Modular forms and multiple zeta values

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$$\Delta(q) = q \prod_{n>0} (1-q^n)^{24}$$

$$28\zeta(3,9) + 150\zeta(5,7) + 168\zeta(7,5) = \frac{5197}{691}\zeta(12)$$

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① MZV - Definition

Definition

For $k_1, \ldots, k_{r-1} \geq 1, k_r \geq 2$ define the **multiple zeta value** (MZV)

$$\zeta(k_1, \ldots, k_r) = \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}\,.$$

By r we denote its **depth** and $k_1 + \cdots + k_r$ will be called its **weight**.

- In the case $r = 1$ these are just the classical Riemann zeta values

$$\zeta(k) = \sum_{n>0} \frac{1}{n^k}\,, \qquad \zeta(2) = \frac{\pi^2}{6}\,, \qquad \zeta(3) \notin \mathbb{Q}\,, \qquad \zeta(4) = \frac{\pi^4}{90}\,, \ldots\,.$$

- MZV were first studied by Euler ($r = 2$) and for general depth, they had their big comeback around 1990 due to works of several people.
- These real numbers appear in various areas of mathematics and physics.

① MZV - Harmonic & shuffle product

There are two different ways to express the product of MZV in terms of MZV.

Harmonic product (coming from the definition as iterated sums)

Example in smallest depth:

$$\begin{aligned}\zeta(a)\cdot\zeta(b) &= \sum_{m>0}\frac{1}{m^a}\sum_{n>0}\frac{1}{n^b}\\ &= \sum_{0<m<n}\frac{1}{m^a n^b}+\sum_{0<n<m}\frac{1}{m^a n^b}+\sum_{m=n>0}\frac{1}{m^{a+b}}\\ &= \zeta(a,b)+\zeta(b,a)+\zeta(a+b)\,.\end{aligned}$$

Shuffle product (coming from the expression as iterated integrals)

Example in smallest depth:

$$\zeta(a)\cdot\zeta(b)=\int\ldots\cdot\int\ldots=\sum_{r+s=a+b}\left(\binom{r-1}{a-1}+\binom{r-1}{b-1}\right)\zeta(r,s)\,.$$

① MZV - Double shuffle relations

These two product expressions give various $\mathbb{Q}$-linear relations between MZV.

Example

$$\zeta(3,2)+3\zeta(2,3)+6\zeta(1,4) \overset{\text{shuffle}}{=} \zeta(2)\cdot\zeta(3) \overset{\text{harmonic}}{=} \zeta(3,2)+\zeta(2,3)+\zeta(5)\,.$$

$$\Longrightarrow 2\zeta(2,3)+6\zeta(1,4) \overset{\text{double shuffle}}{=} \zeta(5)\,.$$

But there are more relations between MZV. e.g.:

$$\sum_{0<m<n}\frac{1}{mn^2}=\zeta(1,2)=\zeta(3)=\sum_{n>0}\frac{1}{n^3}.$$

These follow from regularizing the double shuffle relations and they are called **extended double shuffle relations**.

① MZV - Dimensions

Conjecture

The extended double shuffle relations give all relations between MZV.

Denote by $\mathcal{Z}_k$ the $\mathbb{Q}$-vector space spanned by all MZV of weight k.

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\dim \mathcal{Z}_k \leq$	1	0	1	1	1	2	2	3	4	5	7	9	12	16

Table: Upper bounds given by the extended double shuffle relations

Conjecture (Zagier)

The generating series of the dimension of $\mathcal{Z}_k$ is given by

$$\sum_{k\geq 0} \dim \mathcal{Z}_k X^k = \frac{1}{1 - X^2 - X^3}\,.$$

Relations among $\zeta(\text{odd}, \text{odd})$ and $\zeta(k)$

$$\mathbf{4}: \qquad \zeta(1,3) = \frac{1}{4}\zeta(4)$$

① MZV - Modular forms $\longrightarrow$ relations among double zeta values

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$$\mathbf{10}: \qquad \zeta(1,9) + \zeta(3,7) + \zeta(5,6) + \zeta(7,3) = \frac{1}{4}\zeta(10)$$

$$\mathbf{12}: \qquad \zeta(1,11) + \zeta(3,9) + \zeta(5,7) + \zeta(7,5) + \zeta(9,3) = \frac{1}{4}\zeta(12)$$

$$\mathbf{28\zeta(3,9) + 150\zeta(5,7) + 168\zeta(7,5) = \frac{5197}{691}\zeta(12)}$$

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$$\mathbf{28\zeta(3,9) + 150\zeta(5,7) + 168\zeta(7,5) = \frac{5197}{691}\zeta(12)}$$

$$\mathbf{14}: \qquad \zeta(1,13) + \zeta(3,11) + \zeta(5,9) + \zeta(7,7) + \zeta(9,5) + \zeta(11,3) = \frac{1}{4}\zeta(14)$$

① MZV - Modular forms $\longrightarrow$ relations among double zeta values

- M_k : Modular forms of weight k for $\mathrm{Sl}_2(\mathbb{Z})$.
- Eisenstein series of weight k

$$G_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \,. \qquad \left(q = e^{2\pi i \tau}\right)$$

Theorem - rough version (Gangl-Kaneko-Zagier, 2006)

There are at least $\dim M_k$ (linearly independent) relations among $\zeta(k)$ and the double zeta values $\zeta(a,b)$ with a, b odd and $a + b = k$.

