

多重ゼータ値とそのモジュラー形式との関係

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$$\zeta(k_1, \dots, k_r) = \sum_{m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}$$

G(k₁, ..., k_r) (↑)

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

多元数理談話会 2020年12月2日

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自己紹介

- ハンブルクに生まれました
- 2016: 博士号取得 (ハンブルグ大学)
- 2016 – 2017: JSPS特別研究員 (名古屋大学)
- 2017 – 2019: YLC特任助教 (名古屋大学)
- 2017 – 2018: MPIM Bonn
- 昨年の10月からはG30プログラムの准教授



ハンブルク



多重ゼータ値

定義

自然数の組 $\mathbf{k} = (k_1, \dots, k_r)$ に対し、**多重ゼータ値**を

$$\zeta(\mathbf{k}) = \zeta(k_1, \dots, k_r) = \sum_{m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}$$

で定義する。ただし、収束のため $k_1 \geq 2$ とする。

多重ゼータ値で生成される \mathbb{Q} ベクトル空間を \mathcal{Z} で表す。

注意： \mathcal{Z} は \mathbb{Q} 代数となる。

事実：多重ゼータ値は多くの関係式を満たす。

例 (複シャッフル関係式)

$$\zeta(2, 3) + 3\zeta(3, 2) + 6\zeta(4, 1) = \zeta(2) \cdot \zeta(3) = \zeta(2, 3) + \zeta(3, 2) + \zeta(5)$$

$$\implies 2\zeta(3, 2) + 6\zeta(4, 1) = \zeta(5).$$

数

関数

1重の対象		
多重化された対象		
関係式		

数

関数

1重の対象	Riemannゼータ値 $\zeta(k)$
多重化された対象	多重ゼータ値 $\zeta(k_1, \dots, k_r)$
関係式	複シャッフル関係式 $\begin{aligned}\zeta(2) \cdot \zeta(3) &= \zeta(2, 3) + \zeta(3, 2) + \zeta(5) \\ &= \zeta(2, 3) + 3\zeta(3, 2) + 6\zeta(4, 1)\end{aligned}$

数

関数

1重の対象

Riemannゼータ値

$$\zeta(k)$$

Eisenstein級数
(モジュラー形式)

$$G_k(\tau) = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \frac{n^{k-1} q^n}{1 - q^n}$$

多重化
された対象

多重ゼータ値

$$\zeta(k_1, \dots, k_r)$$

関係式

複シャッフル関係式

$$\begin{aligned}\zeta(2) \cdot \zeta(3) &= \zeta(2, 3) + \zeta(3, 2) + \zeta(5) \\ &= \zeta(2, 3) + 3\zeta(3, 2) + 6\zeta(4, 1)\end{aligned}$$



$$(q = e^{2\pi i\tau})$$

Eisen
stein
鉄
石

数

関数

1重の対象

Riemannゼータ値

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$$G_k(\tau) = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \frac{n^{k-1} q^n}{1 - q^n}$$

多重化
された対象

多重ゼータ値

$$\zeta(k_1, \dots, k_r)$$

多重Eisenstein級数
(Gangl-金子-Zagier, B.)

$$G_{k_1, \dots, k_r}(\tau) = \zeta(k_1, \dots, k_r) + \sum_{n > 0} a_n q^n$$

関係式

複シャッフル関係式

$$\begin{aligned}\zeta(2) \cdot \zeta(3) &= \zeta(2, 3) + \zeta(3, 2) + \zeta(5) \\ &= \zeta(2, 3) + 3\zeta(3, 2) + 6\zeta(4, 1)\end{aligned}$$

?



$$(q = e^{2\pi i \tau})$$

Eisen
stein 鉄石

多重ゼータ値とモジュラー形式の講義

<https://www.henrikbachmann.com/mzv2020.html>

Henrik Bachmann

- Multiple Eisenstein series [[B2], [GKZ]]
- q-analogues of multiple zeta values [[B2]]

Materials

- Lecture notes & Exercises: Multiple zeta values & modular forms (ver. 5.2)
- Lecture videos: Lecture 1, Lecture 2, Lecture 3, Lecture 4, Lecture 5, Lecture 6, Lecture 7, Lecture 8, Lecture 9, Lecture 10, Lecture 11, Lecture 12, Lecture 13, Lecture 14, Lecture 15
- Zoom meeting notes: Meeting 2 (05/01), Meeting 3 (05/08), Meeting 4 (05/15), Meeting 5 (05/22), Meeting 6 (05/29), Meeting 7 (06/05), Meeting 8 (06/12), Meeting 9 (06/19), Meeting 10 (06/26)

Suchen

What objects will we study in this course?

Lectu
Each v
exerci

4:37 / 27:13

Multiple zeta values & modular forms: Lecture 1 - Introduction & Riemann zeta function

484 Aufrufe • 29.04.2020

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英語と日本語の字幕が利用可能です

mzv_mf_2020.pdf
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MZVs and modular forms • Overview & Basics

§1 Overview & Basics

In this section, we will give an overview of the values of the Riemann zeta function, the definition of multiple zeta values, some of the main conjectures concerning their structure, and a glimpse of the connection of (q-analogues of) multiple zeta values and modular forms. The general picture of these concepts will be discussed in detail in the later Sections.

1.1 The values of the Riemann zeta function

The Riemann zeta function is defined for a complex variable $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$ by

$$\zeta(s) = \sum_{n \geq 0} \frac{1}{n^s}.$$

(1.1)

This function appears in various fields of mathematics and theoretical physics and it can be studied from various points of view. It plays a pivotal role in analytic number theory and has applications in physics, probability theory, and applied statistics.

The graph of $\zeta(s+iy)$ near the pole at $s+iy=1$.

For example, it is well-known that the Riemann zeta function can be analytically continued to the whole complex plane with a simple pole at $s = 1$. Even though $\zeta(s)$ was already considered by Euler (1735) and Riemann (1859), who proved its meromorphic continuation and functional equation and established a relation between its zeros and the distribution of prime numbers.

