Multiple zeta values and their relations

Created by Henrik Bachmann (Nagoya University) This poster can be downloaded at **www.henrikbachmann.com**

Notation

For $k_1,\dots,k_{r-1}\geq 1, k_r\geq 2$ the **multiple zeta values** are defined by

$$\zeta(k_1,\dots,k_r)=\sum_{0<m_1<\dots<m_r}\frac{1}{m_1^{k_1}\dots m_r^{k_r}}$$

$\mathcal{Z}$: $\mathbb{Q}$-algebra of multiple zeta values

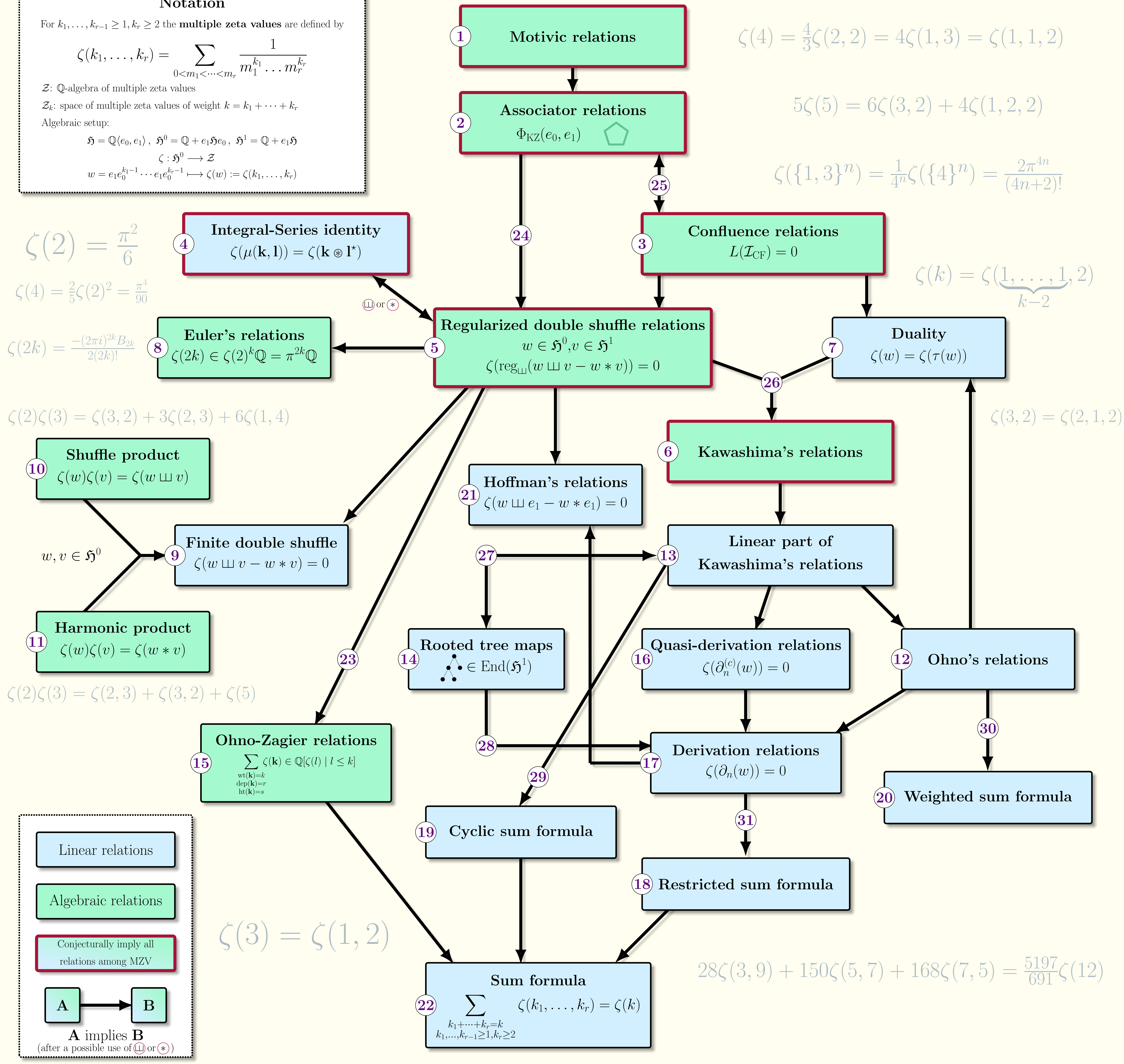
$\mathcal{Z}_k$: space of multiple zeta values of weight $k=k_1+\dots+k_r$

Algebraic setup:

$$\mathfrak{H}=\mathbb{Q}\langle e_0,e_1\rangle\,,\ \mathfrak{H}^0=\mathbb{Q}+e_1\mathfrak{H}e_0\,,\ \mathfrak{H}^1=\mathbb{Q}+e_1\mathfrak{H}$$

$$\zeta:\mathfrak{H}^0\longrightarrow\mathcal{Z}$$

$$w=e_1e_0^{k_1-1}\cdots e_1e_0^{k_r-1}\longmapsto\zeta(w):=\zeta(k_1,\dots,k_r)$$



Conjectured dimensions and number of linearly independent relations

32 Conjecture (Zagier)

The generating series of the dimension of $\mathcal{Z}_k$ is given by

$$\sum_{k\geq0}\dim_{\mathbb{Q}}\mathcal{Z}_kX^k=\frac{1}{1-X^2-X^3}.$$

In particular for $k\geq3$ we have $\dim_{\mathbb{Q}}\mathcal{Z}_k=\dim_{\mathbb{Q}}\mathcal{Z}_{k-2}+\dim_{\mathbb{Q}}\mathcal{Z}_{k-3}$.

weight k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
# of generators	1	0	1	2	4	8	16	32	64	128	256	512	1024	2048	4096
# of relations $\stackrel{?}{=}$	0	0	0	1	3	6	14	29	60	123	249	503	1012	2032	4075
$\dim_{\mathbb{Q}}\mathcal{Z}_k\stackrel{?}{=}$	1	0	1	1	1	2	2	3	4	5	7	9	12	16	21

Number of generators, conjectured number of linearly independent relations and conjectured dimension of $\mathcal{Z}_k$

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