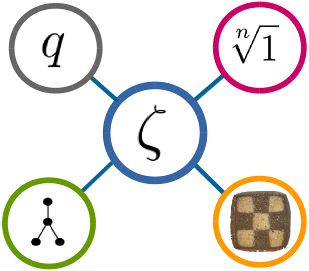
Various aspects of multiple zeta values

Henrik Bachmann



~~New~~ Old guests at the MPIM, 22.02.2018

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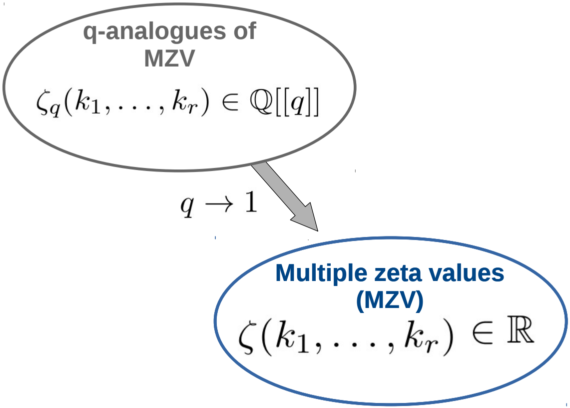
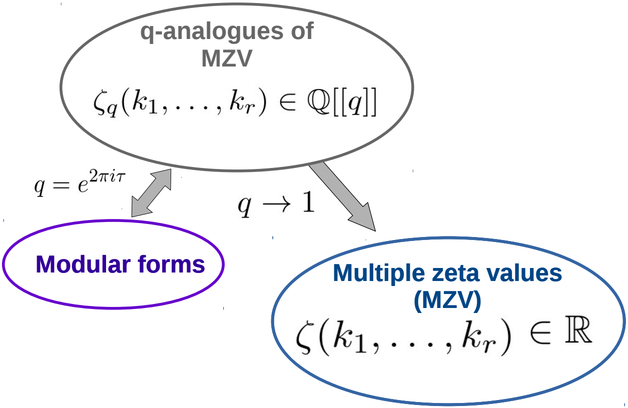
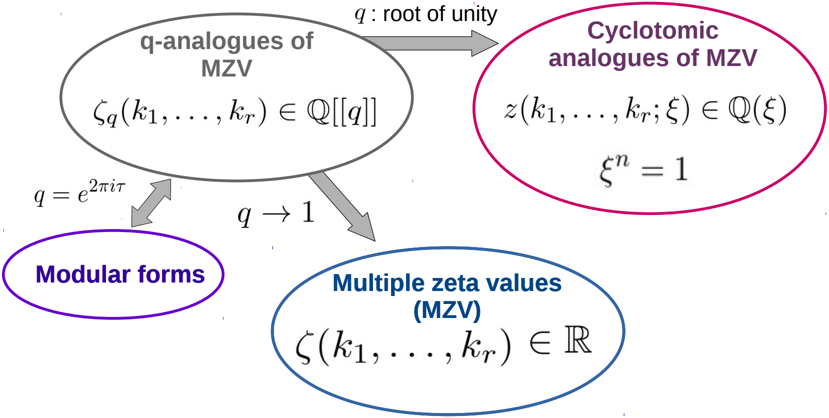
Multiple zeta values (MZV)

$$\zeta(k_1, \ldots, k_r) \in \mathbb{R}$$


q-analogues of MZV
$\zeta_q(k_1, \ldots, k_r) \in \mathbb{Q}[[q]]$
$q \to 1$
Multiple zeta values (MZV)
$\zeta(k_1, \ldots, k_r) \in \mathbb{R}$


q-analogues of MZV
$\zeta_q(k_1,\dots,k_r) \in \mathbb{Q}[[q]]$
$q = e^{2\pi i\tau}$
$q \to 1$
Modular forms
Multiple zeta values (MZV)
$\zeta(k_1,\dots,k_r) \in \mathbb{R}$

