MATHEMATICS
FOR MACHINE LEARNING

Nagoya University, Fall 2023

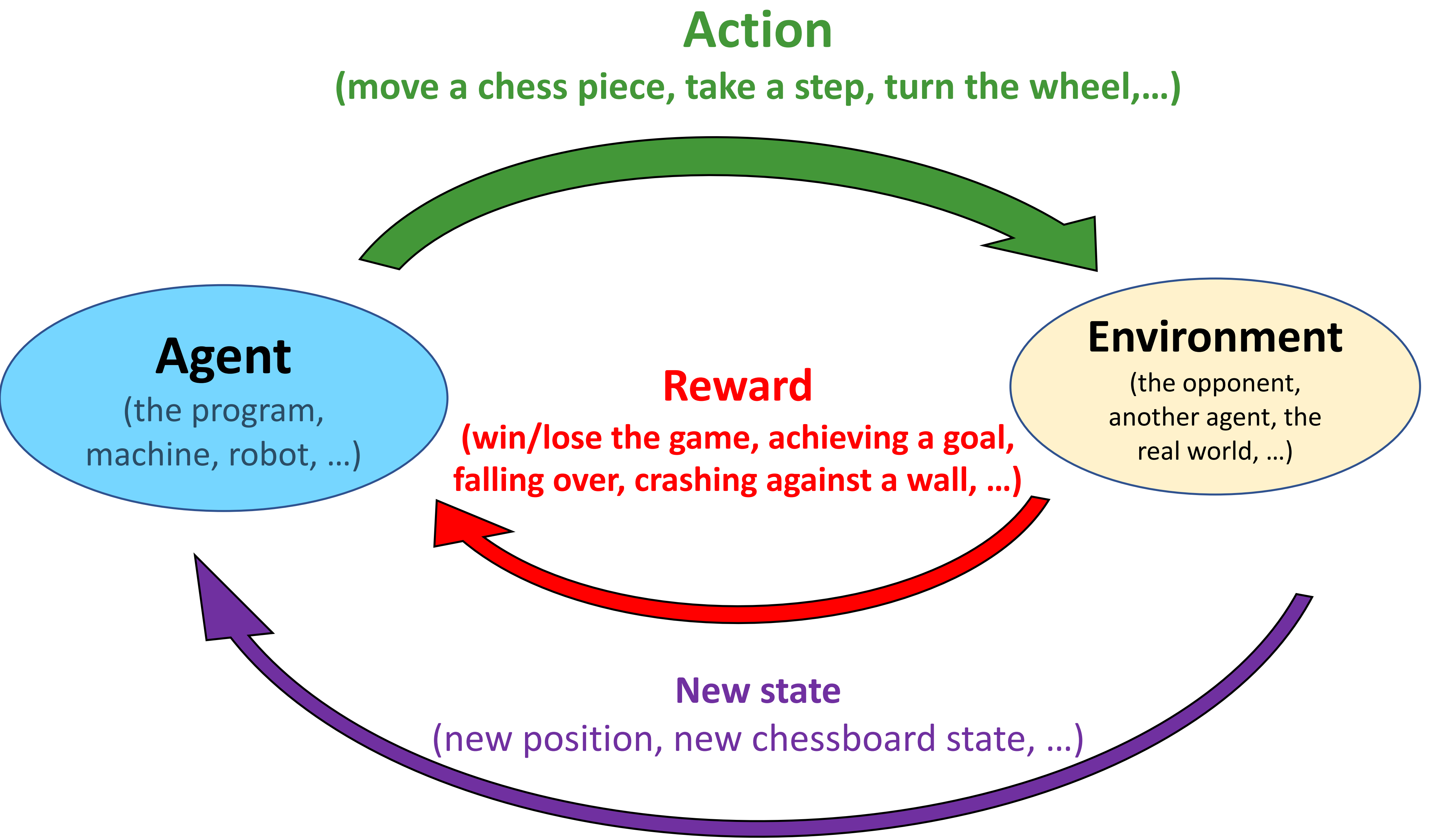
Lecture 9

Reinforcement Learning: Q-Learning II

This week Tutorial: Friday 8th Dec. 6th period

https://www.henrikbachmann.com/mml2023.html

Recall Reinforcement learning: Basic idea



Markov decision process

Definition 5.1. *A* **Markov decision process (MDP)** *is a tuple* (S, A, T, R), *where*

(i) S *is a set of states called the* **state space**,

sometimes

A_s for each $s \in S$

(ii) A *is a set of actions called the* **action space**,

(iii) T *is a map*

$$T : S \times A \times S \to [0,1]\,,$$

called the **transition probability function**,

(iv) R *is a map*

$$R : S \times A \times S \to \mathbb{R}\,,$$

called the **reward function**.

