Modular forms and their combinatorial variants

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Lecture notes and Exercises are available at: http://www.henrikbachmann.com/mf_2023.html

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Contents

1 Introduction & Motivation

Modular forms are functions appearing in several areas of mathematics as well as mathematical physics. We start directly with a rough definition of modular forms and give some explicit applications of them. In this lecture, a **modular form** of weight $k \in \mathbb{Z}$ for some subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ of the **modular group**

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\Big|\, a,b,c,d \in \mathbb{Z}\,, ad-bc=1 \right\}$$

is a holomorphic function $f \in \mathcal{O}(\mathbb{H})$ in the complex upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$, which satisfies the **modular transformation property**

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \tag{1.1}$$

for all $\tau \in \mathbb{H}$ and all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ together with certain growth conditions ("holomorphicity at the cusps"). We will make this notion precise later, but in the case $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ this just means that $f(it)$ is bounded as $t \to \infty$. By $M_k(\Gamma)$ we denote the $\mathbb{C}$-vector space spanned by all modular forms of weight $k \in \mathbb{Z}$ for the subgroup Γ. A common example for subgroups Γ are given for $N \geq 1$ (the **level**) by the **congruence subgroups**

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\Big|\, c \equiv 0 \mod N \right\}. \tag{1.2}$$

Notice that in the level $N=1$ case we have $\Gamma_0(1) = \mathrm{SL}_2(\mathbb{Z})$. One important fact is that (1.1) just needs to be checked for the generators of Γ. For example, $\mathrm{SL}_2(\mathbb{Z})$ is generated by the two matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, a holomorphic function f (bounded at $i\infty$) satisfying

$$f(\tau+1) = f(\tau), \qquad f(-\frac{1}{\tau}) = \tau^k f(\tau)$$

is already a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$.
In general, if $T \in \Gamma$ then $f(\tau+1) = f(\tau)$ and $f \in M_k(\Gamma)$ has a **Fourier expansion**

$$f(\tau) = \sum_{n \geq 0} a_n q^n, \qquad (q = e^{2\pi i \tau})$$

for some $a_n \in \mathbb{C}$, which are called the **Fourier coefficients**. That this sum starts at $n=0$ is a consequence of f being bounded at $i\infty$. One of the many reasons why modular forms are interesting/powerful are given by the following facts:

- Modular forms occur naturally in connection with problems arising in many areas of mathematics and theoretical physics and the Fourier coefficients a_n often encode the arithmetically interesting information about a problem.
- The spaces $M_k(\Gamma)$ are finite dimensional and algorithmically computable.

One common theme is the following: Assume you have some sequence of numbers $a_n \in \mathbb{N}$ (or $\mathbb{Q}, \mathbb{R}, \mathbb{C}$) which arise "naturally" (e.g. they count something) and you want to find an explicit formula for them. Then in a lot of applications, where modular forms appear, the following strategy works:

1. Consider the generating series $G(\tau) = \sum_{n\geq 0} a_n q^n$.

2. Try to find $k \in \mathbb{Z}$, $N \geq 1$ such that $G \in M_k(\Gamma_0(N))$. Often this can be done by showing the modular transformation property (1.1) for the generators of $\Gamma_0(N)$ using well-known tools from calculus (e.g. Poisson summation formula).

3. Determine $\dim M_k(\Gamma_0(N)) = d$. We will see that this can be done explicitly by using complex analysis (residue theorem, Riemann-Roch).

4. Construct a basis of $\dim M_k(\Gamma_0(N))$. Often one can construct d linearly independent elements by using, for example, Eisenstein series. The fourier coefficients of these elements are given explicitly.

5. Write G in terms of this basis. For this one just needs to check the first few Fourier coefficients.

This strategy will then lead to an explicit formula for a_n. We will use this strategy now to prove two well-known theorems in classical number theory.

1.1 The four square theorem

We start by illustrating some applications coming from classical number theory. For this we start with the following well-known theorem:

Theorem 1.1 (Theorem of Lagrange (1770))**.** *Every positive integer is a sum of four*[1] *squares.*

For example $1 = 1^2 + 0^2 + 0^2 + 0^2$ or $30 = 1^2 + 2^2 + 3^2 + 4^2 = 0^2 + 1^2 + 2^2 + 5^2$ and

$$2023 = 1^2 + 2^2 + 13^2 + 43^2 = 3^2 + 3^2 + 18^2 + 41^2 \,.$$

In particular, these examples show that the representation as a sum of four squares is not unique.