B., Y. Takeyama, K. Tasaka: arXiv:1707.05008



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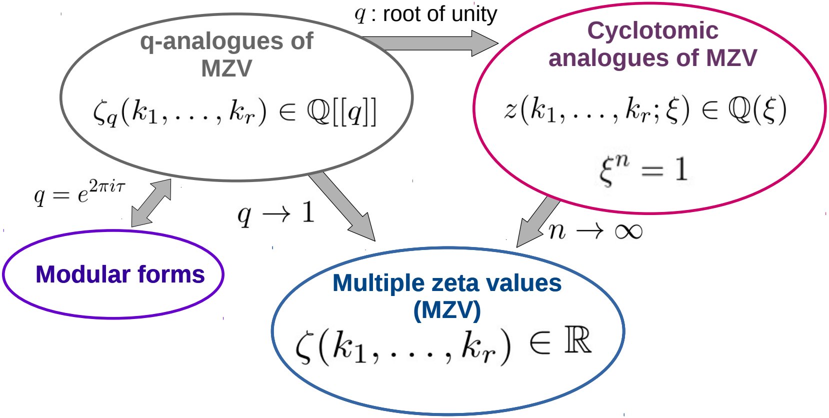

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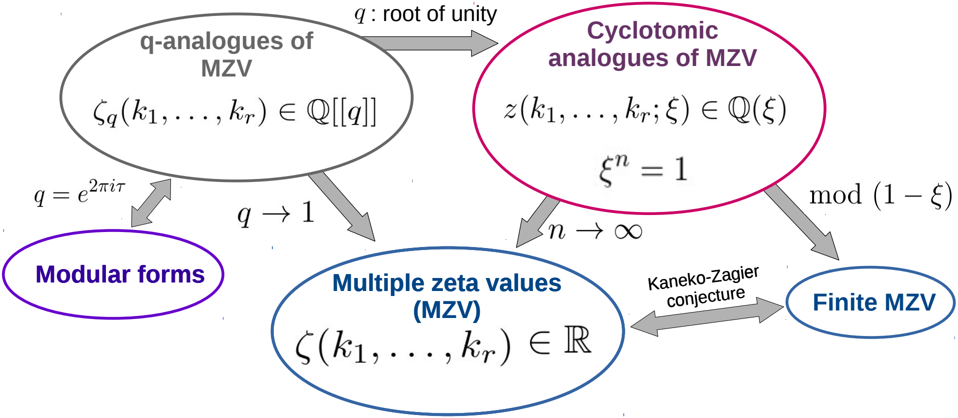

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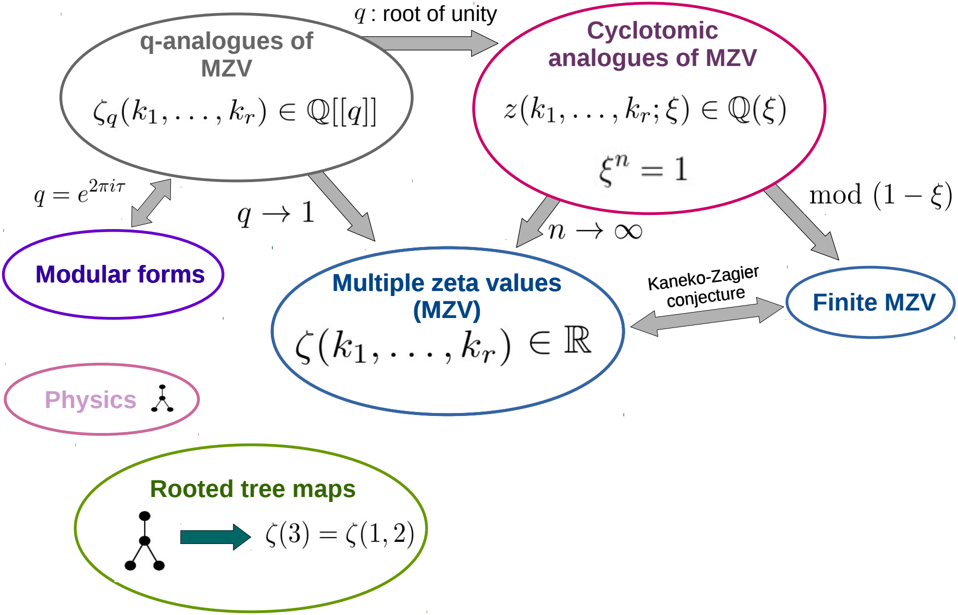
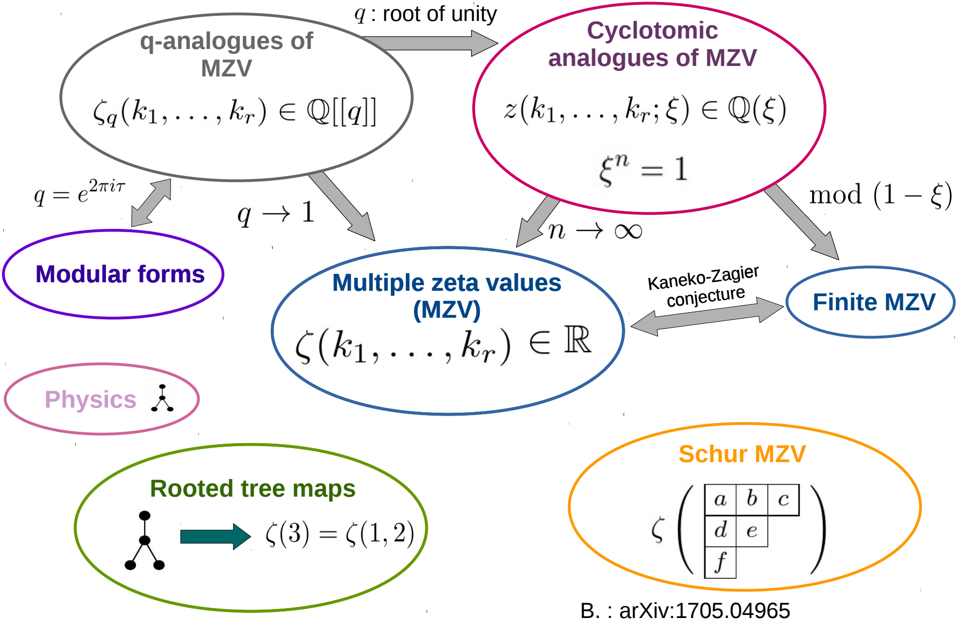