Interpretation:

$T(s, a, s')$: The probability that one reaches state s' when taking action a in state s

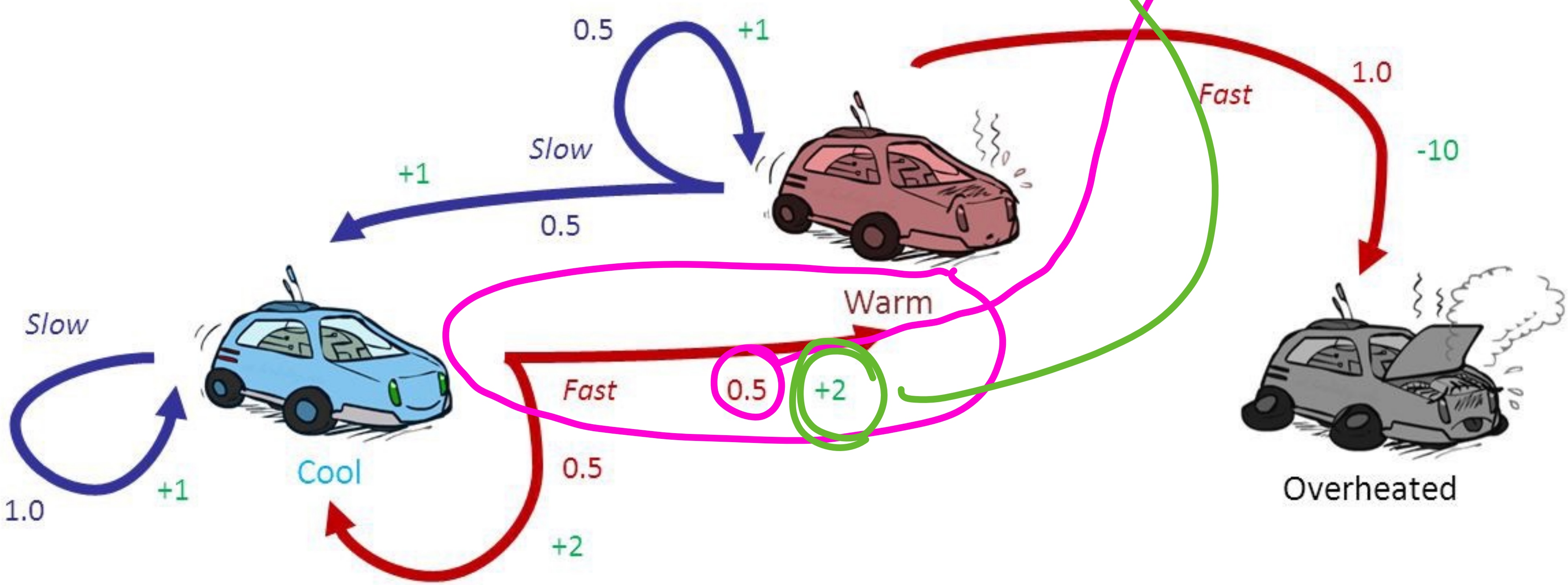
$R(s, a, s')$: The reward that one gets by going from state s to s' by doing action a

Markov decision process: Example

- A robot car wants to travel far, quickly
- Three states: Cool (c, s), Warm (w, f), Overheated
- Two actions: *Slow*, *Fast*
- Going faster gets double reward

$T(c, f, w) = \frac{1}{2}$

$R(c, f, w) = 2$



Markov decision process: Dynamics

i) Start at some state $s_0 \in S$.

ii) Choose an action $a_0 \in A$.

iii) Obtain a new state $s_1 \in S$ (with probability $T(s_0, a_0, s_1)$) and a reward $R(s_0, a_0, s_1)$.

iv) Repeat until one reaches a terminal state or a fixed number of steps N.

($N = \infty$ possible)

Goal: Choose the actions $a_0, a_1, a_2, \ldots$ at each state such that

$$\sum_{j=0}^{N} R(s_j, a_j, s_{j+1})$$

gets big.

Markov decision process: Dynamics

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$$\sum_{j=0}^{N} R(s_j, a_j, s_{j+1})$$

gets big.

Problem: This could be an infinite sum and therefore one introduces a **discount factor** $\gamma \in [0, 1]$ and then considers the discounted total reward:

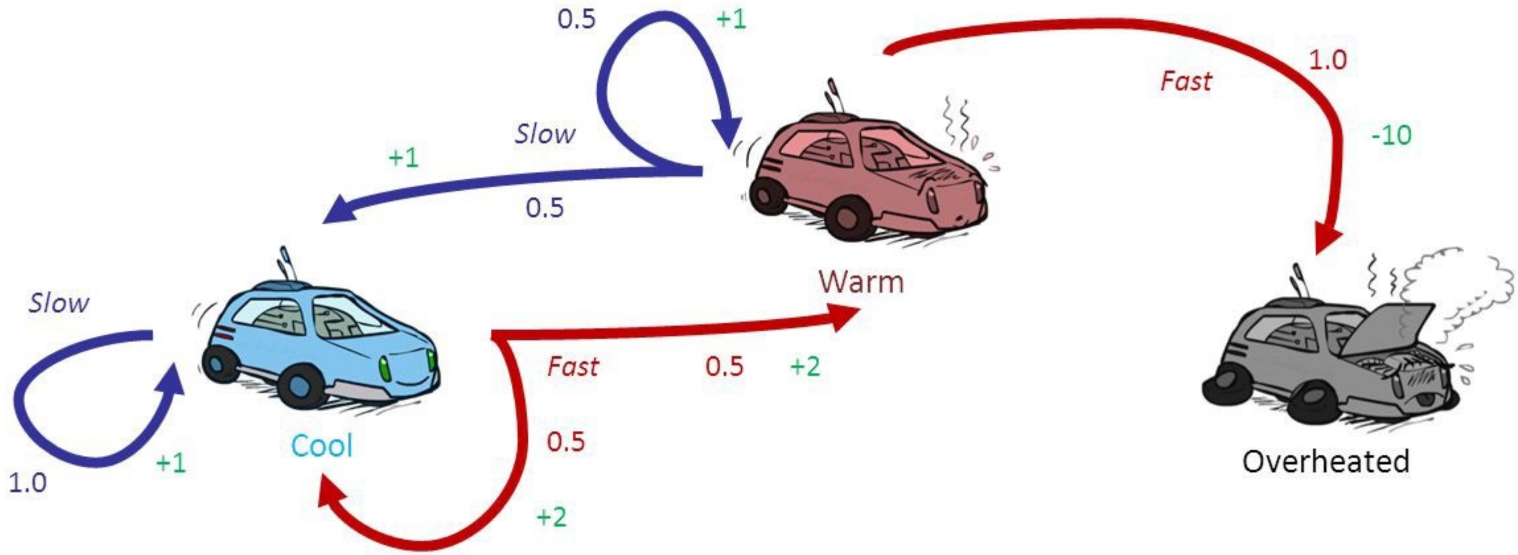
$$\sum_{j \geq 0} \gamma^j R(s_j, a_j, s_{j+1})\,.$$

Another interpretation: Immediate rewards count more than delayed rewards.

Total reward: Car example

For given sequences of states $(s_0, s_1, \dots)$ and actions $(a_0, a_1, \dots)$ the **discounted total reward** (with **discount** $\gamma \in [0,1]$) is given by

$$\sum_{j \geq 0} \gamma^j R(s_j, a_j, s_{j+1}) \, .$$



Markov decision process: Policy

A **policy** is a function $\pi : S \rightarrow A$.

Interpretation: A policy suggests the action you should take when being in a certain state.

Goal: Find the optimal policy which increases the (discounted) total reward.

Markov decision process: Value of a policy

A **policy** is a function $\pi : S \to A$.

Goal: Find the optimal policy which increases the (discounted) total reward.

The **value of a policy** π *at state* $s \in S$ *is defined by*

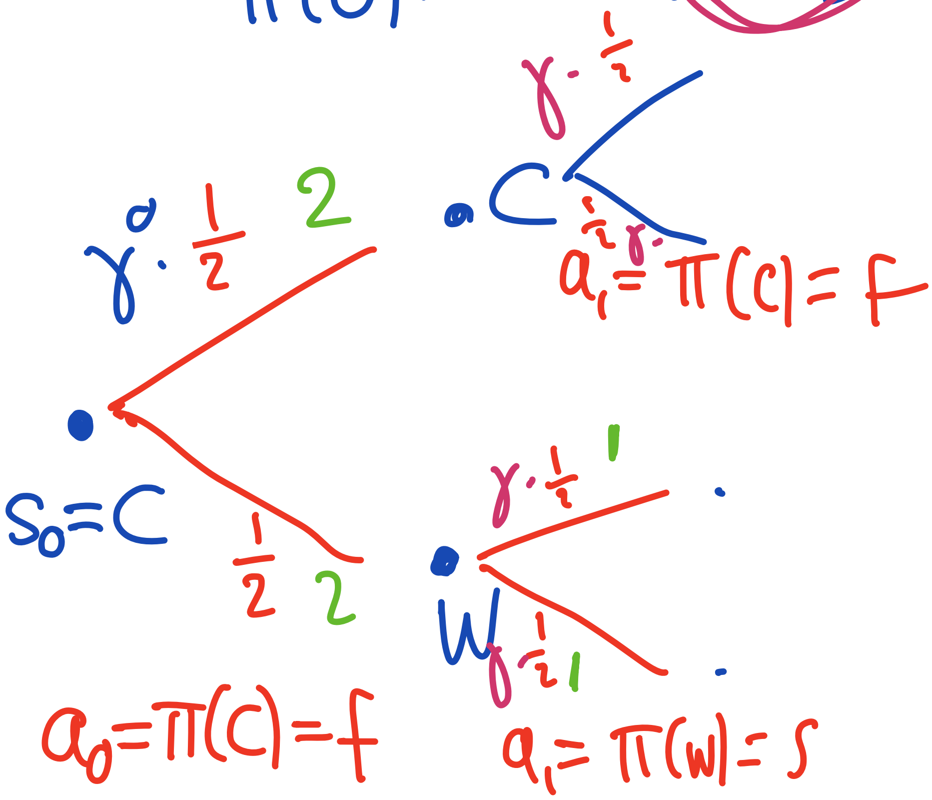
$$V_\pi(s) = E\left[\sum_{j \geq 0} \gamma^j R(s_j, \pi(s_j), s_{j+1}) \mid s_0 = s\right]$$