In particular, he gave his famous conjecture on the location of the non-trivial zeros of the Riemann zeta function, stating that besides the trivial zeros at $s = -2, -4, -6, \dots$ all other zeros have real part $\frac{1}{2}$.

A sculpture of the pole of $\zeta(s)$ at $s=1$ in front of Nagoya station in honor of the

Why "shuffle" & "shuffle" ?

$\zeta(2) = \prod_{m \geq 0} \frac{1}{m^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \frac{\pi^2}{6}$

$\zeta(2k) = \sum_{j_1, j_2, \dots, j_k} \frac{1}{j_1 j_2 \dots j_k} = \sum_{j_1, j_2, \dots, j_k} \left(\binom{j_1-1}{k-1} + \binom{j_2-1}{k-1} + \dots + \binom{j_k-1}{k-1} \right) \zeta(j_1 + j_2 + \dots + j_k)$

$\zeta(2k) = \sum_{j_1, j_2, \dots, j_k} \left(\binom{j_1-1}{k-1} + \binom{j_2-1}{k-1} + \dots + \binom{j_k-1}{k-1} \right) \zeta(j_1 + j_2 + \dots + j_k)$

NU-EMI

NU-EMIは、G30 講義を受講する
一般学生の「英語で学ぶ」を応援します

Plan of this talk

1

Multiple zeta values
&
Double shuffle relations

1

数

1重の対象

Riemannゼータ値
 $\zeta(k)$

関数

Eisenstein級数
(モジュラー形式)

$$G_k(\tau) = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1 - q^n}$$

多重化された対象

多重ゼータ値
 $\zeta(k_1, \dots, k_r)$

多重Eisenstein級数
(Gan-J-Zagier, B.)

$$G_{k_1, \dots, k_r}(\tau) = \zeta(k_1, \dots, k_r) + \sum a_n q^n$$

関係式

複シャッフル関係式
$$\begin{aligned}\zeta(2) - \zeta(3) &= \zeta(2, 3) + \zeta(3, 2) + \zeta(5) \\ &= \zeta(2, 3) + 3\zeta(3, 2) + 6\zeta(5)\end{aligned}$$

?

2

2

Modular forms

3

Multiple Eisenstein series

4

“Double shuffle relations
for functions”

3

4

① MZV & DSH - Definition

Definition

For $k_1 \geq 2, k_2, \dots, k_r \geq 1$ define the **multiple zeta value** (MZV)

$$\zeta(k_1, \dots, k_r) = \sum_{m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}.$$

By r we denote its **depth** and $k_1 + \dots + k_r$ will be called its **weight**.

- \mathcal{Z} : \mathbb{Q} -algebra of MZVs
- \mathcal{Z}_k : \mathbb{Q} -vector space of MZVs of weight k .

MZVs can also be written as **iterated integrals**, e.g.

$$\zeta(2, 3) = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} \int_0^{t_2} \frac{dt_3}{t_3} \int_0^{t_3} \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_5}{1-t_5}.$$

① MZV & DSH - Harmonic & shuffle product

There are two different ways to express the product of MZV in terms of MZV.

Harmonic product (coming from the definition as iterated sums)

Example in depth two ($k_1, k_2 \geq 2$)

$$\begin{aligned}\zeta(k_1) \cdot \zeta(k_2) &= \sum_{m>0} \frac{1}{m^{k_1}} \sum_{n>0} \frac{1}{n^{k_2}} \\ &= \sum_{m>n>0} \frac{1}{m^{k_1} n^{k_2}} + \sum_{n>m>0} \frac{1}{m^{k_1} n^{k_2}} + \sum_{m=n>0} \frac{1}{m^{k_1+k_2}} \\ &= \zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1 + k_2).\end{aligned}$$

Shuffle product (coming from the expression as iterated integrals)

Example in depth two ($k_1, k_2 \geq 2$)

$$\zeta(k_1) \cdot \zeta(k_2) = \sum_{j=2}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) \zeta(j, k_1 + k_2 - j).$$

① MZV & DSH - Double shuffle relations

These two product expressions give various \mathbb{Q} -linear relations between MZV.

Example

$$\begin{aligned}\zeta(2) \cdot \zeta(3) &\stackrel{\text{harmonic}}{=} \zeta(2, 3) + \zeta(3, 2) + \zeta(5) \\ &\stackrel{\text{shuffle}}{=} \zeta(2, 3) + 3\zeta(3, 2) + 6\zeta(4, 1). \\ \implies 2\zeta(3, 2) + 6\zeta(4, 1) &\stackrel{\text{double shuffle}}{=} \zeta(5).\end{aligned}$$

But there are more relations between MZV. e.g.:

$$\sum_{m>n>0} \frac{1}{m^2 n} = \zeta(2, 1) = \zeta(3) = \sum_{m>0} \frac{1}{m^3}.$$