- For each G_k we have $\zeta(1, k-1) + \cdots + \zeta(k-3, 3) = \frac{1}{4}\zeta(k)$.
- How to describe these relations for cusp forms $f \in S_k$?

② GKZ Relations - Period polynomials

Definition

For a modular form $f \in M_k$ define the **even period polynomial** of f by

$$P_f^{ev}(X,Y) = \sum_{\substack{r+s=k \\ r,s\geq 1 \text{ odd}}} (-1)^{\frac{s-1}{2}} \binom{k-2}{s-1} L_f^*(s) X^{r-1} Y^{s-1} \,,$$

with the completed L-function of f given by $L_f^*(s) = \int_0^\infty (f(iy) - a_0)\, y^{s-1} dy$.

Example The even period polynomials of G_k and $\Delta(q) = q\prod_{n>0}(1-q^n)^{24}$ are

$$P_{G_k}^{ev} \doteq X^{k-2} - Y^{k-2} \,,$$

$$P_\Delta^{ev}(X,Y) \doteq \frac{36}{691}(X^{10} - Y^{10}) - X^2Y^2(X^2 - Y^2)^3 \,.$$

($\doteq$: equality up to a factor)

② GKZ Relations - Statement

For a cusp form $f \in S_k$ define the coefficients $q_{r,s}(f) \in \mathbb{C}$ by

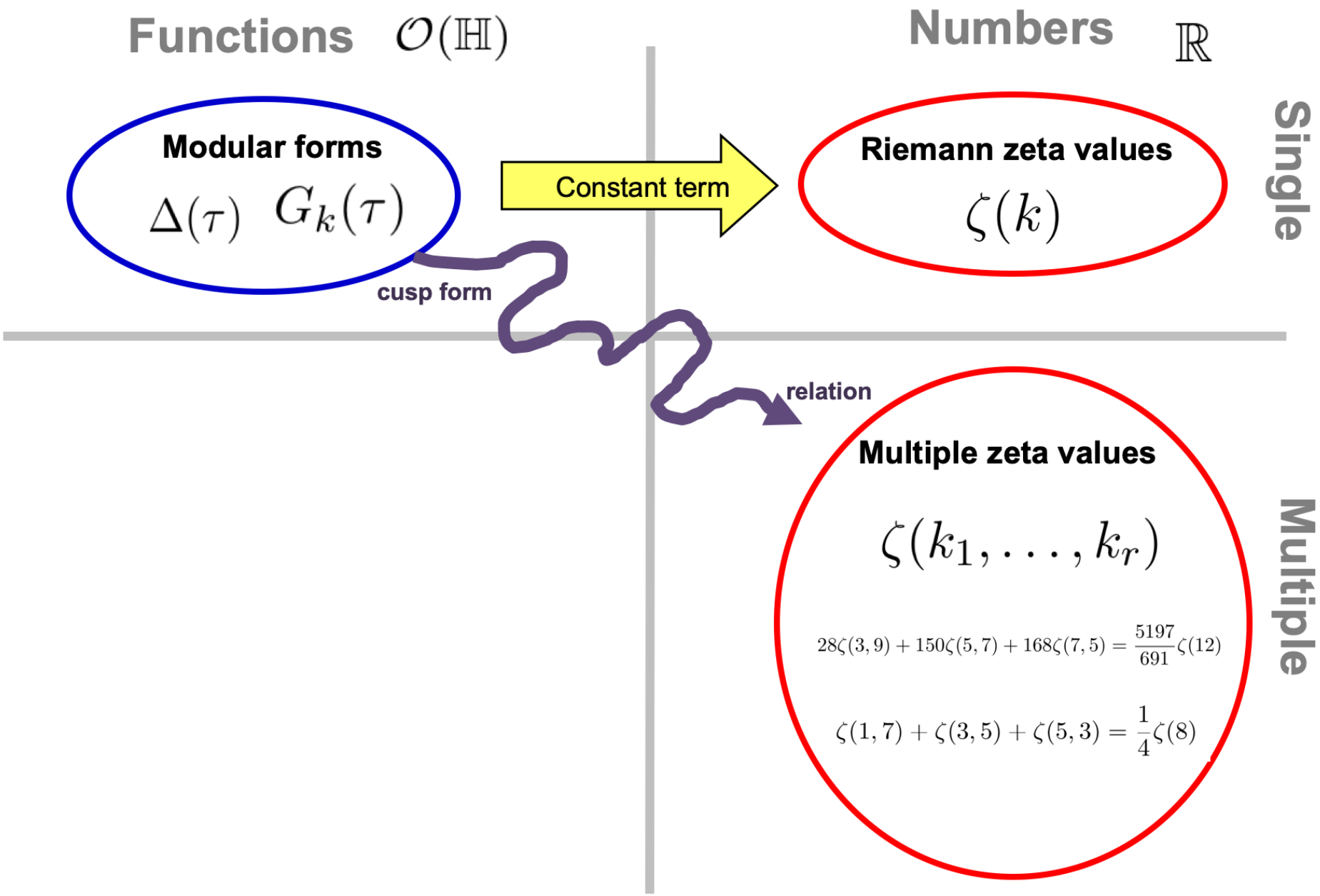
$$P_f^{ev}(X+Y,X) = \sum_{\substack{r+s=k \\ r,s\geq 1}} \binom{k-2}{r-1} q_{r,s}(f) X^{r-1} Y^{s-1}.$$