Our policy: $\pi(C) = f$. What is $V_\pi(C)$

$\pi(W) = s$

$\pi(O) = s$

$\gamma = \frac{1}{2}$



$$V_\pi(s) = E\left[\sum_{j\geq 0} \gamma^j R(s_j, \pi(s_j), s_{j+1}) \mid s_0 = s\right]$$

$j=0$

$$= \sum_{s' \in S} T(s,\pi(s),s') \cdot R(s,\pi(s),s')$$

$$+ \gamma \sum_{s' \in S} T(s,\pi(s),s') \cdot V_\pi(s')$$

For car example:

Our policy:

$\pi(C) = f$

$\pi(W) = s$

$$V_\pi(C) = \overbrace{T(C,\underset{f}{\pi(C)},C)}^{\frac{1}{2}} \cdot \underbrace{R(C,\underset{f}{\pi(C)},C)}_{2}$$

$$+ \overbrace{T(C,f,W)}^{\frac{1}{2}} \overset{2}{\underline{R(C,f,W)}}$$

$$+ \underbrace{T(C,f,O)}_{=0} \cdot R(C,f,O)^{2}$$

$$+\gamma\, \overbrace{T(c,f,c)}^{\frac{1}{2}} \cdot V_\pi(c)$$

$$+\gamma\, \overbrace{T(c,f,w)}^{\frac{1}{2}} \cdot V_\pi(w)$$

$$+\gamma\, \overbrace{T(c,f,o)}^{0} \cdot V_\pi(o)$$

$$= 2 + \frac{\gamma}{2} V_\pi(c) + \frac{\gamma}{2} V_\pi(w).$$

Similar for $s_0 = w$:

$$V_\pi(w) = \underbrace{T(w, \overbrace{\pi(w)}^{s}, c)}_{1/2} \cdot \underbrace{R(w, \overbrace{\pi(w)}^{s}, c)}_{1}$$

$$+ \underbrace{T(w,s,w)}_{\frac{1}{2}} \underbrace{R(w,s,w)}_{1} + \underbrace{T(w,s,o)}_{=0} R(w,s,o)$$

$$+\gamma T(w,s,c)\, V_\pi(c) + \gamma T(w,s,w)\, V_\pi(w)$$

In total: $\gamma = \frac{1}{2}$

$$V_\pi(C) = 2 + \frac{1}{4} V_\pi(C) + \frac{1}{4} V_\pi(W).$$

$$V_\pi(W) = 1 + \frac{1}{4} V_\pi(C) + \frac{1}{4} V_\pi(W).$$

$$\Leftrightarrow \frac{3}{4} V_\pi(C) - \frac{1}{4} V_\pi(W) = 2$$

$$-\frac{1}{4} V_\pi(C) + \frac{3}{4} V_\pi(W) = 1$$

Markov decision process: Value of a policy

A **policy** is a function $\pi : S \to A$.

Goal: Find the optimal policy which increases the (discounted) total reward.

The **value of a policy** π *at state* $s \in S$ *is defined by*

$$V_\pi(s) = E\left[\sum_{j\geq 0} \gamma^j R(s_j, \pi(s_j), s_{j+1}) \mid s_0 = s\right]$$

Interpretation of $V_\pi(s)$: The expected (discounted) total reward when starting in state $s_0 = s$ by using the policy π to choose the action $a_t = \pi(s_t)$ in state s_t.

Goal rephrased: Find a policy which has maximal value at each state.

Markov decision process: Value of a policy

A **policy** is a function $\pi : S \to A$.

Goal: Find the optimal policy which increases the (discounted) total reward.

The **value of a policy** π *at state* $s \in S$ *is defined by*

$$V_\pi(s) = E\left[\sum_{j \geq 0} \gamma^j R(s_j, \pi(s_j), s_{j+1}) \mid s_0 = s\right]$$

