Homework 3

Deadline: 3rd August 2025 (23:55 JST) via TACT.

Exercise 1. We define for $k \geq 1$ the Eulerian polynomials $P_k(X)$ and the power series $R_k(X)$ by

$$R_k(X) = \frac{P_k(X)}{(1-X)^k} = \sum_{d>0} \frac{d^{k-1}}{(k-1)!} X^d \,.$$

(i) Prove [B, Lemma 1.60], i.e., show that we have $P_k(0) = 0$ and $P_k(1) = 1$ for all $k \geq 1$.

(ii) Prove [B, Lemma 1.64], i.e., show that for all $k_1, k_2 \geq 1$

$$R_{k_1}(X) \cdot R_{k_2}(X) = R_{k_1+k_2}(X) + \sum_{j=1}^{k_1+k_2-1} \left(\lambda^j_{k_1,k_2} + \lambda^j_{k_2,k_1} \right) R_j(X) \,,$$

where the rational numbers $\lambda^j_{k_1,k_2}$ are given by

$$\lambda^j_{k_1,k_2} = (-1)^{k_2-1} \binom{k_1+k_2-1-j}{k_1-j} \frac{B_{k_1+k_2-j}}{(k_1+k_2-j)!} \,,$$

and where we use the convention $\binom{n}{k} = 0$ for $k < 0$.

(iii) Define on $\mathbb{Q}A_z$ the product (Recall: $A_z = \{z_1, z_2, z_3, \dots\}$)

$$z_{k_1} \hat{\diamond} z_{k_2} = z_{k_1+k_2} + \sum_{j=1}^{k_1+k_2-1} \left(\lambda^j_{k_1,k_2} + \lambda^j_{k_2,k_1} \right) z_j$$

and denote the corresponding quasi-shuffle product by $\hat{*} = *_{\hat{\diamond}}$. Show that the $\mathbb{Q}$-linear map

$$\begin{aligned} \mathbb{Q}\langle A_z \rangle &\longrightarrow \mathbb{Q}[[q]], \\ z_{k_1} \cdots z_{k_r} &\longmapsto \mathrm{g}(k_1, \dots, k_r) = \sum_{m_1 > \cdots > m_r > 0} \frac{P_{k_1}(q^{m_1})}{(1-q^{m_1})^{k_1}} \cdots \frac{P_{k_r}(q^{m_r})}{(1-q^{m_r})^{k_r}} \end{aligned}$$

is an algebra homomorphism from $(\mathbb{Q}\langle A_z \rangle, \hat{*})$ to $\mathbb{Q}[[q]]$.

Exercise 2.

(i) Show that for any admissible index $\mathbf{k}$ we have $z_1 \sqcup z_{\mathbf{k}} - z_1 * z_{\mathbf{k}} \in \mathfrak{H}^0$.

(ii) Prove Hoffman's relation [B, Proposition 2.41], i.e. show that for an admissible index $\mathbf{k} = (k_1, \dots, k_r)$ we have

$$\sum_{i=1}^{r} \zeta(k_1, \dots, k_{i-1}, k_i + 1, k_{i+1}, \dots, k_r) = \sum_{\substack{1 \leq i \leq r \\ k_i \geq 2}} \sum_{j=0}^{k_i-2} \zeta(k_1, \dots, k_{i-1}, k_i - j, j+1, k_{i+1}, \dots, k_r) \,.$$

References

[B] H. Bachmann, *Multiple zeta values*, Lecture notes V. 21, https://www.henrikbachmann.com/uploads/7/7/6/3/77634444/mzv_2025_lecture_notes_v21.pdf