Question: *In how many ways can a natural number n be written as a sum of four squares?*

In other words, the question asks for an explicit formula for the function

$$r_4(n) = \# \left\{ (a,b,c,d) \in \mathbb{Z}^4 \mid n = a^2 + b^2 + c^2 + d^2 \right\} \,.$$

For example, $r_4(1) = 8$ since

$$1 = 1^2 + 0^2 + 0^2 + 0^2 = 0^2 + 1^2 + 0^2 + 0^2 = \dots = (-1)^2 + 0^2 + 0^2 + 0^2 = \dots \,.$$

The question was answered by Jacobi who gave the following explicit formula for $r_4(n)$.

Theorem 1.2 (Jacobi's four-square theorem (1834))**.** *For all $n \in \mathbb{Z}_{\geq 1}$ we have*

$$r_4(n) = 8 \sum_{\substack{d \mid n \\ 4 \nmid d}} d \,.$$

Here the sum runs over all positive divisors d of n, which are not divisible by 4.

Example 1. *i) If p is prime, then there are $8(p+1)$ ways to write p as a sum of four squares.*

ii) The divisors of 2023 are $1, 7, 17, 119, 289$, and 2023, which are all not divisible by 4 and therefore we have

$$r_4(2023) = 8\,(1 + 7 + 17 + 119 + 289 + 2023) = 19648$$

ways of writing 2023 as a sum of four squares.

[1] Four is the smallest integer with this property, since 5 is not the sum of two squares and 7 is not the sum of three.

We will now use the strategy mentioned before to prove Theorem 1.2:

1. We consider the generating series of $r_4(n)$, i.e. set
$$G(\tau) = \sum_{n\geq 0} r_4(n)q^n = 1 + 8q + 24q^2 + 32q^3 + 24q^4 + 48q^5 + 96q^6 + 64q^7 + 24q^8 + 104q^9 + \dots .$$

2. Now we will sketch how to show that $G \in M_2(\Gamma_0(4))$, i.e. we need to check the modular transformation property
$$G\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 G(\tau)$$
for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$. Since we just need to check this on the generators of $\Gamma_0(4)$ we first show (Homework 1) that $\Gamma_0(4)$ is generated by the matrices $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$. Therefore, we just need to check
$$G(\tau+1) = G(\tau), \qquad G\left(\frac{\tau}{4\tau+1}\right) = (4\tau+1)^2 G(\tau) \tag{1.3}$$
since (1.1) is trivial for the negative identity matrix. Since we defined G by the Fourier expansion we automatically get $G(\tau+1) = G(\tau)$. To show the second equation we define the **theta-function** by
$$\Theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \dots .$$
For this function one can show the following.

Proposition 1.3. *The theta-function satisfies the two functional equations*
$$\Theta(\tau+1) = \Theta(\tau), \qquad \Theta\left(-\frac{1}{4\tau}\right) = \sqrt{\frac{2\tau}{i}}\,\Theta(\tau) \qquad (\tau \in \mathbb{H}). \tag{1.4}$$

Proof. The first equation follows directly from definition and the second follows from the Poisson transformation formula. See [Z, Proposition 9] for details. □

Now notice that we have $G(\tau) = \Theta(\tau)^4$. Using $\frac{\tau}{4\tau+1} = -\frac{1}{4(\frac{-1}{4\tau}-1)}$ in Proposition 3.2 then implies $G\left(\frac{\tau}{4\tau+1}\right) = (4\tau+1)^2 G(\tau)$.