Notice that we have

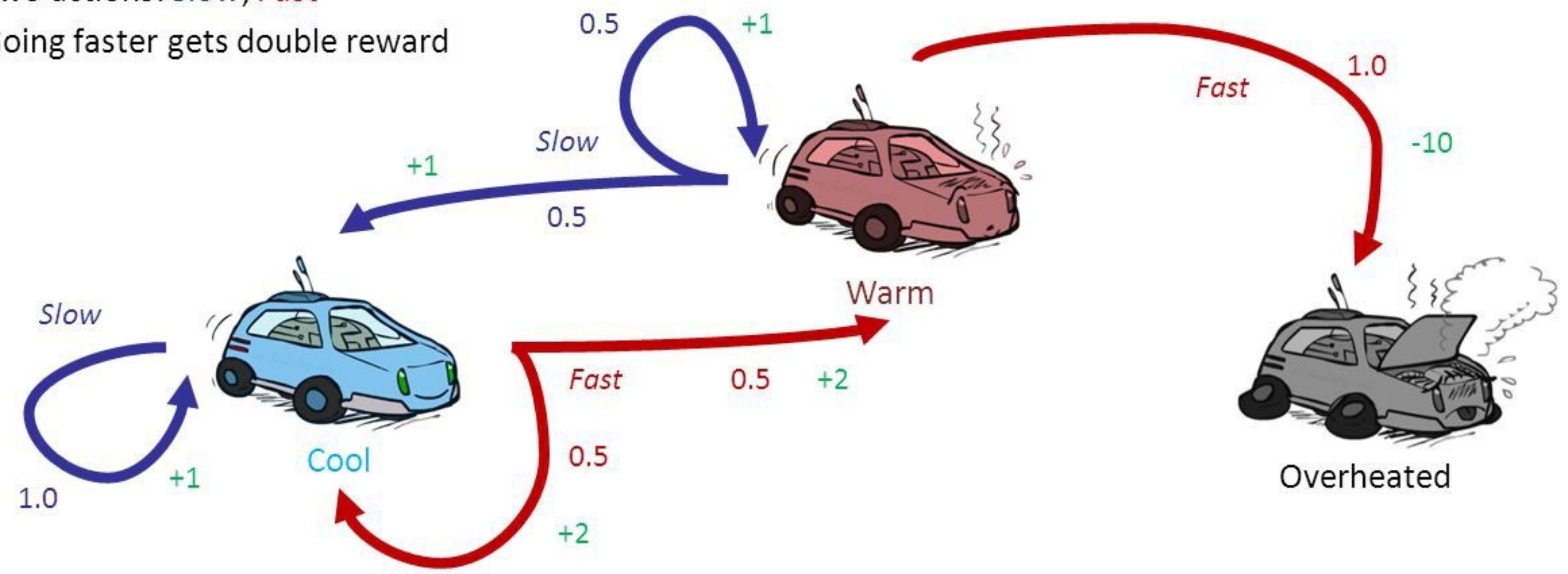
$$V_\pi(s) = \sum_{s' \in S} T(s, \pi(s), s') R(s, \pi(s), s') + \gamma \sum_{s' \in S} T(s, \pi(s), s') V_\pi(s').$$

Optimal policy

A policy π^* is called **optimal** if it has maximal value for all states $s \in S$:

$$V_{\pi^*}(s) = \max_{\pi} V_\pi(s)\,.$$

- Three states: Cool, Warm, Overheated
- Two actions: *Slow*, *Fast*
- Going faster gets double reward



Optimal policy & State-action value function

A policy π^* is called **optimal** if it has maximal value for all states $s \in S$:

$$V_{\pi^*}(s) = \max_{\pi} V_{\pi}(s)\,.$$

The **state-action value function** Q^* is defined for all $(s, a) \in S \times A$ as the expected total reward for taking action $a \in A$ at state $s \in S$ following the optimal policy π^*:

$$Q^*(s,a) = \sum_{s' \in S} T(s,a,s')R(s,a,s') + \gamma \sum_{s' \in S} T(s,a,s')V_{\pi^*}(s')$$

Interpretation: This gives the best possible reward after choosing action a when in state s.

$$Q^*(s, \pi^*(s)) = V_{\pi^*}(s)$$

Optimal policy & State-action value function

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Interpretation: This gives the best possible reward after choosing action a when in state s.

Goal: Find the values of the state-action value function

Having the state-action value function Q^* we can derive the optimal policy by

$$\pi^*(s) = \mathrm{argmax}_{a \in A}\, Q^*(s,a)\,.$$

In car example:

$Q^*(s,a)$	s	f
C	2.7	3.5
W	2.5	-10

Optimal policy & State-action value function

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Interpretation: This gives the best possible reward after choosing action a when in state s.

Goal: Find the values of the state-action value function

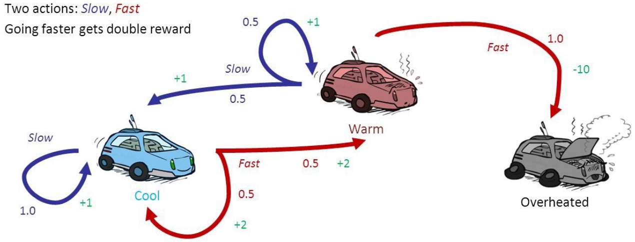
Having the state-action value function Q^* we can derive the optimal policy by

$$\pi^*(s) = \operatorname{argmax}_{a \in A} Q^*(s,a)\,.$$

Goal rephrased: Find a good approximation of the state-action value function. “Learn” the function Q.

Q-learning: Basic idea & Car example

- A robot car wants to travel far, quickly
- Three states: Cool, Warm, Overheated
- Two actions: *Slow*, *Fast*
- Going faster gets double reward



In the car Example:

S = {C,W,O}

A = {f,s}

The (approximated) state-action value function Q can be thought of as a table

State

Action	C (Cool)	W (warm)	O (overheat)
f (fast)	? Q (C,f)	? Q (W,f)	? Q (O,f)
s (slow)	? Q (C,s)	? Q (W,s)	? Q (O,s)

where each entry tells us the expected (best) total reward when taking an action in a certain state.