Theorem ((Gangl-Kaneko-Zagier, 2006), Ma-Tasaka, 2017)

For a cusp form $f \in S_k$ we have

$$\sum_{\substack{r+s=k \\ r,s\geq 1\text{:odd}}} q_{r,s}(f) \left(\zeta(r,s) + \frac{1}{2}\zeta(k) \right) = 0$$

Is there a more "direct" connection of modular forms and MZV?



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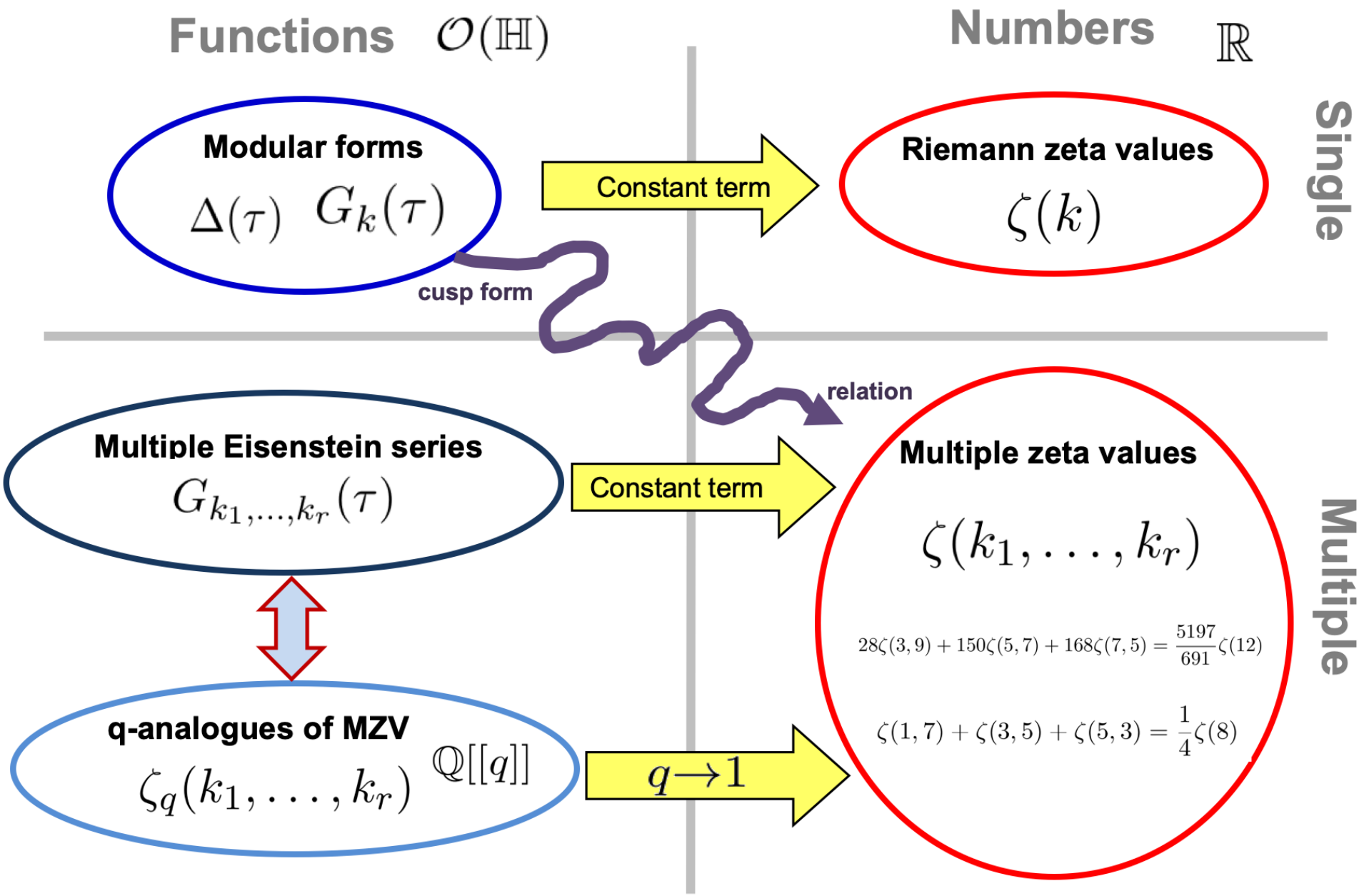


