MODULAR FORMS AND q-ANALOGUES OF MODIFIED DOUBLE ZETA VALUES

HENRIK BACHMANN

Modified double zeta values ($r \geq 1, s \geq 2$):

$$\begin{aligned}\hat{\zeta}(r,s) &= \sum_{0<m<n} \frac{1}{(m+n)^r n^s} \\ &= 2^{s-1}\,\mathrm{Li}_{r,s}(-1) + (2^{s-1}-1)\zeta(r,s) - \zeta(r+s)\,.\end{aligned}$$

Completed L-function and period polynomial of a modular form $f = \sum_{n\geq 0} a_n q^n \in M_k$:

$$L_f^*(s) = \int_0^\infty (f(it)-a_0)\, t^{s-1} dt \overset{\Re(s)\gg 0}{=} (2\pi)^{-s}\Gamma(s) \sum_{n>0} \frac{a_n}{n^s}\,,$$

$$\begin{aligned}P_f^+(X,Y) &= \sum_{\substack{r+s=k\\ r,s\geq 1 \text{ odd}}} (-1)^{\frac{s-1}{2}} \binom{k-2}{s-1} L_f^*(s) X^{r-1} Y^{s-1} \\ &= L_f^*(1)(X^{k-2}-Y^{k-2}) + P_f^{+,0}(X,Y)\,.\end{aligned}$$

Theorem. *For a cuspidal Hecke eigenform $f \in S_k$ define the coefficients $q_{r,s}^f \in \mathbb{C}$ by*

$$P_f^{+,0}(X+Y,X) = \sum_{\substack{r+s=k\\ r,s\geq 1}} \binom{k-2}{r-1} q_{r,s}^f X^{r-1} Y^{s-1}\,.$$

Then f can be written as

$$\frac{L_f^*(1)}{2(k-2)!} f = \sum_{\substack{r+s=k\\ r,s\geq 2}} q_{r,s}^f\, \hat{\zeta}_q(r,s) - \lambda_f \zeta_q(k) - R_f(q)\,,$$

where

$$\lambda_f = \frac{k-1}{2} \left(\sum_{\substack{r+s=k\\ r,s\geq 3 \text{ odd}}} \frac{(-1)^{\frac{s-1}{2}}}{r\, 2^{r-1}} \binom{k-2}{s-1} L_f^*(s) - L_f^*(1) \right),$$

$$\begin{aligned}R_f(q) = &\sum_{\substack{r+s=k\\ r,s\geq 3 \text{ odd}}} (-1)^{\frac{s-1}{2}} \binom{k-2}{s-1} L_f^*(s) \left(\sum_{j=1}^{r-1} \binom{r}{j} \frac{B_j \cdot (k-j-1)!}{r 2^{r-j}(k-2)!}\, \zeta_q(k-j) - \frac{1}{2^r}\, \zeta_q^{\mathbf{e}}(k-1) \right) \\ &+ \sum_{\substack{r+s=k\\ r,s\geq 3 \text{ odd}}} (-1)^{\frac{s-1}{2}} \binom{k-2}{s-1} L_f^*(s) \sum_{\substack{0\leq j\leq r-1\\ 1\leq l\leq r-j}} \binom{r}{j}\binom{r-j}{l} \frac{(-1)^l B_j \cdot (k-j-l-1)!}{r 2^{r-j}(k-2)!}\, \zeta_q^{\mathbf{o}}(k-j-l)\,,\end{aligned}$$

$$\zeta_q^{\mathbf{e}}(k) = \sum_{\substack{a,d>0\\ d \text{ even}}} \frac{d^{k-1}}{(k-1)!} q^{ad}\,, \qquad \zeta_q^{\mathbf{o}}(k) = \sum_{\substack{a,d>0\\ d \text{ odd}}} \frac{d^{k-1}}{(k-1)!} q^{ad}\,, \qquad \zeta_q(k) = \zeta_q^{\mathbf{e}}(k) + \zeta_q^{\mathbf{o}}(k)\,,$$

$$\hat{\zeta}_q(r,s) = \sum_{\substack{0<m<n\\ b,d>0}} \frac{b^{r-1}}{(r-1)!} \frac{d^{s-1}}{(s-1)!} q^{(m+n)b+nd} = \sum_{0<m<n} \frac{Q_r(q^{m+n})}{(1-q^{m+n})^r} \frac{Q_s(q^n)}{(1-q^n)^s}\,.$$

Corollary. *For a cuspidal Hecke eigenform $f \in S_k$ the following relation holds*

$$\sum_{\substack{r+s=k\\ r,s\geq 2}} q_{r,s}^f \hat{\zeta}(r,s) = \lambda_f\, \zeta(k)\,.$$

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There is also a "Eisenstein series"-version of the Theorem before (not part of the talk):

Theorem. *For all even $k \geq 4$ we have*

$$\zeta_q(k) = 2^{k-1} \sum_{\substack{r+s=k \\ r\geq 1, s\geq 2}} \hat{\zeta}_q(r,s) - E_k(q)\,,$$

where the q-series $E_k(q) \in \mathbb{Q}[[q]]$ is given by

$$E_k(q) = 2^{k-2}\zeta_q(k-1) - \frac{2^{k-2}}{(k-2)} q \frac{d}{dq}\zeta_q(k-2) + \sum_{j=2}^{k-2} \frac{2^j B_j}{j!}\zeta_q(k-j)$$
$$+ \sum_{\substack{0\leq j\leq k-2 \\ 1\leq l\leq k-j-1 \\ (l,j)\neq(1,0)}} \frac{(-1)^l 2^j B_j}{j!\,l!}\zeta_q^{\circ}(k-j-l)\,.$$

Again by multiplying with $(1-q)^k$ and taking the limit $q \to 1$ the E_k-part vanishes and we get, for the even weight case (the odd weight can be proved separately), the following sum formula.

Corollary. *For all $k \geq 3$ we have*

$$\zeta(k) = 2^{k-1} \sum_{\substack{r+s=k \\ r\geq 1, s\geq 2}} \hat{\zeta}(r,s)\,.$$

(un)important words appearing in the talk

Deutsch/ドイツ語/German	**Japanisch/日本語/Japanese**	**Englisch/英語/English**
Eisen	鉄(てつ)	iron
Stein	石(いし)	stone
Hecke	垣根(かきね)	hedge
eigen	我(わ)が	own
Modulform(en)	モジュラー形式(けいしき)	modular form(s)
Spitzenform(en)	カスプ形式(けいしき)	cusp form(s)
q-Analoga	q-類似(るいじ)	q-analogue
modifiziert(er,e,es,en)	変形(へんけい)	modified / modification
Doppel Zeta-Wert(e)	二重(じゅう)ゼータ値(ち)	double zeta value(s)