These follow from regularizing the double shuffle relations
~~~**extended double shuffle relations.**

## ① MZV & DSH - Relations conjectures

### Conjecture

All relations among MZVs are consequences of the extended double shuffle relations.

### Conjecture

The space  $\mathcal{Z}$  is graded by weight, i.e.

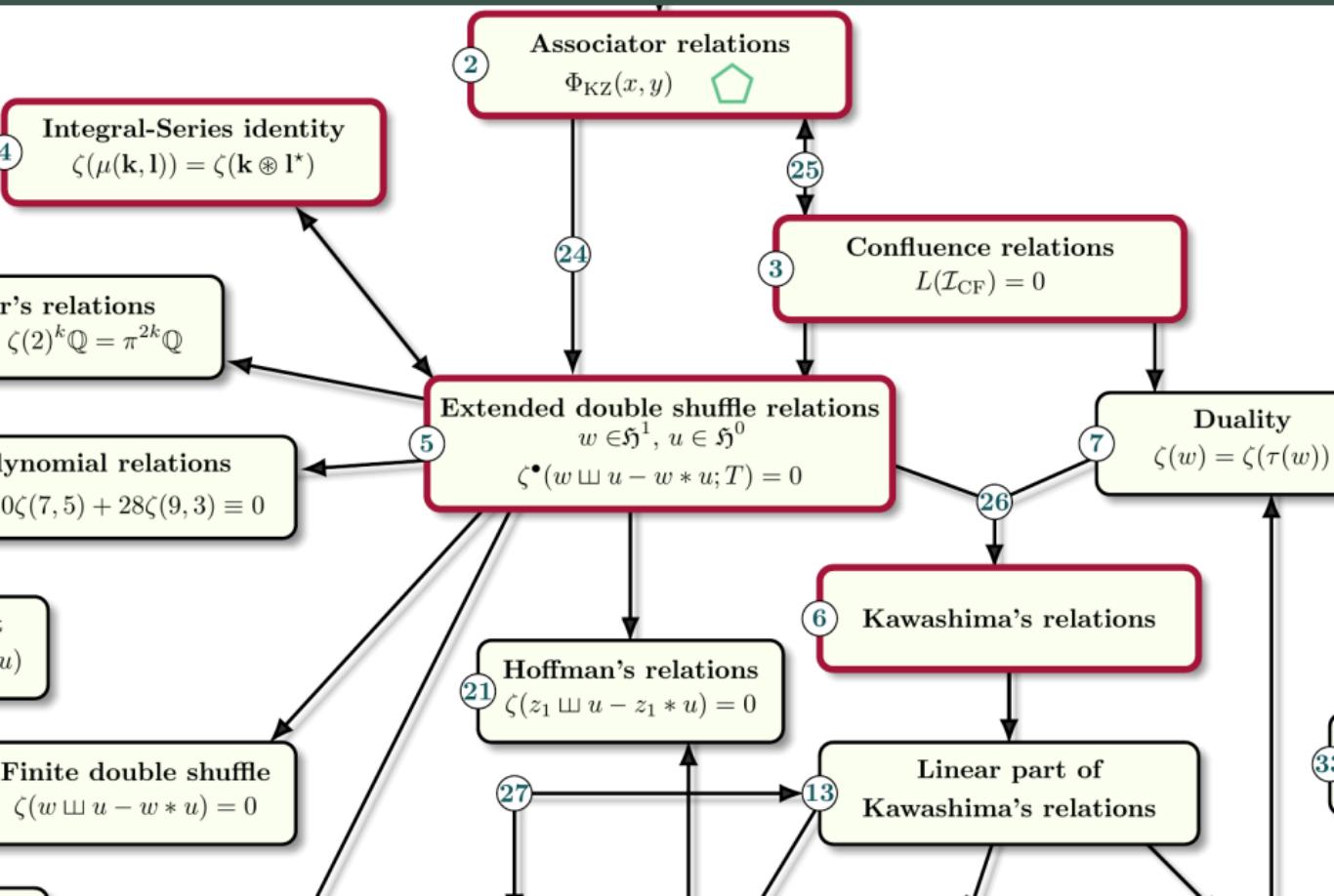
$$\mathcal{Z} = \bigoplus_{k \geq 0} \mathcal{Z}_k .$$

- There are various different families of relations which conjecturally give all relations among MZV.
- Not for all of them it is known if they are equivalent to the extended double shuffle relations.

# Overview of relations among MZV

For details see:

B. "Multiple zeta values & modular forms", Lecture notes



## ① MZV & DSH - Dimension conjectures

Define the numbers  $d_k \in \mathbb{Z}_{\geq 0}$  by

$$\sum_{k \geq 0} d_k X^k = \frac{1}{1 - X^2 - X^3}.$$

Conjecture (Zagier, 1994)

We have  $\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k$  for all  $k \geq 0$ .

| weight $k$                       | 0 | 1 | 2 | 3 | 4 | 5 | 6  | 7  | 8  | 9   | 10  | 11  | 12   |
|----------------------------------|---|---|---|---|---|---|----|----|----|-----|-----|-----|------|
| # of adm. ind.                   | 1 | 0 | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 |
| # of relations $\stackrel{?}{=}$ | 0 | 0 | 0 | 1 | 3 | 6 | 14 | 29 | 60 | 123 | 249 | 503 | 1012 |
| $d_k$                            | 1 | 0 | 1 | 1 | 1 | 2 | 2  | 3  | 4  | 5   | 7   | 9   | 12   |

Theorem (Terasoma (2002), Deligne–Goncharov (2005))

For all  $k \geq 0$  we have  $\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k$ .

## ① MZV & DSH - Conjectures

Conjecture (Hoffman, 1997)

For  $k \geq 0$  the multiple zeta values

$$\{\zeta(k_1, \dots, k_r) \mid r \geq 0, k_1 + \dots + k_r = k, k_1, \dots, k_r \in \{2, 3\}\}$$

form a basis of  $\mathcal{Z}_k$ .

$$\zeta(2n) \in \pi^{2n} \mathbb{Q}, \quad \zeta(2, \dots, 2) = \frac{\pi^{2n}}{(2n+1)!}, \quad \zeta(5) = \frac{6}{5} \zeta(2, 3) + \frac{4}{5} \zeta(3, 2).$$

Theorem (Brown, 2012)

For all  $k \geq 0$  we have

$$\mathcal{Z}_k = \langle \zeta(k_1, \dots, k_r) \mid r \geq 0, k_1 + \dots + k_r = k, k_1, \dots, k_r \in \{2, 3\} \rangle_{\mathbb{Q}}.$$

## ② Modular forms - Definition

Complex upper half plane:  $\mathbb{H} = \{x + iy \in \mathbb{C} \mid x, y \in \mathbb{R}, y > 0\}$ .

### Definition

A holomorphic function  $f \in \mathcal{O}(\mathbb{H})$  is called a **modular form of weight  $k \in \mathbb{Z}$**  if it satisfies

- $f(\tau + 1) = f(\tau)$ ,
- $f(-\frac{1}{\tau}) = \tau^k f(\tau)$ ,

for all  $\tau \in \mathbb{H}$  and if it has a Fourier expansion of the form

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n . \quad (a_n \in \mathbb{C}, q = e^{2\pi i \tau})$$

- $\mathcal{M}_k$  : space of all modular forms of weight  $k$ .
- The space of **cusp forms** of weight  $k$  is defined by

$$\mathcal{S}_k = \left\{ f \in \mathcal{M}_k \mid f = \sum_{n=1}^{\infty} a_n q^n \right\} .$$

## ② Modular forms - Eisenstein series

For even  $k \geq 4$  the **Eisenstein series** are defined by

$$G_k(\tau) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m\tau + n)^k}.$$

These have a Fourier expansion of the form

$$G_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \quad (1)$$

where  $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$  is the divisor sum.

### Proposition

For every even  $k \geq 4$  we have  $G_k \in \mathcal{M}_k$  and

$$\mathcal{M} = \bigoplus_{k=0}^{\infty} \mathcal{M}_k = \mathbb{C}[G_4, G_6].$$

## ② Modular forms - Cusp forms

The first non-trivial cusp form is the **discriminant function**  $\Delta$

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \dots,$$

which is a cusp form of weight 12.

### Theorem

- For  $k \geq 0$  the map  $\mathcal{M}_k \rightarrow \mathcal{S}_{k+12}$  given by  $f \mapsto \Delta \cdot f$  is an isomorphism of  $\mathbb{C}$ -vector spaces.
