Conjecture (Zagier)

The generating series of the dimension of $\mathcal{Z}_k$ is given by

$$\sum_{n \geq 0} \dim \mathcal{Z}_n X^n = \frac{1}{1 - X^2 X^3}.$$ 

In particular for $k \geq 3$ we have $\dim \mathcal{Z}_k = \dim \mathcal{Z}_{k-1} + \dim \mathcal{Z}_{k-3}$.

Conjectured dimensions and number of linearly independent relations

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Number of generators, conjectured number of linearly independent relations and conjectured dimension of $\mathcal{Z}_k$.

References