3. As we will see we have $\dim_{\mathbb{C}} M_2(\Gamma_0(4)) = 2$.

4. Now define the Eisenstein series of weight 2 by
$$E_2(\tau) = 1 - 24\sum_{n\geq 1} \frac{nq^n}{1-q^n} = 1 - 24\sum_{n\geq 1} \sigma_1(n) q^n = 1 - 24q - 72q^2 - 96q^3 - 168q^4 + \dots,$$
where $\sigma_1(n) = \sum_{d|n} d$ is the sum of all divisors of n. Later we will see that E_2 almost transforms like a modular form of weight 2 as we will have
$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) - \frac{6}{\pi} ic(c\tau+d). \tag{1.5}$$
Using this one then checks by direct calculation that $E_2(\tau) - 2E_2(2\tau)$ and $E_2(\tau) - 4E_2(4\tau)$ also satisfy (1.3). Moreover, comparing their first few Fourier coefficients show that they are linearly independent and therefore they form a basis of $M_2(\Gamma_0(4))$. (We ignored the growth condition here, but this one is easy to check for all appearing functions in this example.)

5 Finally we want to write G in terms of the basis we found. Comparing the first few Fourier coefficients (which is sufficient as we will see) we find

$$G(\tau) = -\frac{1}{3}\left(E_2(\tau) - 4E_2(4\tau)\right) = 8\sum_{n\geq 1}\left(\sum_{d|n} d - \sum_{\substack{d|n \\ 4|d}} d\right) q^n,$$

which then gives exactly the formula in 1.2 for $r_4(n)$.

1.2 Sum of divisors and the Hurwitz identity

In this section, we will give another classical example for the application of modular forms. For this we will prove some identities of general divisor sums. For $l \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 1}$ the **l-th divisor sum** $\sigma_l(n)$ is defined by

$$\sigma_l(n) = \sum_{d|n} d^l\,,$$

where the sum runs over all positive divisors d of n. In particular $\sigma_0(n)$ counts the divisors of n and $\sigma_1(n)$ is the sum of all divisor of n. For example since the divisor of 6 are $1, 2, 3, 6$, we have $\sigma_0(6) = 4$ and $\sigma_1(6) = 1 + 2 + 3 + 6 = 12$. A few more examples:

n	$\sigma_1(n)$	$\sigma_3(n)$	$\sigma_5(n)$	$\sigma_7(n)$	$\sigma_9(n)$	$\sigma_{11}(n)$
1	1	1	1	1	1	1
2	3	9	33	129	513	2049
3	4	28	244	2188	19684	177148
4	7	73	1057	16513	262657	4196353
5	6	126	3126	78126	1953126	48828126
6	12	252	8052	282252	10097892	362976252
7	8	344	16808	823544	40353608	1977326744
8	15	585	33825	2113665	134480385	8594130945
9	13	757	59293	4785157	387440173	31381236757

As a generalization of E_2 from the previous section, we define for an even $k \geq 2$ the (normalized) **Eisenstein series of weight k** by

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)q^n\,,$$

where again $q = e^{2\pi i\tau}$ and B_k denotes the **k-th Bernoulli number** defined by the generating function

$$\sum_{k=0}^{\infty}\frac{B_k}{k!}X^k = \frac{X}{e^X - 1}\,. \tag{1.6}$$

A few example for the first Bernoulli numbers are given by the following table The Eisenstein

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
B_k	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0	$-\frac{691}{2730}$	0	$\frac{7}{6}$	0	$-\frac{3617}{510}$

series of weight 4,6 and 8 are given by the following q-series

$$\begin{aligned} E_4(\tau) &= 1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \dots \\ E_6(\tau) &= 1 - 504 \sum_{n\geq 1} \sigma_5(n) q^n = 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 + \dots \\ E_8(\tau) &= 1 + 480 \sum_{n\geq 1} \sigma_7(n) q^n = 1 + 480q + 61920q^2 + 1050240q^3 + 7926240q^4 + \dots . \end{aligned} \tag{1.7}$$

The goal of this section will be to give two different proofs of the following identity. One of these will again use the strategy used before (i.e. we use modular forms) and the other will be done by explicit calculation.