B., T. Tanaka: arXiv:1712.01601
B., T. Tanaka: arXiv:1801.05381

B., Y. Takeyama, **K. Tasaka**: arXiv:1707.05008



B., **T. Tanaka**: arXiv:1712.01601
B., **T. Tanaka**: arXiv:1801.05381

B. : arXiv:1705.04965
B., **Y. Yamasaki**: arXiv:1711.04746
B., **S. Charlton**: work in progress

Definition

For $k_1, \ldots, k_{r-1} \geq 1, k_r \geq 2$ define the **multiple zeta value** (MZV)

$$\zeta(k_1, \ldots, k_r) = \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}\,.$$

weight: $k_1 + \cdots + k_r$, **depth**: r.

- In the case $r = 1$ these are just the classical Riemann zeta values

$$\zeta(k) = \sum_{m>0} \frac{1}{m^k}\,, \qquad \zeta(2) = \frac{\pi^2}{6}\,, \qquad \zeta(3) \notin \mathbb{Q}\,, \qquad \zeta(4) = \frac{\pi^4}{90}\,, \ldots\,.$$

- MZV were first studied by Euler ($r = 2$) and for general depth they had their big comeback around 1990.
- These real numbers appear in various areas of mathematics and physics.

① MZV - $\zeta(k, \ldots, k)$

Eulers formula $\zeta(2) = \frac{\pi^2}{6}$ is a special case of

$$\zeta(\{2\}^n) := \zeta(\underbrace{2, ..., 2}_{n}) = \frac{\pi^{2n}}{(2n+1)!}\,,$$

which is an easy consequence of the product formular of the sine

$$\sum_{n=0}^{\infty} \zeta(\{2\}^n) T^{2n+1} = T \prod_{m=0}^{\infty} \left(1 + \frac{T^2}{m^2}\right) = \frac{\sin(\pi i T)}{\pi i} = \sum_{n=0}^{\infty} \frac{\pi^{2n}}{(2n+1)!} T^{2n+1}\,.$$

Proposition

For all $k \geq 2$ we have

$$\zeta(k, \ldots, k) \in \mathbb{Q}[\zeta(k \cdot m) \mid m \geq 1]\,,$$

i.e. in particular $\zeta(2k, \ldots, 2k) \in \mathbb{Q}[\pi^2]$.