- The generating series for the dimension of cusp forms of weight  $k$  is given by

$$S(X) = \sum_{k \geq 0} \dim_{\mathbb{C}} \mathcal{S}_k X^k = X^{12} \sum_{k \geq 0} \dim_{\mathbb{C}} \mathcal{M}_k X^k = \frac{X^{12}}{(1 - X^4)(1 - X^6)}.$$

① ② MZV & Modular forms - Broadhurst-Kreimer conjecture

$\text{gr}_r^D \mathcal{Z}_k$ : MZV of weight  $k$  and depth  $r$  modulo lower depths MZV.

Conjecture (Broadhurst-Kreimer, 1997)

The generating series of the dimensions of the weight- and depth-graded parts of multiple zeta values is given by

$$\sum_{k,r \geq 0} \dim_{\mathbb{Q}} (\text{gr}_r^D \mathcal{Z}_k) X^k Y^r = \frac{1 + E(X)Y}{1 - O(X)Y + S(X)Y^2 - S(X)Y^4},$$

where

$$E(X) = \frac{X^2}{1 - X^2}, \quad O(X) = \frac{X^3}{1 - X^2}, \quad S(X) = \frac{X^{12}}{(1 - X^4)(1 - X^6)}.$$

Observe that

$$\begin{aligned} & \frac{1 + E(X)Y}{1 - O(X)Y + S(X)Y^2 - S(X)Y^4} \\ &= 1 + (E(X) + O(X))Y + ((E(X) + O(X))O(X) - S(X))Y^2 + \dots. \end{aligned}$$

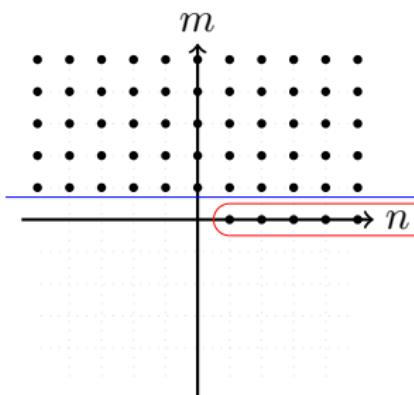
### ③ Multiple Eisenstein series - An order on lattices

Let  $\tau \in \mathbb{H}$ . We define an order  $\succ$  on the lattice  $\mathbb{Z}\tau + \mathbb{Z}$  by setting

$$\lambda_1 \succ \lambda_2 : \Leftrightarrow \lambda_1 - \lambda_2 \in P$$

for  $\lambda_1, \lambda_2 \in \mathbb{Z}\tau + \mathbb{Z}$  and the following set of positive lattice points

$$P := \{m\tau + n \in \mathbb{Z}\tau + \mathbb{Z} \mid m > 0 \vee (m = 0 \wedge n > 0)\} = U \cup R.$$



In other words:  $\lambda_1 \succ \lambda_2$  iff  $\lambda_1$  is above or on the right of  $\lambda_2$ .

### ③ Multiple Eisenstein series - Multiple Eisenstein series

#### Definition

For integers  $k_1 \geq 3, k_2, \dots, k_r \geq 2$ , we define the **multiple Eisenstein series** by

$$G_{k_1, \dots, k_r}(\tau) = \sum_{\substack{\lambda_1 \succ \dots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \cdots \lambda_r^{k_r}}.$$

It is easy to see that these are holomorphic functions in the upper half plane and that they fulfill the **harmonic product**, i.e. it is for example

$$G_2(\tau) \cdot G_3(\tau) = G_{2,3}(\tau) + G_{3,2}(\tau) + G_5(\tau).$$

### ③ Multiple Eisenstein series - Classical Eisenstein series

In depth one we have for  $k \geq 3$

$$G_k(\tau) = \sum_{\substack{\lambda \in \mathbb{Z}\tau + \mathbb{Z} \\ \lambda \succ 0}} \frac{1}{\lambda^k} = \sum_{\substack{m > 0 \\ \vee (m=0 \wedge n>0)}} \frac{1}{(m\tau + n)^k} = \zeta(k) + \sum_{m>0} \sum_{n \in \mathbb{Z}} \frac{1}{(m\tau + n)^k}$$

### ③ Multiple Eisenstein series - Classical Eisenstein series

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By the Lipschitz summation formula we get for  $k \geq 2$  ( $q = e^{2\pi i \tau}$ )

$$\sum_{n \in \mathbb{Z}} \frac{1}{(\tau + n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{d>0} d^{k-1} q^d .$$

This gives

$$G_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{\substack{m > 0 \\ d > 0}} d^{k-1} q^{md} = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n .$$

### ③ Multiple Eisenstein series - $q$ -MZV

#### Definition

For  $k_1, \dots, k_r \geq 1$  we define a  **$q$ -analogue of multiple zeta values** by

$$g(k_1, \dots, k_r) = \sum_{\substack{m_1 > \dots > m_r > 0 \\ d_1, \dots, d_r > 0}} \frac{d_1^{k_1-1}}{(k_1-1)!} \cdots \frac{d_r^{k_r-1}}{(k_r-1)!} q^{m_1 d_1 + \dots + m_r d_r} \in \mathbb{Q}[[q]].$$

These  $q$ -series have a nice combinatorial interpretation

$$g(k_1, \dots, k_r) = \sum_{n>0} \left( \dots \right) q^n.$$

### ③ Multiple Eisenstein series - $q$ -MZV

#### Definition

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#### Proposition

For  $k_1 \geq 2, k_2, \dots, k_r \geq 1$  we have

$$\lim_{q \rightarrow 1} (1-q)^{k_1+\dots+k_r} g(k_1, \dots, k_r) = \zeta(k_1, \dots, k_r).$$