Theorem 1.4 (Hurwitz identity). *For all $n \in \mathbb{Z}_{\geq 1}$ we have*

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{j=1}^{n-1} \sigma_3(j)\sigma_3(n-j) \,.$$

For example for $n = 3$ we have $\sigma_7(3) = 1 + 3^7 = 2188$ and

$$\sigma_3(3) + 120 \sum_{j=1}^{2} \sigma_3(j)\sigma_3(3-j) = 1 + 3^3 + 120(1 \cdot (1 + 2^3) + (1 + 2^3) \cdot 1) = 28 + 120 \cdot 18 = 2188 \,.$$

Until here in lecture 1 (14th April, 2023)

Modular proof of Theorem 1.4. First notice that the identity in question equivalent to the equation

$$E_8(\tau) = E_4(\tau)^2 \,.$$

We will see that for all even $k \geq 4$ we have $E_k \in M_k$ and that (Homework 1) we have $E_4^2 \in M_8$. Further we will show that $\dim M_8 = 1$ and therefore E_4^2 needs to be a multiple of E_8. But this multiple is 1, since both have 1 as the constant term in their Fourier expansion. □

Since we did not prove any of the claims in the above proof, we will also give a combinatorial/elementary proof. These kinds of proofs will also be generalized at a later stage, when we talk about combinatorial modular forms.

Combinatorial proof of Theorem 1.4. We use the elements $F_n \in \mathbb{Z}[[q]] \cap \mathbb{Q}(q)$ defined by

$$F_n = \frac{q^n}{1-q^n} = \sum_{m=1}^{\infty} q^{mn} \qquad (n \in \mathbb{N}),$$

to write

$$\begin{aligned} \Big(\sum_{n>0} \sigma_3(n) q^n\Big)^2 &= \Big(\sum_{u>0} u^3 F_u\Big)^2 = \sum_{u,v>0} \frac{uv}{12} \big((u+v)^4 + (u-v)^4 - 2u^4 - 2v^4\big) F_u F_v \\ &= \sum_{w>0} \frac{w^4}{12} \Big\{ \sum_{u+v=w} uvF_uF_v + 2 \sum_{v-u=w} uvF_uF_v - 4wF_w \sum_{u>0} uF_u \Big\} . \end{aligned}$$

Using the identity $F_aF_b = F_aF_{a+b} + F_bF_{a+b} + F_{a+b}$ we can rewrite the expression in curly brackets as

$$\sum_{0<u<w} u(w-u)F_w\big(2F_u+1\big) + 2\sum_{u>0} u(u+w)\big(F_uF_w - F_{u+w}F_w - F_{u+w}\big) - 4wF_w\sum_{u>0} uF_u$$
$$= 2F_w\left(\sum_{0<n\leq w} + \sum_{n>w} - \sum_{n>0}\right) n(w-n)F_n + \frac{1}{6}(w^3-w)F_w - 2\sum_{u>w} u(u-w)F_u\,.$$

The first term here vanishes identically since every positive $n>0$ satisfies either $0<n\leq w$ or $n>w$ but not both. It follows that

$$\Big(\sum_{n>0}\sigma_3(n)q^n\Big)^2 = \sum_{w>0}\frac{w^7-w^5}{72} - \sum_{u>0} u\Big(\sum_{0<w<u}\frac{w^4(u-w)}{6}\Big)F_u$$
$$= \sum_{w>0}\frac{w^7-w^3}{120}F_w = \sum_{n>0}\frac{\sigma_7(n)-\sigma_3(n)}{120}q^n\,.$$

□

2 Modular forms for $\mathrm{SL}_2(\mathbb{Z})$

The modular forms mentioned in the previous section were given by q-series. But actually modular forms are functions from the upper half plane to the complex numbers. That they can be written as q-series will follow later as a simple implication of their definition. We will start by giving the definition of the upper half plane and the action of the modular group on this space. With this, we will define modular functions and modular forms, before giving (non-trivial) examples.

2.1 The modular group and fundamental domains

The **upper half plane**, denoted $\mathbb{H}$, is the set of all complex numbers with positive imaginary part:

$$\mathbb{H} = \big\{\tau\in\mathbb{C} \mid \mathrm{Im}(\tau)>0\big\} = \big\{x+iy\in\mathbb{C} \mid x,y\in\mathbb{R}\,, y>0\big\}\,.$$

The **modular group** (or special linear group) $\mathrm{SL}_2(\mathbb{Z})$ is the group of 2×2-matrices with integer entries and determinant one:

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \,\middle|\, a,b,c,d\in\mathbb{Z}\,, ad-bc=1\right\}\,.$$

For $\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z})$ and $\tau\in\mathbb{C}$ we define the fractional linear transformation

$$\gamma(\tau) := \frac{a\tau+b}{c\tau+d}\,.$$

This gives a left action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ (Homework 1)
The group $\mathrm{SL}_2(\mathbb{Z})$ contains the following three matrices

$$I = \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\,,\qquad S = \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\,,\qquad T = \begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}\,,$$

which correspond to the identity and the fractional transformation $\tau\mapsto -\frac{1}{\tau}$ and $\tau\mapsto\tau+1$. The latter two fractional transformation will play the major role in our studies, since we have following:

Proposition 2.1. *The matrices S and T generate* $\mathrm{SL}_2(\mathbb{Z})$.

