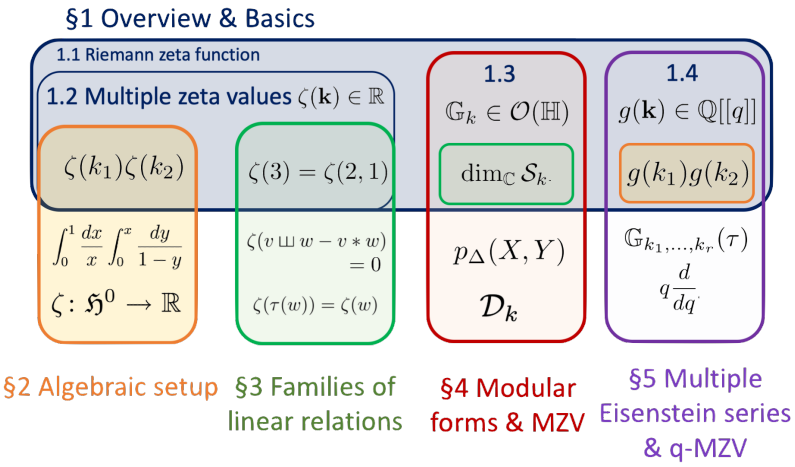
Multiple zeta values & modular forms

Lecture 8: **Extended double shuffle relations,** 5th June 2020



Review:

Corollary 2.32. *We have*

i) $\mathfrak{H}^1_{\sqcup\!\sqcup} = \mathfrak{H}^0_{\sqcup\!\sqcup}[y]$ *and* $\mathfrak{H}_{\sqcup\!\sqcup} = \mathfrak{H}^1_{\sqcup\!\sqcup}[x] = \mathfrak{H}^0_{\sqcup\!\sqcup}[x,y]$.

ii) $\mathfrak{H}^1_* = \mathfrak{H}^0_*[z_1]$.

iii) $\mathfrak{H}^1_{\hat{*}} = \mathfrak{H}^0_{\hat{*}}[z_1]$.

Example:

$$z_1z_1z_2 = \frac{1}{2}z_2 \sqcup\!\sqcup z_1^{\sqcup\!\sqcup 2} - 2z_2z_1 \sqcup\!\sqcup z_1 + 3z_2z_1z_1\,,$$

$$z_1z_1z_2 = \frac{1}{2}z_2 * z_1^{*2} - (z_2z_1 + z_3) * z_1 + \left(z_2z_1z_1 + z_3z_1 + \frac{1}{2}z_4\right)\,,$$

$$z_1z_1z_2 = \frac{1}{2}z_2 \,\hat{*}\, z_1^{\hat{*}2} - (z_2z_1 + z_3 - z_2) \,\hat{*}\, z_1 + \left(z_2z_1z_1 + z_3z_1 + \frac{1}{2}z_4 - z_3 - \frac{3}{2}z_2z_1 + \frac{5}{12}z_2\right)\,.$$

$\bullet \in \{\sqcup, *\}$

$$\begin{array}{ccc} \mathfrak{H}_\bullet^1 & \xrightarrow{\mathrm{reg}_\bullet^T} & \mathfrak{H}_\bullet^0[T] \\ & \searrow^{\zeta^\bullet} & \downarrow \zeta \\ & & \mathcal{Z}[T] \end{array}$$

$$\mathrm{reg}_\bullet^T\Big(\sum_{j=0}^{m} w_j \bullet z_1^{\bullet j}\Big) = \sum_{j=0}^{m} w_j \overline{T}^{\,j}, \quad w_j \in \mathfrak{h}^0$$

shuffle regularized multiple zeta value

$$\zeta^{\sqcup}(\mathbf{k};T) = \zeta^{\sqcup}(k_1,\dots,k_r;T) := \zeta^{\sqcup}(z_{\mathbf{k}}) \in \mathcal{Z}[T]$$

stuffle regularized multiple zeta value

$$\zeta^{*}(\mathbf{k};T) = \zeta^{*}(k_1,\dots,k_r;T) := \zeta^{*}(z_{\mathbf{k}}) \in \mathcal{Z}[T]$$

Example:

$$\zeta^{\sqcup}(1,1,2;T) = \frac{1}{2}\zeta(2)T^2 - 2\zeta(2,1)T + 3\zeta(2,1,1)\,,$$
$$\zeta^{*}(1,1,2;T) = \frac{1}{2}\zeta(2)T^2 - (\zeta(2,1)+\zeta(3))T + \zeta(2,1,1) + \zeta(3,1) + \frac{1}{2}\zeta(4)\,.$$

$\zeta^{\sqcup}$ and ζ^{*} are different.