Q-learning: Algorithm to learn the entries of this table

Q-learning: Basic idea & Car example

Step 1: Start with some values, e.g. choose random values (or all 0):

State

Action	C (Cool)	W (warm)	O (overheat)
f (fast)	3	2	0
s (slow)	1	4	0

Q-learning: Basic idea & Car example

Step 1: Start with some values, e.g. choose random values (or all 0):

State

Action	C (Cool)	W (warm)	O (overheat)
f (fast)	3	2	0
s (slow)	1	4	0

Step 2: Choose a starting state, e.g. $s_0 = C$.

Q-learning: Basic idea & Car example

Step 0: Start with some values, e.g. choose random values (or all 0):

State

Action	C (Cool)	W (warm)	O (overheat)
f (fast)	3	2	0
s (slow)	1	4	0

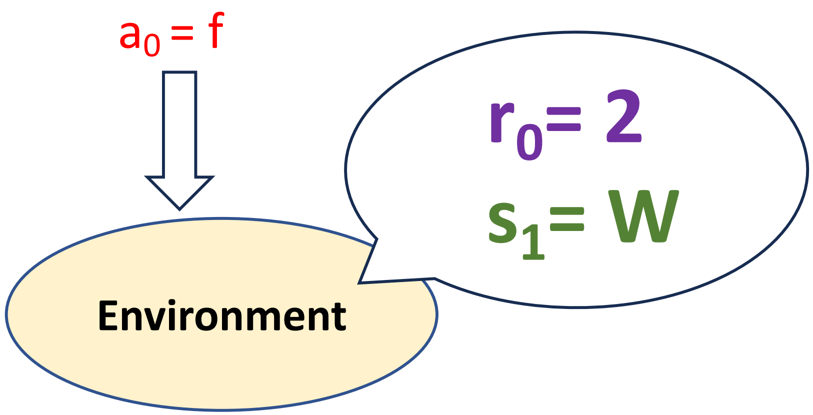
Step 1: Choose a starting state, e.g. $s_0 = C$.

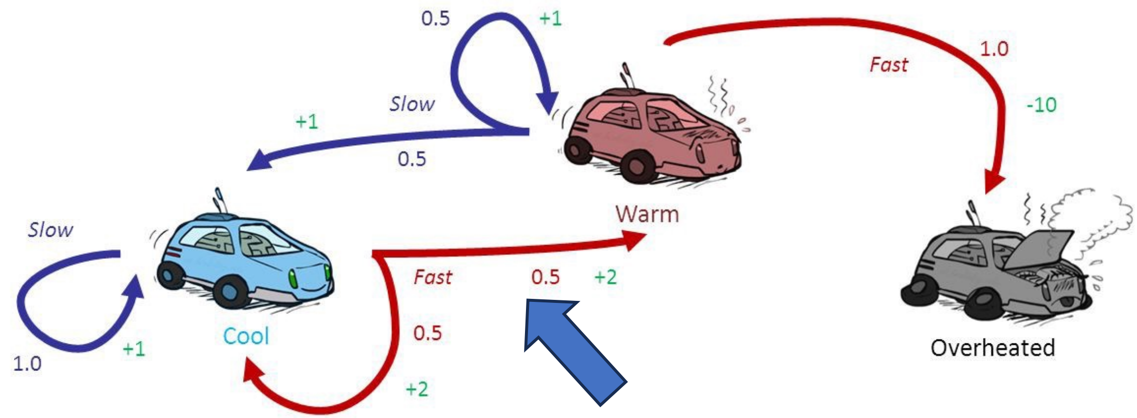
Step 2: Choose the current best action according to the table, i.e. find the maximal entry and set the first action a_0 to the action of the corresponding row.

		C (Cool)	W (warm)	O (overheat)
$a_0 = f$	f (fast)	3	2	0
	s (slow)	1	4	0

Q-learning: Basic idea & Car example

Step 3: Take action $a_0 = f$ and receive a **reward r_0** and a **new state s_1** from the environment. For example





This happened in our case

Question: How could we update the value Q(C,f) now?

$s_0 = C$ $\qquad$ $s_1 = W$

$a_0 = f$

	C (Cool)	W (warm)	O (overheat)
f (fast)	**3**	2	0
s (slow)	1	4	0

Q-learning: Basic idea & Car example

	$s_0 = C$	$s_1 = W$	
	C (Cool)	W (warm)	O (overheat)
$a_0 = f$ f (fast)	**3**	2	0
s (slow)	1	4	0

$r_0 = 2$

Step 4: Update Q(C,f)

We know that Q(C,f)=3 is random and might not be a good value.

But in later steps, we should assume that this is the currently best possible approximation. We should therefore also not overwrite it completely.

Idea:

$$Q(C,f) = (1-\alpha)Q(C,f) + \alpha(something\ new...)$$

new value

old value

some number based on what just happened

$$Here\ \alpha \in [0,1]\ \ is\ the\ \textbf{learning rate}.$$

Q-learning: Basic idea & Car example

$$Q(C, f) = (1-\alpha)Q(C, f) + \alpha(something\ new...)$$

Should give an approximation of Q(C,f) just based on what happened:

- We received a reward $\mathbf{r_0 = 2}$
- We ended in state $s_1 = W$

Question: Based on this information what would be a good guess for Q(C,f)? Assume that the other values in the table below are good approximations.