### ③ Multiple Eisenstein series - Fourier expansion

Theorem (B., 2012)

The multiple Eisenstein series  $G_{k_1, \dots, k_r}(\tau)$  have a Fourier expansion of the form

$$G_{k_1, \dots, k_r}(\tau) = \zeta(k_1, \dots, k_r) + \sum_{n>0} a_n q^n \quad (q = e^{2\pi i \tau})$$

and they can be written explicitly as a  $\mathcal{Z}[2\pi i]$ -linear combination of  $q$ -analogues of multiple zeta values  $g$ . In particular,  $a_n \in \mathcal{Z}[2\pi i]$ .

#### Examples

$$G_k(\tau) = \zeta(k) + (-2\pi i)^k g(k),$$

$$G_{3,2}(q) = \zeta(3, 2) + 3\zeta(3)(-2\pi i)^2 g(2) + 2\zeta(2)(-2\pi i)^3 g(3) + (-2\pi i)^5 g(3, 2).$$

### ③ Multiple Eisenstein series - Do they satisfy double shuffle?

We saw the following example:

#### Example

$$\begin{aligned}\zeta(2) \cdot \zeta(3) &\stackrel{\text{harmonic}}{=} \zeta(2, 3) + \zeta(3, 2) + \zeta(5) \\ &\stackrel{\text{shuffle}}{=} \zeta(2, 3) + 3\zeta(3, 2) + 6\zeta(4, 1). \\ \implies 2\zeta(3, 2) + 6\zeta(4, 1) &\stackrel{\text{double shuffle}}{=} \zeta(5).\end{aligned}$$

#### Question

Are these relations also satisfied by the multiple Eisenstein series?

### ③ Multiple Eisenstein series - Do they satisfy double shuffle?

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#### Example

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#### Question

Are these relations also satisfied by the multiple Eisenstein series?

Problem: No definition of  $G_2$ ,  $G_{2,3}$  and  $G_{4,1}$ !

### ③ Multiple Eisenstein series - Do they satisfy double shuffle?

There are different ways to extend the definition of  $G_{k_1, \dots, k_r}$  to  $k_1, \dots, k_r \geq 1$

- Formal double zeta space realization  $G_{r,s}$  (Gangl-Kaneko-Zagier, 2006)

$$\begin{aligned} G_{k_1} \cdot G_{k_2} + (\delta_{k_1,2} + \delta_{k_2,2}) \frac{G'_{k_1+k_2-2}}{2(k_1+k_2-2)} &= G_{k_1,k_2} + G_{k_2,k_1} + G_{k_1+k_2} \\ &= \sum_{j=2}^{k_1+k_2-1} \left( \binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) G_{j,k_1+k_2-j}, \quad (k_1+k_2 \geq 3). \end{aligned}$$

- Finite double shuffle version  $G_{r,s}$  (Kaneko, 2007).
- Shuffle regularized multiple Eisenstein series  $G_{k_1, \dots, k_r}^{\sqcup\sqcup}$  (B.-Tasaka, 2017)
- Harmonic regularized multiple Eisenstein series  $G_{k_1, \dots, k_r}^*$  (B., 2019)

#### Observation

- No version of these objects satisfy the double shuffle relations for all indices/weights.
- The derivative is always somewhere as an extra term.

### ③ Multiple Eisenstein series - Do they satisfy double shuffle?

Theorem (Gangl-Kaneko-Zagier +  $\epsilon$ )

For all  $k \geq 0$  there exists a basis of  $S_k$  given by explicit linear combinations of  $G_{\text{odd}, \text{odd}}$ .

Corollary (taking constant term)

For each cusp form there is a relation among  $\zeta(\text{odd}, \text{odd})$ .

**Example** There exist a  $c \in \mathbb{C}$  with

$$c\Delta = G_{3,9} - \frac{23825}{5197}G_{5,7} - \frac{41431}{10394}G_{7,5} + \frac{360}{5197}G_{9,3} + G_{11,1},$$

which implies the relation

$$0 = \zeta(3, 9) - \frac{23825}{5197}\zeta(5, 7) - \frac{41431}{10394}\zeta(7, 5) + \frac{360}{5197}\zeta(9, 3) + \zeta(11, 1).$$

Conjecturally these are the only relations among  $\zeta(\text{odd}, \text{odd})$

$\rightsquigarrow$  Explanation of  $O(X)O(X) - S(X)$  in the Broadhurst-Kreimer conjecture.

## ④ Extension of the DSH relations - General Idea

### Questions

- What are the relations satisfied by multiple Eisenstein series?
- Can we formalize these relation?

The double shuffle relations can also be stated in terms of **generating series**:

$$Z^*(X_1, \dots, X_r) = \sum_{k_1, \dots, k_r \geq 1} \zeta^*(k_1, \dots, k_r) X_1^{k_1-1} \dots X_r^{k_r-1}$$

Then the extended double shuffle relations in lowest depths can be written as

$$\begin{aligned} Z^*(X_1)Z^*(X_2) &= Z^*(X_1, X_2) + Z^*(X_2, X_1) + \frac{Z^*(X_1) - Z^*(X_2)}{X_1 - X_2} \\ &= Z^*(X_1 + X_2, X_2) + Z^*(X_1 + X_2, X_1) + \zeta(2). \end{aligned}$$