$\zeta(2,1) = \zeta(3)$ } "duality"
$\zeta(2,1,1) = \zeta(4)$ }

$\zeta(3,1) = \frac{1}{4}\zeta(4)$ finite dsh.

$$\zeta^{\sqcup}(1,1,2;T) - \zeta^{*}(1,1,2;T) = \frac{5}{4}\zeta(2,1) = \frac{5}{4}\zeta(4) = \frac{1}{2}\zeta(2)^2$$

$\frac{5}{4}\zeta(2,1) = 3 - 1 - \frac{1}{4} - \frac{1}{2}$

$\frac{5}{4}\zeta(4) = \frac{1}{2}\zeta(2)^2$ ← Euler

Comparison of $\zeta^{ш}$ and ζ^*

$$A(u) = \exp\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n}\zeta(n)u^n\right) = \Gamma(1+u)e^{\gamma u} \quad |u|<1$$

$$= 1 + \frac{\zeta(2)}{2}u^2 - \frac{\zeta(3)}{3}u^3 + \left(\frac{\zeta(4)}{4} + \frac{\zeta(2)^2}{8}\right)u^4 - \left(\frac{\zeta(5)}{5} + \frac{\zeta(2)\zeta(3)}{6}\right)u^5 + \dots$$

$$=: \sum_{k\geq 0} \gamma_k u^k .$$

$\zeta(n) = \zeta(\overbrace{1+1+\dots+1}^{n})$

$\zeta(z_1^{\circ n})$

Define the $\mathbb{R}$-linear map. $\rho : \mathbb{R}[T] \longrightarrow \mathbb{R}[T]$

$$\rho(e^{Tu}) = A(u)e^{Tu} = \dots \quad u^n$$

$$\rho(e^{Tu}) = \rho\left(1 + Tu + T^2\frac{u^2}{2} + \dots\right) = \rho(1) + \rho(T)u + \rho(T^2)\frac{u^2}{2} + \dots$$

$$\rho(T^m) = m!\sum^{m} \gamma_k \frac{T^{m-k}}{(m-k)!} .$$

$$\begin{aligned}
\rho(1) &= 1 , \\
\rho(T) &= T , \\
\rho(T^2) &= T^2 + \zeta(2) , \\
\rho(T^3) &= T^3 + 3\zeta(2)T - 2\zeta(3) , \\
\rho(T^4) &= T^4 + 6\zeta(2)T^2 - 8\zeta(3)T + 6\zeta(4) + 3\zeta(2)^2 , \\
\rho(T^5) &= T^5 + 10\zeta(2)T^3 - 20\zeta(3)T^2 + \left(30\zeta(4) + 15\zeta(2)^2\right)T - 24\zeta(5) - 20\zeta(2)\zeta(3)
\end{aligned}$$

$$\zeta^*(1,1,2;T) = \frac{1}{2}\zeta(2)T^2 - (\zeta(2,1) + \zeta(3))T + \zeta(2,1,1) + \zeta(3,1) + \frac{1}{2}\zeta(4)$$

$$\begin{aligned}
\rho\left(\zeta^*(1,1,2;T)\right) &= \frac{1}{2}\zeta(2)T^2 - 2\zeta(2,1)T + 3\zeta(2,1,1) \\
&= \zeta^{ш}(1,1,2;T) ,
\end{aligned}$$

IKZ

Theorem 2.35. *For all* $\mathbf{k} \in \mathbb{Z}^r_{\geq 1}$ *we have*

$$\zeta^{ш}(\mathbf{k};T) = \rho\left(\zeta^*(\mathbf{k};T)\right) .$$

Or equivalently, when viewed as maps from $\mathfrak{H}^1$ *to* $\mathbb{R}[T]$, *we have* $\zeta^{ш} = \rho \circ \zeta^*$.