Proof. Homework 1. □

Remark 2.2. Some authors denote by the modular group the group of transformations generated by $\gamma(\,.\,)$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Since $(-I)(\tau) = \tau$ this group is isomorphic to $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$.

Definition 2.3. *Two points $\tau_1, \tau_2 \in \mathbb{H}$ are called* $\mathrm{SL}_2(\mathbb{Z})$**-equivalent** *if there exists a $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $\gamma(\tau_1) = \tau_2$. A* **fundamental domain** *$\mathcal{F}$ for $\mathrm{SL}_2(\mathbb{Z})$ is a closed subset of $\mathbb{H}$, such that*

i) every $\tau \in \mathbb{H}$ is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to a point in $\mathcal{F}$.

ii) no two points in the interior of $\mathcal{F}$ are $\mathrm{SL}_2(\mathbb{Z})$-equivalent.

Proposition 2.4. *The following set is a fundamental domain for the action of* $\mathrm{SL}_2(\mathbb{Z})$

$$\mathcal{F} = \left\{ \tau \in \mathbb{H} \mid |\tau| \geq 1 \text{ and } |\operatorname{Re}(\tau)| \leq \frac{1}{2} \right\}.$$

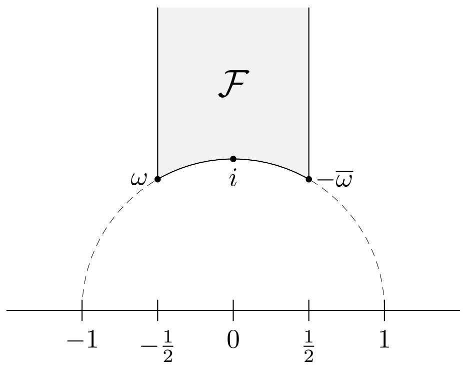

Figure 1: Fundamental domain $\mathcal{F}$ and the points $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$ and $-\overline{\omega} = S(\omega) = \frac{1}{2} + \frac{\sqrt{3}}{2} i$.

Proof. We first show that every element $\tau \in \mathbb{H}$ is equivalent to a point in $\mathcal{F}$: For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have (see Homework 1)

$$\operatorname{Im}(\gamma(\tau)) = \frac{\operatorname{Im}(\tau)}{|c\tau + d|^2}\,. \tag{2.1}$$

Since c, d are integers, we can find a matrix $\gamma_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, such that $|c\tau + d|$ is minimal. In particular we get by (2.1) that

$$\operatorname{Im}(\gamma_0(\tau)) \geq \operatorname{Im}(\gamma(\tau)) \qquad \text{for all } \gamma \in \mathrm{SL}_2(\mathbb{Z})\,. \tag{2.2}$$

Since the action of T corresponds to a horizontal translation, we can find a $j \in \mathbb{Z}$, such that $\gamma_1 = T^j \gamma_0$ satisfies $-\frac{1}{2} \leq \operatorname{Re}(\gamma_1(\tau)) \leq \frac{1}{2}$. We now already have $\gamma_1(\tau) \in \mathcal{F}$ because otherwise we would have $|\gamma_1(\tau)| < 1$ and therefore

$$\operatorname{Im}(S\gamma_1(\tau)) = \frac{\operatorname{Im}(\gamma_1(\tau))}{|\gamma_1(\tau)|^2} > \operatorname{Im}(\gamma_1(\tau)) = \operatorname{Im}(\gamma_0(\tau))\,,$$

which is not possible by (2.2).