Definition

For $k_1, \ldots, k_r \geq 1$ define the q-series

$$\zeta_q(k_1, \ldots, k_r) = \sum_{0<m_1<\cdots<m_r} \frac{P_{k_1}(q^{m_1}) \cdots P_{k_r}(q^{m_r})}{(1-q^{m_1})^{k_1} \cdots (1-q^{m_r})^{k_r}} \in \mathbb{Q}[[q]]\,,$$

where the (Eulerian) polynomials P_k are given by

$$\frac{P_k(x)}{(1-x)^k} = \frac{1}{(k-1)!} \sum_{d>0} d^{k-1} x^d\,.$$

Denote by $\mathcal{Z}_q$ the $\mathbb{Q}$-vector space spanned by all these q-series.

In depth 1 these are just the generating series of the divisor-sums:

$$\zeta_q(k) = \sum_{m>0} \frac{P_k(q^m)}{(1-q^m)^k} = \frac{1}{(k-1)!} \sum_{\substack{m>0 \\ d>0}} d^{k-1} q^{md} = \frac{1}{(k-1)!} \sum_{n>0} \sigma_{k-1}(n) q^n\,.$$

Definition (reminder)

$$\zeta_q(k_1,\dots,k_r) = \sum_{0<m_1<\cdots<m_r} \frac{P_{k_1}(q^{m_1})\cdots P_{k_r}(q^{m_r})}{(1-q^{m_1})^{k_1}\cdots(1-q^{m_r})^{k_r}}\,.$$

Since we have $P_k(1) = 1$ for all k and

$$\lim_{q\to 1}\frac{(1-q)^k}{(1-q^m)^k} = \lim_{q\to 1}\frac{1}{(1+q+\cdots+q^{m-1})^k} = \frac{1}{m^k}$$

these q-series can be seen as **q-analogues of multiple zeta values**,i.e.

For $k_2,\dots,k_{r-1}\geq 1$,$k_r\geq 2$ we have

$$\lim_{q\to 1}(1-q)^{k_1+\cdots+k_r}\zeta_q(k_1,\dots,k_r) = \zeta(k_1,\dots,k_r)\,.$$

Theorem (B.-Kühn, 2013)

- The space $\mathcal{Z}_q$ is a $\mathbb{Q}$-algebra.
- It contains the space of (quasi-)modular forms with rational coefficients.
- It is closed under the operator $q\frac{d}{dq}$.

Similar to the double shuffle relations for MZV we can prove relations in $\mathcal{Z}_q$, e.g.

$$\frac{1}{2^6 \cdot 5 \cdot 691}\Delta = 28\zeta_q(3,9) + 150\zeta_q(5,7) + 168\zeta_q(7,5) - \frac{5197}{691}\zeta_q(12)$$
$$+\frac{1}{1408}\zeta_q(2) - \frac{83}{14400}\zeta_q(4) + \frac{187}{6048}\zeta_q(6) - \frac{7}{120}\zeta_q(8)\,.$$

Multipliying this equation by $(1-q)^{12}$ and taking the limit $q \to 1$ gives another explanation for the relation $28\zeta(3,9) + 150\zeta(5,7) + 168\zeta(7,5) = \frac{5197}{691}\zeta(12)$.

Question

How can we explain these extra lower weight terms?

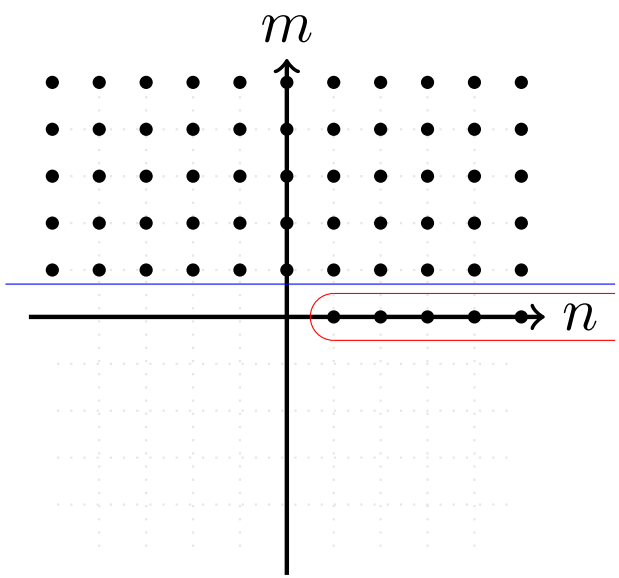
Answer: Multiple Eisenstein series.