Euler-Mascheroni const.

$$\zeta_M(1) = 1 + \frac{1}{2} + \cdots + \frac{1}{M-1} = \log(M) + \gamma + \mathrm{O}\left(\frac{1}{M}\right) \qquad M \to \infty$$

$$\zeta_M(\mathbf{k}) = \zeta^*(\mathbf{k}; \log(M) + \gamma) + \mathrm{O}(M^{-1}\log^J(M)) \qquad (\text{as } M \to \infty)$$

$$\mathrm{Li}_1(z) = \log\left(\tfrac{1}{1-z}\right) = \sum_{m>0} \frac{z^m}{m}$$

k adm.

$$\mathrm{Li}_k(1) - \mathrm{Li}_k(z) = O(1-z)$$

$$\mathrm{Li}_{\mathbf{k}}(z) = \zeta^{\sqcup\!\sqcup}\left(\mathbf{k}, \log\left(\frac{1}{1-z}\right)\right) + \mathrm{O}\left((1-z)\log^J\left(\frac{1}{1-z}\right)\right)$$

$z \nearrow 1$

$$\mathrm{Li}_{\mathbf{k}}(z) = \mathrm{Li}_{k_1,\ldots,k_r}(z) = \sum_{m_1>\cdots>m_r>0} \frac{z^{m_1}}{m_1^{k_1}\cdots m_r^{k_r}} = \sum_{m=1}^{\infty}\left(\sum_{m>m_2>\cdots>m_r>0} \frac{1}{m^{k_1} m_2^{k_2}\cdots m_r^{k_r}}\right) z^m$$
$$= \sum_{m=1}^{\infty} (\zeta_{m+1}(\mathbf{k}) - \zeta_m(\mathbf{k}))\, z^m = (1-z)\sum_{m=1}^{\infty} \zeta_m(\mathbf{k}) z^{m-1}.$$

$z = t$

$P \in \mathbb{R}[T]$, $Q(T) = P(P(T))$

$$(1-z)\sum_{m=1}^{\infty} P(\log(m) + \gamma)\, z^{m-1} = Q\left(\log\left(\frac{1}{1-z}\right)\right) + \mathrm{O}\left((1-z)\log^J\left(\frac{1}{1-z}\right)\right) \qquad (\text{as } z \to 1)$$

$$\sum_{m=1}^{\infty} \frac{\log^l(m)}{m} z^{m-1} = \mathrm{O}\left(\log^{l+1}\left(\frac{1}{1-z}\right)\right)$$

The q-series g for adm. indices

$$\mathrm{g}(\mathbf{k}) = \mathrm{g}(k_1,\dots,k_r) = \sum_{m_1>\cdots>m_r>0} \frac{P_{k_1}(q^{m_1})}{(1-q^{m_1})^{k_1}} \cdots \frac{P_{k_1}(q^{m_r})}{(1-q^{m_r})^{k_r}}$$
$$= \sum_{\substack{m_1>\cdots>m_r>0 \\ d_1,\dots,d_r>0}} \frac{d_1^{k_1-1}}{(k_1-1)!} \cdots \frac{d_r^{k_r-1}}{(k_r-1)!}\, q^{m_1 d_1+\cdots+m_r d_r}.$$

$\mathbf{k}$ adm. : $\lim_{q\to 1} (1-q)^{\mathrm{wt}(\mathbf{k})}\, g(\mathbf{k}) \simeq \zeta(\mathbf{k})$ $\qquad g(1) \in \mathbb{Q}(i\pi)$

$$\mathcal{G}^0 = \langle\, \mathrm{g}(\mathbf{k}) \mid \mathbf{k} \text{ admissible index} \rangle_{\mathbb{Q}},$$
$$\mathcal{G}^0_{\le k} = \langle\, \mathrm{g}(\mathbf{k}) \mid \mathbf{k} \text{ admissible index}, \mathrm{wt}(\mathbf{k}) \le k \rangle_{\mathbb{Q}}.$$

$k \ge 0$

$$\mathrm{Z}_k : \mathcal{G}^0_{\le k} \twoheadrightarrow \mathcal{Z}_k$$
$$f \longmapsto \lim_{q\to 1} (1-q)^k f$$

Want to extend Z_k to

$$\mathcal{G}_{\le k} = \langle\, \mathrm{g}(\mathbf{k}) \mid \mathbf{k} \text{ index}, \mathrm{wt}(\mathbf{k}) \le k \rangle_{\mathbb{Q}}$$

$\mathfrak{H}^1_* = \mathfrak{H}^0_*[z_1]$

$\downarrow g$

$\mathcal{G}$

Proposition 2.36. *i) We have* $\mathcal{G} = \mathcal{G}^0[\mathrm{g}(1)]$.

ii) $\mathrm{g}(1)$ *is algebraically independent over* $\mathcal{G}^0$.