We now prove that no two points in the interior of $\mathcal{F}$ are $\mathrm{SL}_2(\mathbb{Z})$-equivalent: Let $\tau \in \mathcal{F}$ and assume we have a $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that also $\gamma(\tau) \in \mathcal{F}$. Without loss of generality we can assume that $\mathrm{Im}(\gamma(\tau)) \geq \mathrm{Im}(\tau)$ (otherwise replace γ by γ^{-1}). By (2.1) we therefore have $|c\tau + d| \leq 1$. Since $c, d \in \mathbb{Z}$ and $\tau \in \mathcal{F}$ this can just be the case if $|c| \leq 1$, which leaves us with the following cases:

i) $c = 0$, $d = \pm 1$: In this case we have $\gamma = \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}$ and therefore we either have $\gamma = I$ or $\mathrm{Re}(\tau) = \pm \frac{1}{2}$, i.e. τ is on one of the vertical boundary lines of $\mathcal{F}$.

ii) $c = \pm 1$, $d = 0$ and $|\tau| = 1$: In this case we have $\gamma = \begin{pmatrix} a & \mp 1 \\ \pm 1 & 0 \end{pmatrix} = \pm T^a S$. This gives either $a = 0$ with τ and $\gamma(\tau)$ on the unit circle (and symmetrically located with respect to the imaginary axis), $a = -1$ with $\tau = \gamma(\tau) = \omega$ or $a = 1$ with $\tau = \gamma(\tau) = -\overline{\omega}$.

iii) $c = d = \pm 1$ and $\tau = \omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$: In this case we have $\gamma = \begin{pmatrix} a & a \mp 1 \\ \pm 1 & \pm 1 \end{pmatrix} = \pm T^a S T$ which gives either $a = 0$ and $\gamma(\tau) = \omega$ or $a = 1$ and $\gamma(\tau) = -\overline{\omega}$.

iv) $c = -d = \pm 1$ and $\tau = -\overline{\omega} = \frac{1}{2} + \frac{\sqrt{3}}{2} i$: This case is similar to case iii).

In all cases we conclude that either $\gamma = I$ or τ and $\gamma(\tau)$ are on the boundary of $\mathcal{F}$. $\square$

Remark 2.5. The following diagram shows how the fundamental domain $\mathcal{F}$ is translated by different matrices in $\mathrm{SL}_2(\mathbb{Z})$.

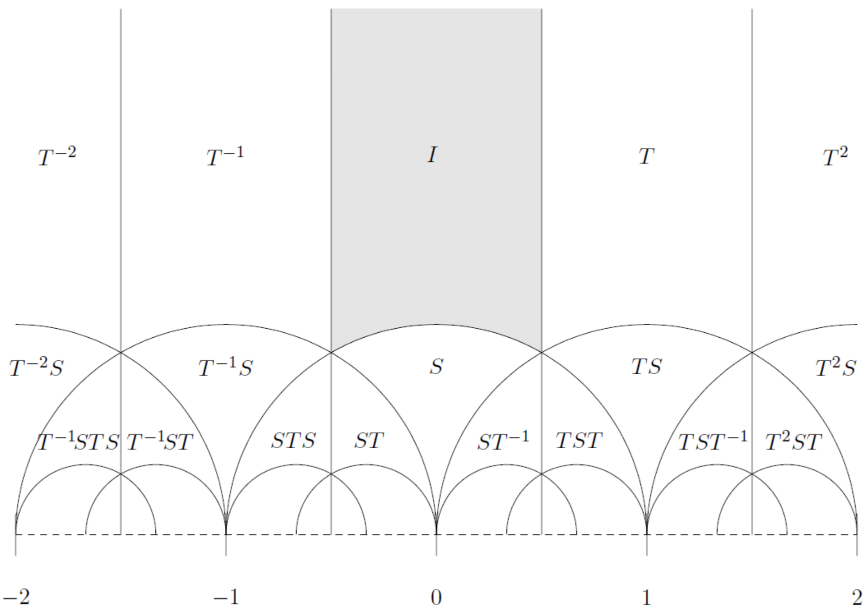

Figure 2: Translations of the fundamental domain $\mathcal{F}$.

2.2 Modular functions and modular forms

We will now recall some basic definitions from complex analysis. For details, we refer to [SS] and [FB].