Recall: Q(C,f) tells us the expected total reward we get when taking action f in state C (and afterwards following the best possible policy).

$s_0 = C$ $\quad$ $s_1 = W$

$a_0 = f$

	C (Cool)	W (warm)	O (overheat)
f (fast)	**3**	2	0
s (slow)	1	4	0

Q-learning: Basic idea & Car example

$$Q(s_0, a_0) = (1-\alpha)Q(s_0, a_0) + \alpha(r_0 + \gamma \max_{a \in A} Q(s_1, a))$$

$$= (1-\alpha)3 + \alpha(2 + \gamma 4).$$

s_0 = C s_1 = W

a_0 = f

	C (Cool)	W (warm)	O (overheat)
f (fast)	3	2	0
s (slow)	1	4	0

For example, if $\alpha = \frac{1}{2}$ *and* $\gamma = \frac{3}{4}$ we get

$$Q(C, f) = \frac{3}{2} + \frac{1}{2}(2 + \frac{3}{4}4) = \mathbf{4}.$$

And the new table

	C (Cool)	W (warm)	O (overheat)
f (fast)	**4**	2	0
s (slow)	1	4	0

From here continue with Step 2...

Q-learning

Q-learning algorithm: Find for all $s \in S$ and $a \in A$ a function $Q(s, a)$, which gives a good approximation for $Q^*(a, s)$.

1. Start with random values for $Q(s, a)$. (e.g. all zero)
2. Choose a starting state $s_0 \in S$.
3. Look up the current best action in that state, i.e. $a_0 = \operatorname{argmax}_{a \in A} Q(s_0, a)$.
4. Apply this action and get a new state s_1 and reward $r_0 = R(s_0, a_0, s_1)$.
5. Update the value $Q(s_0, a_0)$ as follows (**Bellman equation**)

$$Q(s_0, a_0) = (1 - \alpha)Q(s_0, a_0) + \alpha \left(r_0 + \gamma \max_{a \in S} Q(s_1, a) \right) .$$

   Here $\alpha \in [0, 1]$ is the **learning rate**.

6. If s_1 is not a terminal state repeat with step 3.

Q-learning + Epsilon-Greedy

Q-learning algorithm: Find for all $s \in S$ and $a \in A$ a function $Q(s,a)$, which gives a good approximation for $Q^*(a,s)$.

1. Start with random values for $Q(s,a)$. (e.g. all zero)

2. Choose a starting state $s_0 \in S$.

3. Look up the current best action in that state, i.e. $a_0 = \text{argmax}_{a \in A} Q(s_0, a)$ or choose a random action $a_0 \in A$ with probability $\epsilon \in [0,1]$ (Epsilon-Greedy Algorithm).

4. Apply this action and get a new state s_1 and reward $r_0 = R(s_0, a_0, s_1)$.

5. Update the value $Q(s_0, a_0)$ as follows (**Bellman equation**)

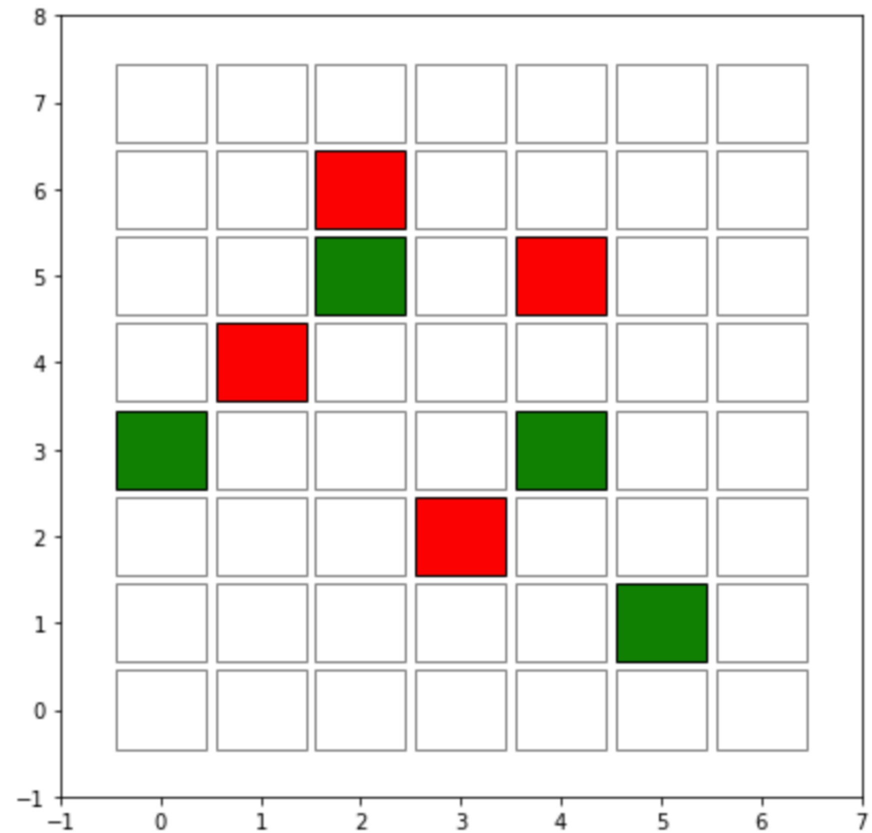
$$Q(s_0, a_0) = (1-\alpha)Q(s_0, a_0) + \alpha \left(r_0 + \gamma \max_{a \in S} Q(s_1, a) \right) .$$

   Here $\alpha \in [0,1]$ is the **learning rate**.