$f \simeq g$ as $q \to 1$

$\Leftrightarrow \exists A, B \in \mathbb{R}_{>0}$

$A \le \frac{|f(q)|}{|g(q)|} \le B$ $\quad q \to 1$

Proof: i) follows from $\mathfrak{H}^1_* = \mathfrak{H}^0_*[z_1]$

ii) $g(1) = \sum_{m,d\ge 1} q^{md} \simeq -\frac{\log(1-q)}{1-q}$ $\quad q \to 1.$

$\mathbf{k}$ adm. $g(\mathbf{k}) \simeq \frac{1}{(1-q)^{\mathrm{wt}(\mathbf{k})}}$

$\Rightarrow$ ii)

Pupyrev

$(1-q)\, g(1) = \sum \frac{q^m (1-q)}{\cdots}$

$$\prod_{m\geq 0}(1-q^m)^{-1} = \sum_{m\geq 0} \frac{q^m}{1+q+\cdots+q^{m-1}}$$

$$\mathrm{Z}_k^T : \mathcal{G}_{\leq k} \longrightarrow \mathcal{Z}[T]$$

$$\sum_{j=0}^{k} f_j \; \mathrm{g}(1)^{k-j} \longmapsto \sum_{j=0}^{k} \mathrm{Z}_j(f_j)\, T^{k-j}$$

$f_j \in \mathcal{G}^0_{\leq j}$

$\|"q\to 1"$

$\sum_{m\geq 0} \frac{q^m}{m}$

$-\log(1-q)$

Proposition 2.37. *For any index* $\mathbf{k}$ *we have*

$$\mathrm{Z}^T_{\mathrm{wt}\,(\mathbf{k})}\left(\mathrm{g}(\mathbf{k})\right) = \zeta^*(\mathbf{k}; T)\,.$$

Proof: $w \in \mathfrak{h}^1$: $w = \sum_{j=0}^{k} u_j * z_1^{*k-j} = \sum_{j=0}^{k} v_j \hat{*} z_1^{\hat{*}k-j}$

$\mathrm{wt}(w) = k$

$u_j, v_j \in \mathfrak{h}^0$

$z_{k_1} \diamond z_{k_2} = z_{k_1+k_2}$

$$z_{k_1} \hat{\diamond} z_{k_2} = z_{k_1+k_2} + \sum_{j=1}^{k_1+k_2-1} \left(\lambda^j_{k_1,k_2} + \lambda^j_{k_2,k_1}\right) z_j$$

$$a *_\diamond a^{m-1}v = m\, a^m v + a^{m-1} b_1 (a \sqcup\!\sqcup b_2 \cdots b_l) + \sum_{j=0}^{m-2} a^j (a \diamond a) a^{m-2-j} v + a^{m-1} \sum_{i=1}^{l} b_1 \cdots (a \diamond b_i) \cdots b_l\,.$$

$\Rightarrow u_j - v_j$ is a lin. comb. of $z_{k'}$ with $\mathrm{wt}(k') < j$

$\mathcal{G}_{\leq k-1} \subset \ker \mathrm{Z}_k$

3 Families of lin. relations

$$\zeta : \mathfrak{H}^0 \longrightarrow \mathcal{Z}$$

Question: What is $\ker \zeta$?