Theorem (Borwein-Bradley-Broadhurst-Lisonek)

For all $n \geq 1$ we have

$$\zeta(\{1,3\}^n) = \frac{2\pi^{4n}}{(4n+2)!} = \frac{1}{4^n}\zeta(\{4\}^n)\,.$$

Theorem (Bowman-Bradley)

For all $n \geq 1$ we have

$$\zeta(3, \{1,3\}^n) = \frac{1}{4^n}\sum_{k=0}^{n}(-1)^k \zeta(4k+3)\zeta(\{4\}^{n-k})\,.$$

Definition

For $k_1, \ldots, k_{r-1} \geq 1, k_r \geq 2$ define the **multiple zeta-star value** (MZSV)

$$\zeta^\star(k_1, \ldots, k_r) = \sum_{0 < m_1 \leq \cdots \leq m_r} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}\,.$$

Theorem (Muneta)

For all $n \geq 1$ we have

$$\zeta^\star(\{1,3\}^n) = \text{complicated but explicit coefficient} \cdot \pi^{4n} \in \mathbb{Q}\pi^{4n}\,.$$

Goal

- Introduce Schur multiple zeta values as a generalization of MZV and MZSV.
- Show 13-formulas for Schur multiple zeta values.

② Schur MZV - Definition

Definition by example I

For $a \geq 1$ and $b, c \geq 2$ we define the **Schur multiple zeta value**

$$\zeta\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & \\ \cline{1-1}\end{array}\right) = \sum_{\substack{m_a \leq m_b \\ \wedge \\ m_c}} \frac{1}{m_a^a \cdot m_b^b \cdot m_c^c}.$$

Definition by example II

For $a, b, d \geq 1$ and $c, e, f \geq 2$ we define the **Schur multiple zeta value**

$$\zeta\left(\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & \\ \cline{1-2} f & & \\ \cline{1-1}\end{array}\right) = \sum_{\substack{m_a \leq m_b \leq m_c \\ \wedge \quad\;\; \wedge \\ m_d \leq m_e \\ \wedge \\ m_f}} \frac{1}{m_a^a \cdot m_b^b \cdot m_c^c \cdot m_d^d \cdot m_e^e \cdot m_f^f}.$$

Definition by example III (Skew type)

② Schur MZV - 13-Stairs

With the notion of Schur MZV the first 13-identity reads

$$\zeta\left(\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \vdots \\ \hline 1 \\ \hline 3 \\ \hline \end{array}\right) = \frac{1}{4^n}\zeta(\{4\}^n)\,,$$

where the coloring is just for optical reasons and n is the number of $\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array}$.

Theorem (B.-Yamasaki)

For any $n \geq 1$ we have

$$\zeta\left(\begin{array}{ccc} \cline{3-3} & & \multicolumn{1}{|c|}{1} \\ \cline{2-3} & \multicolumn{1}{|c|}{\iddots} & \multicolumn{1}{c|}{3} \\ \cline{1-3} \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{\iddots} & \\ \cline{1-2} \multicolumn{1}{|c|}{3} & & \\ \cline{1-1} \end{array}\right) = \frac{1}{4^n}\zeta^\star(\{4\}^n)\,.$$

n: number of $\begin{ytableau} 1 \\ 3 \end{ytableau}$.

Theorem (B.-Yamasaki)

For any $n \geq 1$ we have

$$\zeta\left(\begin{ytableau} \none & \none & \none & 1 \\ \none & \none & \iddots & 3 \\ \none & 1 & \iddots \\ 1 & 3 \end{ytableau}\right) = \frac{2}{4^n}\zeta(4n+1), \qquad \zeta\left(\begin{ytableau} \none & \none & 1 & 3 \\ \none & \iddots & 3 \\ 1 & \iddots \\ 3 \end{ytableau}\right) = \frac{1}{4^n}\zeta(4n+3).$$

Example

$$\zeta\left(\begin{ytableau} \none & 1 & 3 \\ 1 & 3 \\ 3 \end{ytableau}\right) = \sum_{\substack{ b_2 \leq a_3 \\ \wedge \\ b_1 \leq a_2 \\ \wedge \\ a_1 }} \frac{1}{(a_1 a_2 a_3)^3\, b_1 b_2} = \frac{1}{16}\zeta(11).$$

Question: Elementary proof for this?

② Schur MZV - Full 13-Stairs

Using the Theorem for stairs we can write any "full" stair as (Hankel) determinants in odd single zeta values.