## ④ Extension of the DSH relations - Formal double shuffle relations

- $A$ :  $\mathbb{Q}$ -algebra.
- For  $z_{k_1, \dots, k_r} \in A$  for  $k_1, \dots, k_r \geq 1$  we write

$$Z(X_1, \dots, X_r) = \sum_{k_1, \dots, k_r \geq 1} z_{k_1, \dots, k_r} X_1^{k_1-1} \dots X_r^{k_r-1}.$$

- A collection  $Z = (Z(X_1, \dots, X_r))_{r \geq 1}$  will be called a **mould**.

### Definition

A mould  $Z$  satisfies the **double shuffle relations** (in depth 2) if

$$\begin{aligned} Z(X_1)Z(X_2) &= Z(X_1, X_2) + Z(X_2, X_1) + \frac{Z(X_1) - Z(X_2)}{X_1 - X_2} \\ &= Z(X_1 + X_2, X_1) + Z(X_1 + X_2, X_2) + z_2. \end{aligned}$$

## ④ Extension of the DSH relations - Known solutions

- $A = \mathbb{R}$ : Harmonic regularized multiple zeta values

$$z_{k_1, \dots, k_r} = \zeta^*(k_1, \dots, k_r).$$

- $A = \mathbb{Q}$ : Explicit solutions are known up to depth 3 (Brown, Ecalle, Gangl-Kaneko-Zagier, Tasaka). In depth 1 they are all given by

$$z_k = \begin{cases} -\frac{B_k}{2k!} = \frac{\zeta(k)}{(2\pi i)^k}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}.$$

- $A = \mathbb{Q}$ : Solution exist in all depths (Drinfel'd + Furusho, Racinet).

## ④ Extension of the DSH relations - General Idea

### General idea

- Include also (arbitrary) derivatives as objects.
- Instead of series  $Z(X_1, \dots, X_r)$  we will consider generating series with two types of variables  $X_i$  and  $Y_i$ .
- Roughly:  $X_i$ : weight,  $Y_i$ : derivative.
- In the case  $Y_i = 0$ , we get back our original story.

$A$ :  $\mathbb{Q}$ -algebra

$$B\binom{X_1, \dots, X_r}{Y_1, \dots, Y_r} \in A[[X_1, Y_1, \dots, X_r, Y_r]].$$

### Definition

A collection  $B = \left( B\binom{X_1, \dots, X_r}{Y_1, \dots, Y_r} \right)_{r \geq 1}$  will be called a **bimould**.

## ④ Extension of the DSH relations - Symmetril

### Definition

A bimould  $B$  is **symmetril** (up to depth 2), if

$$B\binom{X_1}{Y_1}B\binom{X_2}{Y_2} = B\binom{X_1, X_2}{Y_1, Y_2} + B\binom{X_2, X_1}{Y_2, Y_1} + \frac{B\binom{X_1}{Y_1+Y_2} - B\binom{X_2}{Y_1+Y_2}}{X_1 - X_2}.$$

### Remark

- Can be written down explicitly in arbitrary depth.
- This corresponds to the harmonic product of MZV, i.e. compare it to

$$Z(X_1)Z(X_2) = Z(X_1, X_2) + Z(X_2, X_1) + \frac{Z(X_1) - Z(X_2)}{X_1 - X_2}.$$

## ④ Extension of the DSH relations - Swap

### Definition

A bimould  $B$  is called **swap invariant** if

$$B\begin{pmatrix} X_1, \dots, X_r \\ Y_1, \dots, Y_r \end{pmatrix} = B\begin{pmatrix} Y_1 + \dots + Y_r, Y_1 + \dots + Y_{r-1}, \dots, Y_1 + Y_2, Y_1 \\ X_r, X_{r-1} - X_r, \dots, X_2 - X_3, X_1 - X_2 \end{pmatrix}.$$

$$B\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \stackrel{\text{sw}}{=} B\begin{pmatrix} Y_1 \\ X_1 \end{pmatrix}, \quad B\begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix} \stackrel{\text{sw}}{=} B\begin{pmatrix} Y_1 + Y_2, Y_1 \\ X_2, X_1 - X_2 \end{pmatrix}.$$

## ④ Extension of the DSH relations - q-shuffle

Recall **symmetrility** and **swap** in low depth

$$B\binom{X_1}{Y_1} \stackrel{\text{sw}}{=} B\binom{Y_1}{X_1}, \quad B\binom{X_1, X_2}{Y_1, Y_2} \stackrel{\text{sw}}{=} B\binom{Y_1 + Y_2, Y_1}{X_2, X_1 - X_2},$$

$$B\binom{X_1}{Y_1}B\binom{X_2}{Y_2} \stackrel{\text{IL}}{=} B\binom{X_1, X_2}{Y_1, Y_2} + B\binom{X_2, X_1}{Y_2, Y_1} + \frac{B\binom{X_1}{Y_1+Y_2} - B\binom{X_2}{Y_1+Y_2}}{X_1 - X_2}.$$

### Definition

Swap + Symmtril + Swap = **q-shuffle**

$$\begin{aligned} B\binom{X_1}{Y_1}B\binom{X_2}{Y_2} &\stackrel{\text{sw}}{=} B\binom{Y_1}{X_1}B\binom{Y_2}{X_2} \\ &\stackrel{\text{IL}}{=} B\binom{Y_1, Y_2}{X_1, X_2} + B\binom{Y_2, Y_1}{X_2, X_1} + \frac{B\binom{Y_1}{X_1+X_2} - B\binom{Y_2}{X_1+X_2}}{Y_1 - Y_2} \\ &\stackrel{\text{sw}}{=} B\binom{X_1 + X_2, X_1}{Y_2, Y_1 - Y_2} + B\binom{X_1 + X_2, X_2}{Y_1, Y_2 - Y_1} + \frac{B\binom{X_1 + X_2}{Y_1} - B\binom{X_1 + X_2}{Y_2}}{Y_1 - Y_2}. \end{aligned}$$