Let $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$ be a lattice with $\tau \in \mathbb{H}$. We define an order $\prec$ on Λ_τ by setting

$$\lambda_1 \prec \lambda_2 :\Leftrightarrow \lambda_2 - \lambda_1 \in P$$

for $\lambda_1, \lambda_2 \in \Lambda_\tau$ and the following set of positive lattice points

$$P := \{m\tau + n \in \Lambda_\tau \mid m > 0 \vee (m = 0 \wedge n > 0)\}$$



In other words: $\lambda_1 \prec \lambda_2$ **iff** λ_2 **is above or on the right of** λ_1.

With this order on Λ_τ we have for even $k > 2$:

$$G_k(\tau) = \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k} = \sum_{\substack{\lambda\in\Lambda_\tau \\ 0\prec\lambda}} \frac{1}{\lambda^k}.$$

Since we are not summing over all lattice points the odd Eisenstein series don't vanish anymore and we get for **all** k:

$$\sum_{\substack{\lambda\in\Lambda_\tau \\ 0\prec\lambda}} \frac{1}{\lambda^k} = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \,.$$

This order now allows us to define a multiple version of these series in an obvious way.

Definition

For integers $k_1, \ldots, k_r \geq 2$, we define the **multiple Eisenstein series** $G_{k_1,\ldots,k_r}(\tau)$ by

$$G_{k_1,\ldots,k_r}(\tau) = \sum_{\substack{0 \prec \lambda_1 \prec \cdots \prec \lambda_r \prec 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \cdots \lambda_r^{k_r}} .$$

It is easy to see that these are holomorphic functions in the upper half plane and that they fulfill the **harmonic product**, i.e. it is for example

$$G_2(\tau) \cdot G_3(\tau) = G_{2,3}(\tau) + G_{3,2}(\tau) + G_5(\tau) .$$

Remark

Use Eisenstein summation in the case $k_r = 2$. (c.f. B. Kane Lecture yesterday)

Theorem (B., 2012)

The multiple Eisenstein series $G_{k_1,\ldots,k_r}(\tau)$ have a Fourier expansion of the form

$$G_{k_1,\ldots,k_r}(\tau) = \zeta(k_1,\ldots,k_r) + \sum_{n>0} a_n q^n \,. \qquad \left(q = e^{2\pi i\tau}\right)$$

and they can be written explicitly as a $\mathcal{Z}[2\pi i]$-linear combination of $\zeta_q(l_1,\ldots,l_d)$.

Examples

$$G_k(\tau) = \zeta(k) + (-2\pi i)^k \zeta_q(k)\,,$$

$$G_{2,3}(\tau) = \zeta(2,3) + 3\zeta(3)(2\pi i)^2\zeta_q(2) + 2\zeta(2)(-2\pi i)^3\zeta_q(3) + (-2\pi i)^5\zeta_q(2,3)\,.$$

There are different ways to extend the definition of $G_{k_1,\ldots,k_r}$ to $k_1,\ldots,k_r \geq 1$

- (depth 2) Formal double zeta space realization $G_{r,s}$ (Gangl-Kaneko-Zagier, 2006)
- Shuffle regularized multiple Eisenstein series $G^{\sqcup\!\sqcup}_{k_1,\ldots,k_r}$ (B.-Tasaka, 2017)
- Harmonic regularized multiple Eisenstein series $G^{*}_{k_1,\ldots,k_r}$ (B., 2019)

Theorem (Tasaka, 2018)

For a cuspidal Hecke Eigenform $f \in S_k$ we have

$$\sum_{\substack{r+s=k \\ r,s\geq 1:\text{odd}}} q_{r,s}(f)\left(G_{r,s}+\frac{1}{2}G_k\right)=\frac{L_f^*(1)}{4(k-2)!}f$$

④ Higher level & other cases - Euler sums / alternating MZV

Question

- What about higher level modular forms?
- What are the "higher level" multiple zeta values?

In this talk: Just level 2.

Definition

For $k_1, \ldots, k_r \geq 1, e_1, \ldots, e_r \in \{1, -1\}$ define the **Euler sums** by

$$\zeta\begin{pmatrix} e_1, \ldots, e_r \\ k_1, \ldots, k_r \end{pmatrix} = \sum_{0<m_1<\cdots<m_r} \frac{e_1^{m_1} \cdots e_r^{m_r}}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}\,.$$

(converges for $(k_r, e_r) \neq (1, 1)$). Denote by $\mathcal{E}$ the space of all Euler sums.