3.1 Extended double shuffle relations

$w, u \in \mathfrak{H}^1$ $\qquad \mathrm{ds}(w,u) := w \sqcup\!\sqcup u - w * u \in \mathfrak{H}^1$

Finite double shuffle: $w, u \in \mathfrak{h}^0 \Rightarrow \mathrm{ds}(w,u) \in \ker \zeta$

$\mathrm{reg}_\bullet = \mathrm{reg}^0_\bullet$ $\quad \mathrm{reg}^T_\bullet : \mathfrak{h}^1 \longrightarrow \mathfrak{h}^0[T]$ $\quad \bullet \in \{*, \sqcup\!\sqcup\}$

$\mathrm{reg}_\bullet : \mathfrak{h}^1 \longrightarrow \mathfrak{h}^0$ $\quad \zeta^\bullet(u;T) =$

Theorem 3.1 (Extended double shuffle relations). *For $w \in \mathfrak{H}^1$, $u \in \mathfrak{H}^0$ and $\bullet \in \{\sqcup\!\sqcup, *\}$ we have*

$$\zeta^\bullet(w \sqcup\!\sqcup u - w * u; T) = 0\,,$$

i.e. in particular $\mathrm{reg}_\bullet(\mathrm{ds}(w,u)) \in \ker \zeta$.

$\mathrm{ds}(w,w) \in \mathfrak{h}^1$ $\quad \zeta^{\sqcup\!\sqcup}(u) = \zeta^*(u)$

Proof: $\quad \zeta^{\sqcup\!\sqcup}(w;T) = \rho(\zeta^*(w;T))$ $\quad , u \in \mathfrak{h}^0$, $\zeta(u) \in \mathbb{R}$ $\quad \zeta^{\sqcup\!\sqcup}(u;T) = \zeta^*(u;T)$

$\zeta(u) =$ $\Rightarrow$ $\zeta^{\sqcup\!\sqcup}(u \sqcup\!\sqcup w;T) = \rho(\zeta^*(u * w;T))$ $\quad \zeta^{\sqcup\!\sqcup}(u;T)$

$\zeta^{\sqcup\!\sqcup}(u * w;T)$

$= \zeta(u * w; 1)$

$\Rightarrow \quad \zeta^{ш}(w ш u - u * w; T) = 0$

$\rho^{-1} \Rightarrow \quad \zeta^{*}(w ш u - u * w; T) = 0$

Conjecture 3.2. *The kernel of $\zeta : \mathfrak{H}^0 \to \mathcal{Z}$ is given by*

$$\begin{aligned}\ker \zeta &= \left\langle \operatorname{reg}_{ш}(w ш u - w * u) \mid w \in \mathfrak{H}^1,\, u \in \mathfrak{H}^0 \right\rangle_{\mathbb{Q}} \\ &= \left\langle \operatorname{reg}_{*}(w ш u - w * u) \mid w \in \mathfrak{H}^1,\, u \in \mathfrak{H}^0 \right\rangle_{\mathbb{Q}},\end{aligned}$$

i.e. the extended double shuffle relations give all $\mathbb{Q}$-linear relations among multiple zeta values.

Conjecture 1.13 (Zagier, 1994). *We have $\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k$ for all $k \geq 0$.*

$$\sum_{k \geq 0} d_k X^k = \frac{1}{1 - X^2 - X^3}$$

Theorem 1.15 (Terasoma (2002), Deligne–Goncharov (2005)). *For all $k \geq 0$ we have $\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k$.*

weight k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
# of adm. ind.	1	0	1	2	4	8	16	32	64	128	256	512	1024	2048	4096
# of relations $\overset{?}{=}$	0	0	0	1	3	6	14	29	60	123	249	503	1012	2032	4075
$\left(\dim_{\mathbb{Q}} \mathcal{Z}_k \overset{?}{=}\right) d_k$	1	0	1	1	1	2	2	3	4	5	7	9	12	16	21

Open problem 3.3. *Count the number of linearly independent extended double shuffle relations. Define for $k \geq 0$ and $\bullet \in \{ш, *\}$ the spaces*

$$\mathsf{eds}^{\bullet}_k = \left\langle \operatorname{reg}^{\bullet}(w ш u - w * u) \mid w \in \mathfrak{H}^1,\, u \in \mathfrak{H}^0, \operatorname{wt}(w) + \operatorname{wt}(u) = k \right\rangle_{\mathbb{Q}} \subset \mathfrak{g}'$$

Show that for all $k \geq 2$ and any $\bullet \in \{ш, *\}$ we have

$$\dim_{\mathbb{Q}} \mathsf{eds}^{\bullet}_k = 2^{k-2} - d_k\,. \tag{3.1}$$

Checked up to $k = 21$ — T. Machide, T. Sonobe

Here the d_k are the conjectured dimensions (Conjecture 1.13) of $\mathcal{Z}_k$, defined by

$$\sum_{k \geq 0} d_k X^k = \frac{1}{1 - X^2 - X^3},$$

and 2^{k-2} is the number of admissible indices of weight k.