## ④ Extension of the DSH relations - q-double shuffle

### Definition

A bimould satisfies **q-double shuffle** (in depth 2) if

$$\begin{aligned} B\binom{X_1}{Y_1}B\binom{X_2}{Y_2} &= B\binom{X_1, X_2}{Y_1, Y_2} + B\binom{X_2, X_1}{Y_2, Y_1} + \frac{B\binom{X_1}{Y_1+Y_2} - B\binom{X_2}{Y_1+Y_2}}{X_1 - X_2} \\ &= B\binom{X_1 + X_2, X_1}{Y_2, Y_1 - Y_2} + B\binom{X_1 + X_2, X_2}{Y_1, Y_2 - Y_1} + \frac{B\binom{X_1 + X_2}{Y_1} - B\binom{X_1 + X_2}{Y_2}}{Y_1 - Y_2}, \end{aligned}$$

i.e.  $B$  is symmetril and satisfies the  $q$ -shuffle product formula.

- Clearly: Symmetril + Swap invariant  $\implies$  q-double shuffle.
- Compare this to the double shuffle relations

$$\begin{aligned} Z(X_1)Z(X_2) &= Z(X_1, X_2) + Z(X_2, X_1) + \frac{Z(X_1) - Z(X_2)}{X_1 - X_2} \\ &= Z(X_1 + X_2, X_1) + Z(X_1 + X_2, X_2) + z_2. \end{aligned}$$

## ④ Extension of the DSH relations - "Constant function" $\rightsquigarrow$ Sol. to dsh

Solution to q-dsh  $\Rightarrow$  solution to dsh

### Proposition

Let  $B$  be **symmetril** and **swap invariant** with  $\frac{d}{dX} \frac{d}{dY} B\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right) = 0$ . Then

$$Z(X) = B\left(\begin{smallmatrix} X \\ 0 \end{smallmatrix}\right), \quad Z(X_1, X_2) = B\left(\begin{smallmatrix} X_1, X_2 \\ 0, 0 \end{smallmatrix}\right)$$

satisfies the double shuffle relations.

### Proof:

$$\begin{aligned} \frac{B\left(\begin{smallmatrix} X_1 + X_2 \\ Y_1 \end{smallmatrix}\right) - B\left(\begin{smallmatrix} X_1 + X_2 \\ Y_2 \end{smallmatrix}\right)}{Y_1 - Y_2} & \Big|_{Y_1=Y_2=0} = \sum_{k \geq 1} b\binom{k}{1} (X_1 + X_2)^{k-1} \\ & = b\binom{1}{1} \stackrel{\text{SW}}{=} b\binom{2}{0} = z_2. \end{aligned}$$

**Interpretation:** "When the derivative vanishes (constant function) then we obtain a solution to classical dsh (equations for numbers)".

#### ④ Extension of the DSH relations - Sol. to dsh $\rightsquigarrow$ Sol. to $q$ -dsh

Theorem (B.-Matthes-Kühn, 2020+)

Let  $Z$  satisfy the double shuffle relations (in all depths), then there exists an explicit construction of a symmetril and swap invariant bimould  $B$ .

For example, in lowest depth the bimould

$$B\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = Z(X_1) + Z(Y_1),$$

$$B\begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix} = Z(X_1, X_2) + Z(Y_1 + Y_2, Y_1) + Z(X_2)Z(Y_1) + \frac{1}{2}z_2$$

is **symmetril** and **swap invariant**

**Interpretation:** "Numbers can be viewed as constant functions".

## ④ Extension of the DSH relations - Combinatorial MES

Theorem (B.-Kühn-Matthes 2020+, B.-Burmester 2020+)

There exist a **symmetril** and **swap invariant** bimould  $\mathfrak{G}$  (up to depth 3) which in depth one is given by the generating series of derivatives of Eisenstein series.

### Remark

- The construction of this bimould is inspired by the Fourier expansion of multiple Eisenstein series.
- We have a conjectured construction for all depths (j.w. A. Burmester).

## ④ Extension of the DSH relations - Combinatorial MES

### Definition

We define the **combinatorial multiple Eisenstein series**  $G$  in depth  $\leq 2$  by

$$\mathfrak{G}\left(\frac{X}{Y}\right) =: \sum_{\substack{k \geq 1 \\ d \geq 0}} G\binom{k}{d} X^{k-1} \frac{Y^d}{d!},$$

$$\mathfrak{G}\left(\frac{X_1, X_2}{Y_1, Y_2}\right) =: \sum_{\substack{k_1, k_2 \geq 1 \\ d_1, d_2 \geq 0}} G\binom{k_1, k_2}{d_1, d_2} X_1^{k_1-1} X_2^{k_2-1} \frac{Y_1^{d_1}}{d_1!} \frac{Y_2^{d_2}}{d_2!}.$$

- In depth one  $G\binom{k}{d}$  is basically the  $d$ -th derivative of  $G_{k-d}$ .
- In depth two the  $G\binom{k_1, k_2}{0, 0}$  are (almost) the double Eisenstein series.
- The symmetrility  $\mathfrak{G}$  of gives

$$G\binom{k_1}{d_1} G\binom{k_2}{d_2} = G\binom{k_1, k_2}{d_1, d_2} + G\binom{k_2, k_1}{d_2, d_1} + G\binom{k_1 + k_2}{d_1 + d_2}.$$