Definition 2.6. *Let $U \subset \mathbb{C}$ be an open subset of the complex numbers. A function $f : U \to \mathbb{C}$ is called* **holomorphic** *on U, if for all $z_0 \in U$ the limit*

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

exists. If it exists, it is denoted by $f'(z_0)$. By $\mathcal{O}(U)$, we denote the set of all holomorphic functions on the open set U.

A basic fact from complex analysis is that holomorphic functions are also analytic. This means that if f is holomorphic on U, then for each $z_0 \in U$ there exists a $\epsilon > 0$, such that f can be written as a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \,,$$

for all $z \in U$ with $|z - z_0| < \epsilon$ and some $a_n \in \mathbb{C}$.

Definition 2.7. *A function f is* **meromorphic** *on U, if there exists a discrete subset $P \subset U$ with*

i) f is holomorphic on $U \backslash P$.

ii) f has poles at $p_j \in P$.

If f is meromorphic on U, then for each $p \in U$, there exists a unique integer $v_p(f) \in \mathbb{Z}$, a $\epsilon > 0$ and a non-vanishing holomorphic function g on the neighborhood $|z - p| < \epsilon$, such that

$$f(z) = (z-p)^{v_p(f)} g(z)$$

for all $0 < |z - p| < \epsilon$. The $v_p(f) \in \mathbb{Z}$ is called **the order of f at the point** $p \in U$ and we have:

i) If $v_p(f) < 0$ then f has a pole of order $|v_p(f)|$ at p.

ii) If $v_p(f) = 0$ then f has no pole and no zero at p.

iii) If $v_p(f) > 0$ then f has a zero of order $v_p(f)$ at p.

Equivalent to above condition is that f has a **Laurent expansion** in all $p \in U$ of the form

$$f(z) = \sum_{n=v_p(f)}^{\infty} a_n (z-p)^n \,,$$

for $0 < |z - p| < \epsilon$ and $a_n \in \mathbb{C}$ with $a_{v_p(f)} \neq 0$.

Example 2. *i) The rational function $f(z) = \frac{z-2}{(z-1)(z+1)^2}$ is meromorphic on $\mathbb{C}$ with $v_2(f) = 1, v_1(f) = -1$ and $v_{-1}(f) = -2$. Its Laurent expansion around $z = 1$ is given by*

$$f(z) = -\frac{1}{4}(z-1)^{-1} + \frac{1}{2} - \frac{7}{16}(z-1) + \frac{5}{16}(z-1)^2 + \dots .$$

ii) The function $e^{\frac{1}{z}}$ is holomorphic on $\mathbb{C}\backslash\{0\}$, but it is not meromorphic on $\mathbb{C}$, since it has an essential singularity at $z = 0$.

Definition 2.8. *For a function $f : \mathbb{H} \to \mathbb{C}$ and a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we define the* **slash operator of weight** *$k \in \mathbb{Z}$ by*

$$(f|_k\gamma)(\tau) := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) .$$

This gives a right action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{O}(\mathbb{H})$ (Homework 1).

Definition 2.9. *A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a* **weakly modular function of weight** $k \in \mathbb{Z}$*, if it satisfies*

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)\,, \tag{2.3}$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and all $\tau \in \mathbb{H}$. In other words, if it is invariant under the slash operator, i.e., $f|_k\gamma = f$.

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Remark 2.10. (i) Since $-I \in \mathrm{SL}_2(\mathbb{Z})$, a weakly modular function of weight k satisfies $f(\tau) = (-1)^k f(\tau)$. This shows that there are no non-trivial weakly modular functions of odd weight.

(ii) A weakly modular functions f is uniquely determined by its values $f(\tau)$ for $\tau \in \mathcal{F}$.

If f is a weakly modular function of weight k, we have

$$\begin{aligned} f(\tau + 1) &= f(\tau)\,, \\ f(-1/\tau) &= \tau^k f(\tau)\,, \end{aligned} \tag{2.4}$$

by choosing the matrices T and S for (2.3). These two conditions are already sufficient for f to be a weakly modular function of weight k.