$\omega \in \mathfrak{g}^0$

$\tau : \mathfrak{g}^0 \to \mathfrak{g}^0$ anti-automorph. $\quad \tau(x) = y,\ \tau(y) = x$

$\zeta(2,1,\dots,1) = \zeta(k)$

$\zeta(\tau(\omega) - \omega) = 0$

$\tau(\omega) - \omega \in \ker \zeta$

Open problem 3.4. *Show that for any $v \in \mathfrak{H}^0$ we have for $\bullet \in \{ш, *\}$*

$$\tau(v) - v \in \left\langle \operatorname{reg}_\bullet(w ш u - w * u) \mid w \in \mathfrak{H}^1,\, u \in \mathfrak{H}^0 \right\rangle_{\mathbb{Q}},$$

i.e. the duality relation is a consequence of the extended double shuffle relations.

Proposition 3.5 (Hoffman's relation ([H1])). *For an admissible index $\mathbf{k} = (k_1, \dots, k_r)$ we have*

$$\sum_{i=1}^{r} \zeta(k_1, \dots, k_{i-1}, k_i + 1, k_{i+1}, \dots, k_r) = \sum_{\substack{1 \le i \le r \\ k_i \ge 2}} \sum_{j=0}^{k_i - 2} \zeta(k_1, \dots, k_{i-1}, k_i - j, j+1, k_{i+1}, \dots, k_r).$$

Proof: $w = z_1$, $u = z_k$ // extended dsh. (Ex 10).

$$\underbrace{\zeta(z_1 ш z_k - z_1 * z_k)}_{\in \mathfrak{H}^0} = 0$$

Conjecture 3.6 (H. N. Minh, M. Petitot et al. ([MJOP])). *We have*

$$\ker \zeta = \left\langle w ш u - w * u \mid w \in \mathfrak{H}^0 \cup \{z_1\},\, u \in \mathfrak{H}^0 \right\rangle_{\mathbb{Q}},$$

i.e. Hoffman's relation and the finite double shuffle relations give all linear relations among multiple zeta values.

Conjecture 3.7 (M. Kaneko, M.Noro and K.Tsurumaki ([KNT])). *We have*

$$\ker \zeta = \left\langle w ш u - w * u \mid w \in \{z_1, z_2, z_3, z_2 z_1\},\, u \in \mathfrak{H}^0 \right\rangle_{\mathbb{Q}}.$$

Open problem 3.8. *Show that*

$$\begin{aligned} \left\langle \operatorname{reg}_{ш}(w ш u - w * u) \mid w \in \mathfrak{H}^1,\, u \in \mathfrak{H}^0 \right\rangle_{\mathbb{Q}} &\overset{?}{=} \left\langle \operatorname{reg}_*(w ш u - w * u) \mid w \in \mathfrak{H}^1,\, u \in \mathfrak{H}^0 \right\rangle_{\mathbb{Q}} \\ &\overset{?}{=} \left\langle w ш u - w * u \mid w \in \mathfrak{H}^0 \cup \{z_1\},\, u \in \mathfrak{H}^0 \right\rangle_{\mathbb{Q}} \\ &= \left\langle w ш u - w * u \mid w \in \{z_1, z_2, z_3, z_2 z_1\},\, u \in \mathfrak{H}^0 \right\rangle_{\mathbb{Q}}. \end{aligned}$$

Open problem 3.9. *Show that for* $\bullet_1, \bullet_2 \in \{\sqcup\!\sqcup, *\}$ *we have*

$$\mathfrak{H}^0 \bullet_1 \left\langle \mathrm{reg}_{\bullet_2}(w \sqcup\!\sqcup u - w * u) \mid w \in \mathfrak{H}^1,\, u \in \mathfrak{H}^0 \right\rangle_{\mathbb{Q}} \subset \left\langle \mathrm{reg}_{\bullet_2}(w \sqcup\!\sqcup u - w * u) \mid w \in \mathfrak{H}^1,\, u \in \mathfrak{H}^0 \right\rangle_{\mathbb{Q}}.$$