## ④ Extension of the DSH relations - The space of CMES

### Definition

Space of double combinatorial multiple Eisenstein series of weight  $K \geq 1$ :

$$\mathfrak{D}_K = \left\langle G\binom{k}{d}, G\binom{k_1, k_2}{d_1, d_2} \mid \begin{array}{l} k+d=k_1+k_2+d_1+d_2=K \\ k, k_1, k_2 \geq 1, d, d_1, d_2 \geq 0 \end{array} \right\rangle_{\mathbb{Q}}$$

### Proposition

$$\begin{aligned} q \frac{d}{dq} \mathfrak{G}\binom{X_1}{Y_1} &= \frac{d}{dX_1} \frac{d}{dY_1} \mathfrak{G}\binom{X_1}{Y_1}, \\ q \frac{d}{dq} \mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2} &= \left( \frac{d}{dX_1} \frac{d}{dY_1} + \frac{d}{dX_2} \frac{d}{dY_2} \right) \mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2}. \end{aligned}$$

### Corollary

Combinatorial multiple Eisenstein series are closed under  $q \frac{d}{dq}$ . In particular

$$q \frac{d}{dq} \mathfrak{D}_K \subset \mathfrak{D}_{K+2}.$$

## ④ Extension of the DSH relations - The space of CMES

$$\mathfrak{D}_K = \left\langle G\binom{k}{d}, G\binom{k_1, k_2}{d_1, d_2} \mid \begin{array}{l} k+d=k_1+k_2+d_1+d_2=K \\ k, k_1, k_2 \geq 1, d, d_1, d_2 \geq 0 \end{array} \right\rangle_{\mathbb{Q}}$$

$$\mathfrak{D}_K^0 = \left\langle G\binom{k}{0}, G\binom{k_1, k_2}{0, 0} \in \mathfrak{D}_K \right\rangle_{\mathbb{Q}}$$

### Proposition

- $\mathfrak{D}_K$  contains the space of **quasi modular forms**  $\mathbb{Q}[\tilde{G}_2, \tilde{G}_4, \tilde{G}_6]_K$  of weight  $K$ .
- $\mathfrak{D}_K^0$  contains the space of **modular forms**  $\mathbb{Q}[\tilde{G}_4, \tilde{G}_6]_K$  of weight  $K$

Numerical computer calculation give:

| $k$                                     | 1 | 2 | 3 | 4  | 5  | 6  | 7  | 8  |
|-----------------------------------------|---|---|---|----|----|----|----|----|
| $\dim \mathfrak{D}_K \stackrel{?}{=}$   | 1 | 2 | 3 | 5  | 7  | 11 | 14 | .. |
| $\dim \mathfrak{D}_K^0 \stackrel{?}{=}$ | 1 | 2 | 3 | 3  | 4  | 4  | 5  | 5  |
| # generators of $\mathfrak{D}_K$        | 1 | 3 | 7 | 14 | 25 | 41 | 63 | 92 |

# Summary

## 数

## 関数

1重の対象

Riemannゼータ値

$$\zeta(k)$$

Eisenstein級数  
(モジュラー形式)

$$G_k(\tau) = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \frac{n^{k-1} q^n}{1-q^n}$$

多重化された対象

多重ゼータ値

$$\zeta(k_1, \dots, k_r)$$

多重Eisenstein級数  
(Gangl-金子-Zagier, B.)

$$G_{k_1, \dots, k_r}(\tau) = \zeta(k_1, \dots, k_r) + \sum_{n>0} a_n q^n$$

関係式

複シャッフル関係式

$$\begin{aligned} Z(X_1)Z(X_2) &= Z(X_1, X_2) + Z(X_2, X_1) + \frac{Z(X_1) - Z(X_2)}{X_1 - X_2} \\ &= Z(X_1 + X_2, X_1) + Z(X_1 + X_2, X_2) + z_2. \end{aligned}$$

Symmetril & Swap invariant

$$\begin{aligned} B\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} &\stackrel{\text{sw}}{=} B\begin{pmatrix} Y_1 \\ X_1 \end{pmatrix}, \quad B\begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix} \stackrel{\text{sw}}{=} B\begin{pmatrix} Y_1+Y_2, Y_1 \\ X_2, X_1-X_2 \end{pmatrix}, \\ B\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}B\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} &\stackrel{\text{a}}{=} B\begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix} + B\begin{pmatrix} X_2, X_1 \\ Y_2, Y_1 \end{pmatrix} + \frac{B\begin{pmatrix} X_1 \\ Y_1+Y_2 \end{pmatrix} - B\begin{pmatrix} X_2 \\ Y_1+Y_2 \end{pmatrix}}{X_1 - X_2}. \end{aligned}$$

## ⑤ Bonus - Symmetril in depth 3

### Definition

A bimould  $B$  is **symmetril** (up to depth 3), if

$$B\binom{X_1}{Y_1}B\binom{X_2}{Y_2} = B\binom{X_1, X_2}{Y_1, Y_2} + B\binom{X_2, X_1}{Y_2, Y_1} + \frac{B\binom{X_1}{Y_1+Y_2} - B\binom{X_2}{Y_1+Y_2}}{X_1 - X_2},$$

$$\begin{aligned} B\binom{X_1}{Y_1}B\binom{X_2, X_3}{Y_2, Y_3} &= B\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3} + B\binom{X_2, X_1, X_3}{Y_2, Y_1, Y_3} + B\binom{X_2, X_3, X_1}{Y_2, Y_3, Y_1} \\ &+ \frac{B\binom{X_1, X_3}{Y_1+Y_2, Y_3} - B\binom{X_2, X_3}{Y_1+Y_2, Y_3}}{X_1 - X_2} + \frac{B\binom{X_2, X_1}{Y_2, Y_1+Y_3} - B\binom{X_2, X_3}{Y_2, Y_1+Y_3}}{X_1 - X_3}. \end{aligned}$$