Homework 2

Deadline: 8th June 2025 (23:55 JST) via TACT.

In [B, Chapter 2] we define the space $\mathfrak{H}^1 = \mathbb{Q}\langle z_1, z_2, \dots\rangle$ and call a monomial $z_{k_1}\cdots z_{k_r} \in \mathfrak{H}^1$ a word. On $\mathfrak{H}^1$ we then define the **stuffle product** $*$ as the $\mathbb{Q}$-bilinear product, which satisfies $1*w = w*1 = w$ for any word $w \in \mathfrak{H}^1$ and

$$z_i w_1 * z_j w_2 = z_i(w_1 * z_j w_2) + z_j(z_i w_1 * w_2) + z_{i+j}(w_1 * w_2)$$

for any $i, j \geq 1$ and words $w_1, w_2 \in \mathfrak{H}^1$. (For more details see [B, Sections 2.1.1 & 2.2.1])

Exercise 1.

(i) Calculate $z_a z_b * z_c z_d$ for $a, b, c, d \geq 1$.

(ii) Show that $\mathfrak{H}^1_* := (\mathfrak{H}^1, *)$ is a commutative $\mathbb{Q}$-algebra.

(iii) Show that the $\mathbb{Q}$-linear map $d : \mathfrak{H}^1 \longrightarrow \mathfrak{H}^1$ defined on words by[1]

$$d : z_{k_1}\cdots z_{k_r} \longmapsto \begin{cases} z_{k_2}\cdots z_{k_r} & , \text{ if } k_1 = 1, \\ 0 & , \text{ else} \end{cases}$$

is a derivation on $\mathfrak{H}^1_*$.

(iv) Define $\mathfrak{H}^0 := \ker(d)$. Show that for every $w \in \mathfrak{H}^1$ there exist unique $w_j \in \mathfrak{H}^0$ such that

$$w = \sum_{j=0}^{l} w_j * z_1^{*j}.$$

In other words, show that $\mathfrak{H}^1_* = \mathfrak{H}^0_*[z_1]$, where $\mathfrak{H}^0_* := (\mathfrak{H}^0, *)$.

(v) For $w = z_1 z_1 z_3 z_2$ determine the $w_j \in \mathfrak{H}^0$ in (iv).

Exercise 2.

(i) Prove [B, Proposition 1.32], i.e., show that for all $m \geq 1$ and $k_1, \dots, k_r \geq 1$ we have

$$S_m(k_1, \dots, k_r) = \sum_{j=0}^{r} (-1)^{k_1+\cdots+k_j} H_m(k_j, k_{j-1}, \dots, k_1) H_m(k_{j+1}, \dots, k_r),$$

where S_m is defined in [B, (1.16)] and H_m in [B, (1.8)].

(ii) Show that for a fixed $m \geq 1$ the $\mathbb{Q}$-linear map defined on a word by

$$h_m : z_{k_1}\cdots z_{k_r} \longmapsto H_m(k_1, \dots, k_r)$$

is a $\mathbb{Q}$-algebra homomorphism $h_m : \mathfrak{H}^1_* \to \mathbb{Q}$.

(iii) Use the results[2] from Exercise 1 to show that for any $k_1, \dots, k_r \geq 1$ we have

$$\sum_{j=0}^{r} (-1)^{k_1+\cdots+k_j} z_{k_j} z_{k_{j-1}} \cdots z_{k_1} * z_{k_{j+1}} \cdots z_{k_r} \in \mathfrak{H}^0.$$

Conclude, by using (ii), that $\lim_{m\to\infty} S_m(k_1, \dots, k_r) \in \mathcal{Z}$ for all $k_1, \dots, k_r \geq 1$.

Exercise 3. Calculate [B, (2.2)] by hand, i.e. show

$$\zeta(2,3) = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} \int_0^{t_2} \frac{dt_3}{t_3} \int_0^{t_3} \frac{dt_4}{t_4} \int_0^{t_5} \frac{dt_5}{1-t_5}$$

without using [B, Corollary 2.4].

References

[B] H. Bachmann, *Multiple zeta values*, Lecture notes V. 13, `https://www.henrikbachmann.com/uploads/7/7/6/3/77634444/mzv_2025_lecture_notes_v13.pdf`

[1] Notice that for $r = 1$ we view $z_{k_2}\cdots z_{k_r}$ as the empty word, i.e. we define $d(z_1) = 1$.

[2] You are also allowed to use the statements from Exercise 1 if you did not prove them.