Example

$$\zeta\left(\begin{array}{|c|c|c|}\hline 3 & 1 & 3 \\ \hline 1 & 3 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}\right) = \frac{1}{4^2}\begin{vmatrix}\zeta(3) & \zeta(7) \\ \zeta(7) & \zeta(11)\end{vmatrix},$$

$$\zeta\left(\begin{array}{|c|c|c|c|c|}\hline 3 & 1 & 3 & 1 & 3 \\ \hline 1 & 3 & 1 & 3 \\ \cline{1-4} 3 & 1 & 3 \\ \cline{1-3} 1 & 3 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}\right) = \frac{1}{4^6}\begin{vmatrix}\zeta(3) & \zeta(7) & \zeta(11) \\ \zeta(7) & \zeta(11) & \zeta(15) \\ \zeta(11) & \zeta(15) & \zeta(19)\end{vmatrix}.$$

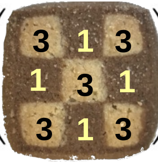
Theorem (B.-Charlton)

Every Schur MZV with alternating entries in 1 and 3 is an element in $\mathbb{Q}[\pi^4, \zeta(3), \zeta(5), \zeta(7), \dots]$.

- We can give explicit formulas for a lot of shapes as determinants in odd zeta values and powers of π^4.
- Have also results for arbitrary numbers a and b instead of 1 and 3.

Example For all $k \geq 2$ we have

$$\zeta\left(\begin{array}{|c|c|}\hline k & 1 \\ \hline 1 & k \\ \hline\end{array}\right) = \begin{vmatrix} \zeta\left(\begin{array}{|c|}\hline k \\ \hline\end{array}\right) & \zeta\left(\begin{array}{|c|}\hline 1 \\ \hline k \\ \hline\end{array}\right) \\ \zeta\left(\begin{array}{|c|c|}\hline 1 & k \\ \hline\end{array}\right) & \zeta\left(\begin{array}{c|c|}\cline{2-2} & 1 \\ \hline \multicolumn{1}{|c|}{1} & k \\ \hline\end{array}\right) \end{vmatrix}.$$



$$\zeta\begin{pmatrix}3 & 1 & 3\\ 1 & 3 & 1\\ 3 & 1 & 3\end{pmatrix} = \frac{1}{32}\begin{vmatrix}\zeta(3) & \frac{\pi^4}{180} & \zeta(7)\\ \frac{\pi^4}{72} & \zeta(5) & \frac{17\pi^8}{90720}\\ \zeta(7) & \frac{13\pi^8}{226800} & \zeta(11)\end{vmatrix}$$

Thank you for your attention!

Slides can be found on my homepage
www.henrikbachmann.com

③ Bonus - More 13-Stuff

$$\zeta\left(\begin{array}{cccc} & & 1 & 3 \\ & \iddots & \iddots & \\ 1 & 3 & & \end{array}\right) = \sum_{k=0}^{n} \frac{1}{4^k} \zeta^\star(\{4\}^k)\zeta(\{4\}^{n-k})\,,$$

$$\zeta\left(\begin{array}{cccc} 1 & 3 & 1 & 3 \\ 3 & 1 & 3 & \\ 1 & 3 & & \\ 3 & & & \end{array}\right) = \frac{1}{4^4}\begin{vmatrix} \zeta(7) & \zeta(11) \\ \zeta(11) & \zeta(15) \end{vmatrix}\,,$$

$$\zeta\left(\begin{array}{ccccc} & & 3 & 1 & 3 \\ & 3 & 1 & 3 & \\ 3 & 1 & 3 & & \\ 1 & 3 & & & \\ 3 & & & & \end{array}\right) = -\frac{1}{4^4}\begin{vmatrix} 0 & 0 & \zeta(3) & \zeta(7) \\ 0 & \zeta(3) & \zeta(7) & \zeta(11) \\ \zeta(3) & \zeta(7) & \zeta(11) & \zeta(15) \\ \zeta(7) & \zeta(11) & \zeta(15) & \zeta(19) \end{vmatrix}\,.$$

③ Bonus - 12-Stairs

We have

$$\zeta\left(\begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & \iddots & 2 \\ \hline & 1 & \iddots & \\ \hline 1 & 2 & & \\ \hline \end{array}\right) = 3\zeta(3n+1)$$

but in general it is

$$\zeta\left(\begin{array}{|c|c|c|c|} \hline & & 1 & 2 \\ \hline & \iddots & 2 & \\ \hline 1 & \iddots & & \\ \hline 2 & & & \\ \hline \end{array}\right) \notin \mathbb{Q}[\zeta(k) \mid k \geq 2]\,.$$

Also easy to check:

$$\zeta\left(\begin{array}{|c|c|c|} \hline & & 1 \\ \hline & \iddots & 2 \\ \hline 1 & \iddots & \\ \hline 2 & & \\ \hline \end{array}\right) = \zeta^{\star}(\{3\}^{n})\,.$$