Notation: Instead of double indices we write $\overline{k_j}$ when $e_j = -1$, e.g.

$$\zeta(2, \overline{3}) = \zeta\begin{pmatrix} 1, -1 \\ 2, 3 \end{pmatrix}$$

④ Higher level & other cases - Euler sums and $\Gamma_0(2)$

Theorem (Gangl-Kaneko-Zagier, 2006)

$$\dim_{\mathbb{Q}} \left\langle \zeta(r,s) \mid r+s=k,\ r \geq 3, s \geq 1 \text{ odd} \right\rangle_{\mathbb{Q}} \leq \frac{k}{2} - 1 - \dim_{\mathbb{Q}} S_k$$

Kaneko and Tasaka considered an odd variant of the double zeta values

$$\zeta^{\mathbf{oo}}(r,s) = \sum_{\substack{0<m<n \\ m,n \text{ odd}}} \frac{1}{m^r n^s} = \frac{1}{4} \sum_{0<m<n} \frac{(1-(-1)^m)(1-(-1)^n)}{m^r n^s}$$

$$= \frac{1}{4}\big(\zeta(r,s) - \zeta(\overline{r},s) - \zeta(r,\overline{s}) + \zeta(\overline{r},\overline{s})\big) \in \mathcal{E}\,.$$

Theorem (Kaneko-Tasaka, 2013)

$$\dim_{\mathbb{Q}} \left\langle \zeta^{\mathbf{oo}}(r,s) \mid r+s=k,\ r,s \geq 2 \text{ even} \right\rangle_{\mathbb{Q}} \leq \frac{k}{2} - 1 - \dim_{\mathbb{Q}} S_k(\Gamma_0(2))$$

Definition

Define for $r \geq 1, s \geq 2$ the following modified version of double zeta values by

$$\hat{\zeta}(r,s) = \sum_{0<m<n} \frac{1}{(m+n)^r n^s}$$

and their q-analogues

$$\hat{\zeta}_q(r,s) = \sum_{0<m<n} \frac{P_r(q^{m+n})}{(1-q^{m+n})^r} \frac{P_s(q^n)}{(1-q^n)^s}.$$

These are also Euler sums, since one can check

$$\hat{\zeta}(r,s) = 2^{s-1}\zeta(r,\overline{s}) + (2^{s-1}-1)\zeta(r,s) - \zeta(r+s) \in \mathcal{E}.$$

④ Higher level & other cases - Modified double zeta values

The **even restricted period polynomial** $P_f^{ev,0}$ of a modular form $f \in M_k$ is given by

$$P_f^{ev,0}(X,Y) = \sum_{\substack{r+s=k \\ r,s\geq 3 \text{ odd}}} (-1)^{\frac{s-1}{2}} \binom{k-2}{s-1} L_f^*(s) X^{r-1} Y^{s-1} .$$

Define the coefficients $q_{r,s}^0(f) \in \mathbb{C}$ by

$$P_f^{ev,0}(X+Y,X) = \sum_{\substack{r+s=k \\ r,s\geq 1}} \binom{k-2}{r-1} q_{r,s}^0(f) X^{r-1} Y^{s-1} .$$

Theorem (B., 2018)

For a cuspidal Hecke eigenform $f \in S_k$ we have

$$\frac{L_f^*(1)}{2(k-2)!} f(q) = \sum_{\substack{r+s=k \\ r,s\geq 2}} q_{r,s}^0(f)\, \hat{\zeta}_q(r,s) - \lambda_f \zeta_q(k) - R_f(q) ,$$

with an explicitly given "lower weight" q-series $R_f(q) \in \mathbb{C}[[q]]$ and $\lambda_f \in \mathbb{C}$.

Theorem (reminder)

Let $f \in S_k$ be a cuspidal Hecke eigenform

$$\frac{L_f^*(1)}{2(k-2)!} f(q) = \sum_{\substack{r+s=k \\ r,s\geq 2}} q_{r,s}^0(f)\, \hat{\zeta}_q(r,s) - \lambda_f \zeta_q(k) - R_f(q)\,.$$

By "lower weight q-series $R_f(q)$", we mean that

$$\lim_{q\to 1} (1-q)^k R_f(q) = 0\,.$$

Corollary

For any cusp form $f \in S_k$ the following relation holds

$$\sum_{\substack{r+s=k \\ r,s\geq 2}} q_{r,s}^0(f) \hat{\zeta}(r,s) = \lambda_f\, \zeta(k)\,.$$

⑤ Remarks - More cases...

Remark

- There are level N analogues of multiple Eisenstein series (Yuan-Zhao, 2015).
- There are relations among $\zeta(\text{odd}, \text{even})$ in odd weight k, which have connections to period polynomials in weight $k-1$ and $k+1$ (Ma, 2016).
- Hirose (2019) recently announced a new results on connection of cusp forms for the congruence subgroup

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Gamma(2) \sqcup \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \Gamma(2) \sqcup \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \Gamma(2)$$

  and relations among generalized double zeta values $\zeta_n(r, s)$.