6. If s_1 is not a terminal state repeat with step 3.

Combini & Lecture Example

Goal: Find the nearest Combini to get food. Avoid lecture halls so the professor does not know that you are actually on campus!



Lecture hall

Combini

States: 7 * 8 = 56 positions
(terminal states:)

Actions:

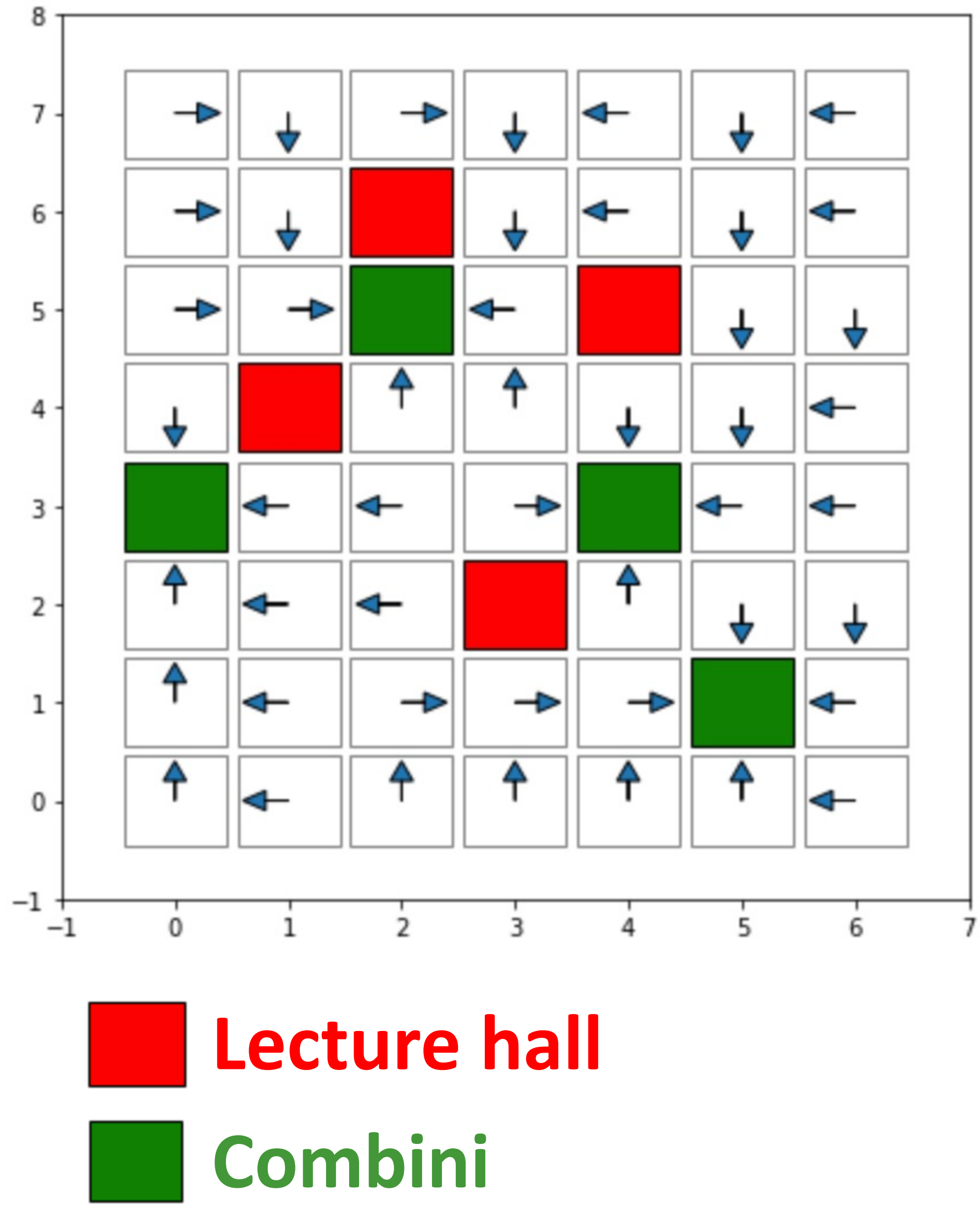
Rewards:

- Reaching Combini: +1
- Reaching Lecture hall: -1
- Making a step to an empty position: -0.1

Optimal policy: Gives the direction we should go at each state to get to the nearest Combini

Combini & Lecture Example: Possible optimal policy

Goal: Find the nearest Combini to get food. Avoid lecture halls so the professor does not know that you are actually on campus!



Optimal policy: Gives the direction we should go at each state to get to the nearest Combini