Homework 1

Deadline: 8th May 2025 (23:55 JST) via TACT.

Submission: Create one PDF file which contains your name on the first page and submit it before the deadline on TACT under the 'Homework 1' assignment. Please use the following format as a filename: `FAMILYNAME_Givenname_MZV_HW1.pdf` (e.g. `EULER_Leonhard_MZV_HW1.pdf`). Your PDF can be either handwritten (scanned or tablet) or written in LaTeX. You can submit your homework solutions in English or Japanese (or German).

Exercise 1.

(i) Prove Proposition 1.10, i.e., show that for $k_1, k_2 \geq 2$ we have

$$\zeta(k_1)\zeta(k_2) = \sum_{j=2}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) \zeta(j, k_1+k_2-j).$$

Hint: Use the partial fraction expansion for $a, b \geq 1$:

$$\frac{1}{x^a y^b} = \sum_{j=1}^{a+b-1} \left(\frac{\binom{j-1}{a-1}}{(x+y)^j y^{a+b-j}} + \frac{\binom{j-1}{b-1}}{(x+y)^j x^{a+b-j}} \right).$$

(This formula may be used without proof.)

(ii) Use (i) together with the formula $\zeta(k_1)\zeta(k_2) = \zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1 + k_2)$ to prove the relations:

$$\zeta(6) = 6\zeta(3,3) - 3\zeta(4,2),$$
$$\zeta(7) = 4\zeta(3,4) + 3\zeta(4,3) - 2\zeta(5,2).$$

Exercise 2. Find explicit expressions for the number of indices and admissible indices of a given weight k (and depth r). Specifically, for $k, r \geq 0$, find formulas for the following four quantities:

$$I_k = \sum_{r \geq 0} I_{k,r}, \qquad I_{k,r} = \left| \{ \mathbf{k} \in \mathbb{Z}_{\geq 1}^r \mid \mathrm{wt}(\mathbf{k}) = k \} \right|,$$
$$I_k^0 = \sum_{r \geq 0} I_{k,r}^0, \qquad I_{k,r}^0 = \left| \{ \mathbf{k} \in \mathbb{Z}_{\geq 1}^r \mid \mathrm{wt}(\mathbf{k}) = k, \mathbf{k} \text{ is admissible} \} \right|.$$

Exercise 3.

(i) Show that Conjecture 1.13, together with Proposition 1.12, would imply that all multiple zeta values (except for $\zeta(\emptyset) = 1$) are transcendental.

(ii) Show that Conjecture 1.18 (Hoffman) would imply Conjecture 1.15 (Zagier).

Exercise 4.

(i) Show that there exists an injective ring homomorphism

$$\iota : \mathbb{Q} \to \mathcal{A},$$

i.e., prove that $\mathcal{A}$ is a $\mathbb{Q}$-algebra. Furthermore, show that ι cannot be surjective.

(ii) Show that $\sqrt{2} \notin \mathcal{A}$, i.e., prove that there does not exist any $a \in \mathcal{A}$ such that $a^2 - 2 = 0$ in $\mathcal{A}$.

Exercise 5. Show that for any $k_1, \ldots, k_r \in \mathbb{Z}$, we have $\zeta_{\mathcal{A}}(k_1, \ldots, k_r) \in \mathcal{Z}^{\mathcal{A}}$. i.e., show that $\zeta_{\mathcal{A}}(k_1, \ldots, k_r)$ can be written as a $\mathbb{Q}$-linear combination of finite multiple zeta values with positive integer arguments.