⑤ Remarks - Open problems

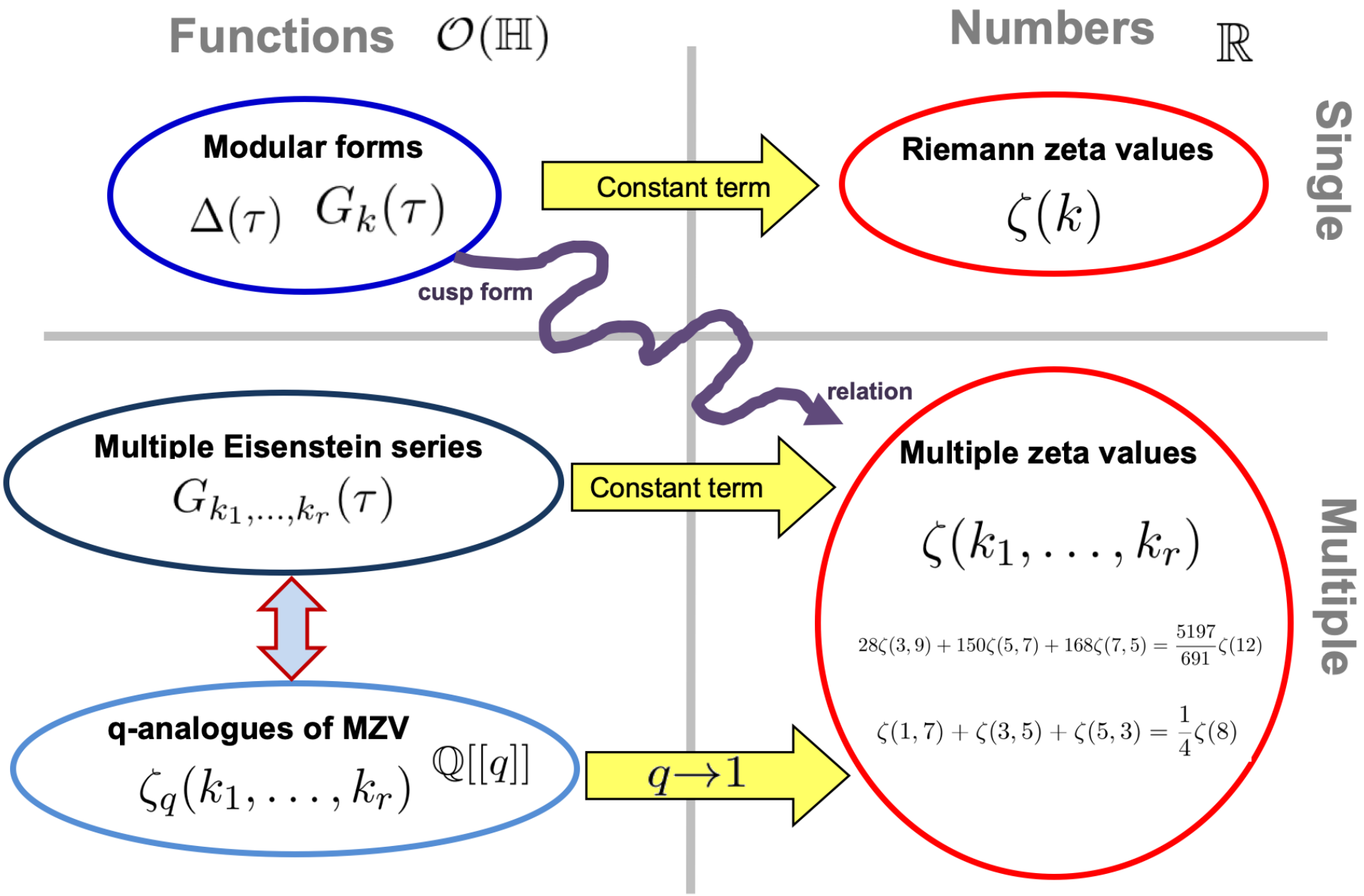
- What about higher levels (≥ 3)?
- What about higher depths?
- When is a linear combination of multiple Eisenstein series modular?

$$\sum_{\substack{r\geq 0\\ k_1,\ldots,k_r\geq 2\\ k_1+\cdots+k_r=k}} \alpha_{k_1,\ldots,k_r} G_{k_1,\ldots,k_r} \overset{?}{\in} M_k$$

- Is the space of multiple Eisenstein series closed under the operator $q\frac{d}{dq}$?
- There seem to be modular phenomena for "finite multiple zeta values" (Kaneko-Zagier).

$$\zeta_{\mathcal{A}}(k_1,\ldots,k_r) = \left(\sum_{0<m_1<\cdots<m_r<p} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \mod p\right)_{p \text{ prime}}$$

Thank you very much for your attention



⑥ Bonus - MZV iterated integral representation

Proposition (Integral representation)

The $\zeta(k_1, ..., k_r)$ of weight $k = k_1 + ... + k_r$ can be written as an iterated integral

$$\zeta(k_1, ..., k_r) = \int_{0<t_1<\cdots<t_k<1} \omega_1(t_1) \dots \omega_k(t_k)\,,$$

where

$$\omega_j(t) = \begin{cases} \frac{dt}{1-t} & \text{if } j \in \{1, k_1+1, k_1+k_2+1, \dots, k_1+\cdots+k_{r-1}+1\} \\ \frac{dt}{t} & \text{else} \end{cases}.$$

Examples

$$\zeta(2) = \int_{0<t_1<t_2<1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}$$

$$\zeta(2,3) = \int_{0<t_1<\cdots<t_5<1} \underbrace{\frac{dt_1}{1-t_1}\cdot\frac{dt_2}{t_2}}_{2}\cdot\underbrace{\frac{dt_3}{1-t_3}\cdot\frac{dt_4}{t_4}\cdot\frac{dt_5}{t_5}}_{3}\,.$$

