{(\pi i)^2}{1*2} + \frac{(\pi i)^3}{1*2*3} + \frac{(\pi i)^4}{1*2*3*4} +$$



Thank you very much for your attention!