⑥ Bonus - Original GKZ relations

The **even restricted period polynomial** $P_f^{ev,0}$ of a modular form $f \in M_k$ is given by

$$P_f^{ev,0}(X,Y) = \sum_{\substack{r+s=k \\ r,s\geq 3 \text{ odd}}} (-1)^{\frac{s-1}{2}} \binom{k-2}{s-1} L_f^*(s) X^{r-1} Y^{s-1} .$$

Define the coefficients $q_{r,s}^0(f) \in \mathbb{C}$ by

$$P_f^{ev,0}(X+Y,X) = \sum_{\substack{r+s=k \\ r,s\geq 1}} \binom{k-2}{r-1} q_{r,s}^0(f) X^{r-1} Y^{s-1} .$$

Theorem (Gangl-Kaneko-Zagier, 2006)

For a cusp form $f \in S_k$ the following relation holds

$$\sum_{\substack{r+s=k \\ r,s\geq 3:\text{odd}}} q_{r,s}^0(f)\zeta(r,s) \equiv 0 \mod \zeta(k) .$$

⑥ Bonus - GKZ Example

Example The even period polynomial of $f = -28L^*_\Delta(9)^{-1}\Delta$ is given by

$$P_f^{ev}(X,Y) = -\frac{45360}{691}(X^{10} - Y^{10}) + 1260X^2Y^2(X^2 - Y^2)^3$$

and therefore the even restricted period polynomial is

$$P_f^{ev,0}(X,Y) = 1260X^2Y^2(X^2 - Y^2)^3\,.$$

Expanding $P_f^{ev,0}(X+Y,X)$ out we get

$$\binom{10}{2}28X^2Y^8 + \binom{10}{3}84X^3Y^7 + \binom{10}{4}150X^4Y^6 + \binom{10}{5}190X^5Y^5 + \binom{10}{6}168X^6Y^4 + \binom{10}{7}84X^7Y^3\,.$$

Which by the Theorem gives the relation

$$28\zeta(3,9) + 150\zeta(5,7) + 168\zeta(7,5) = \frac{5197}{691}\zeta(12)\,.$$

(The coefficient $\frac{5197}{691}$ can also be described explicitly (Ma-Tasaka, 2017)).

⑥ Bonus - Modified double zeta values

Theorem (B., 2018)

For a cuspidal Hecke eigenform $f \in S_k$ we have

$$\frac{L_f^*(1)}{2(k-2)!} f(q) = \sum_{\substack{r+s=k \\ r,s\geq 2}} q_{r,s}^0(f)\, \hat{\zeta}_q(r,s) - \lambda_f \zeta_q(k) - R_f(q)\,,$$

where $R_f(q) \in \mathbb{C}[[q]]$ is an explicitly given "lower weight" q-series and where

$$\lambda_f = \frac{k-1}{2} \left(\sum_{\substack{r+s=k \\ r,s\geq 3 \text{ odd}}} \frac{(-1)^{\frac{s-1}{2}}}{r\, 2^{r-1}} \binom{k-2}{s-1} L_f^*(s) - L_f^*(1) \right) .$$

Proof ingredients: Explicit description of Hecke operators on the space of period polynomials.

⑥ Bonus - Modified double zeta values

For a cusp form $f \in S_k$ with even restricted period polynomial

$$P_f^{ev,0}(X,Y) = \sum_{\substack{r+s=k \\ r,s\geq 3 \text{ odd}}} c_{r,s} X^{r-1} Y^{s-1}$$

the q-series $R_f(q)$ is given by

$$R_f(q) = \sum_{\substack{r+s=k \\ r,s\geq 3 \text{ odd}}} c_{r,s} \left(\sum_{j=1}^{r-1} \binom{r}{j} \frac{B_j \cdot (k-j-1)!}{r 2^{r-j} (k-2)!} \zeta_q(k-j) - \frac{1}{2^r} \zeta_q^{\mathbf{e}}(k-1) \right)$$

$$+ \sum_{\substack{r+s=k \\ r,s\geq 3 \text{ odd}}} c_{r,s} \sum_{\substack{0\leq j\leq r-1 \\ 1\leq l\leq r-j}} \binom{r}{j} \binom{r-j}{l} \frac{(-1)^l B_j \cdot (k-j-l-1)!}{r 2^{r-j} (k-2)!} \zeta_q^{\mathbf{o}}(k-j-l)$$

where

$$\zeta_q^{\mathbf{e}}(k) = \sum_{\substack{a,d>0 \\ d \text{ even}}} \frac{d^{k-1}}{(k-1)!} q^{ad}, \qquad \zeta_q^{\mathbf{o}}(k) = \sum_{\substack{a,d>0 \\ d \text{ odd}}} \frac{d^{k-1}}{(k-1)!} q^{ad}.$$