Proposition 2.11. *A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ which satisfies* (2.4) *is already a weakly modular function of weight k.*

Proof. This is Homework 1, Exercise 2 (iv). □

Now consider the following holomorphic map from the upper half plane to the punctured unit disc

$$\begin{aligned} \mathbb{H} &\longrightarrow \mathbb{D}^* := \{z \in \mathbb{C} \mid 0 < |z| < 1\}\,, \\ \tau &\longmapsto q_\tau := e^{2\pi i \tau}\,. \end{aligned}$$

First notice that this is indeed a map from $\mathbb{H}$ to $\mathbb{D}^*$, since if $\tau = x + iy$ then $q_\tau = e^{2\pi i \tau} = e^{-2\pi y} e^{2\pi x i}$, which lies in $\mathbb{D}^*$ because of $y > 0$.
The equation $f(\tau + 1) = f(\tau)$ implies that f can be written in the form

$$f(\tau) = \tilde{f}(q_\tau)\,,$$

where $\tilde{f}$ is a meromorphic function on the punctured unit disc $\mathbb{D}^*$.

Definition 2.12. *i) A weakly modular function f is* **meromorphic (resp. holomorphic) in** ∞*, if the function $\tilde{f}$ extends to a meromorphic (resp. holomorphic) function at 0.*

ii) The **order at** ∞ *of a meromorphic weakly modular function f is defined by $v_\infty(f) := v_0(\tilde{f})$.*

Extending to a meromorphic (resp. holomorphic) function at 0, means that there exists a $N \in \mathbb{Z}$ (resp. $N \in \mathbb{Z}_{\geq 0}$) such that the Laurent expansion of $\tilde{f}$ around 0 has the form

$$\tilde{f}(q) = \sum_{n=N}^{\infty} a_n q^n\,,$$

for some $a_n \in \mathbb{C}$. The smallest such N is given by $v_0(\tilde{f})$.

Definition 2.13. *A weakly modular function (of weight k) f is called* **modular function (of weight** k**)** *if it is meromorphic at ∞.*

Definition 2.14. *A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a* **modular form of weight** k*, if*

i) f is a modular function of weight k,

ii) f is holomorphic at ∞.

By M_k, we denote the space of all modular forms of weight k.

In other words, modular forms of weight k are holomorphic functions $f : \mathbb{H} \to \mathbb{C}$, which satisfy (2.3) and which have a **Fourier expansion** of the form

$$f(\tau) = \sum_{n=0}^{\infty} a_n q_\tau^n$$

for some $a_n \in \mathbb{C}$, which are called the **Fourier coefficients** of f. By abuse of notation, we will, in the following, always write q instead of q_τ.

Definition 2.15. *i) A modular form $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$ is called a* **cusp form***, if $a_0 = 0$.*

ii) By S_k we denote[2] *the space of all cusp forms of weight k.*

In other words, cusp forms are modular forms that vanish as $\tau \to i\infty$ or equivalently have order $v_\infty(f) > 0$ at infinity. We have the decomposition $M_k = \mathbb{C}E_k \oplus S_k$ (Homework 1, Exercise 3 (i)).

Example 3. *i) For all $k \in \mathbb{Z}$ the function $f(\tau) = 0$ is a modular form of weight k.*

ii) There are no non-trivial modular forms of odd weight.

iii) For all $c \in \mathbb{C}$ the constant function $f(\tau) = c$ is a modular form of weight 0.

Of course, other non-trivial examples of modular forms exist, as we will see in the next section.

2.3 Eisenstein series

In this section we will introduce Eisenstein series, which is one of the most important examples of modular forms. These already appeared in the first section as q-series. Here we will give their "correct" definition as a function in a complex variable $\tau \in \mathbb{H}$ and calculate their Fourier expansion. For this, we will also need to recall the **Riemann zeta function**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\,, \qquad (s \in \mathbb{C},\ \mathrm{Re}(s) > 1)$$

which will give the constant term in the Fourier expansion of the Eisenstein series.

Proposition 2.16. *For even $k \geq 4$ the* **Eisenstein series of weight** k*, defined by*

$$G_k(\tau) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m\tau + n)^k}\,, \tag{2.5}$$

is a modular form of weight k.

[2] "S" in S_k stands for the German word "Spitzenform" for cusp form.

Proof. First one can check that for $k > 2$ the above sum is absolutely convergent and uniformly convergent on compacts subset (actually also on $\mathcal{F}$) of $\mathbb{H}$ and therefore G_k is a holomorphic function on $\mathbb{H}$. For proof of this fact, we refer to the literature (see for example [Ki, Lemma 2.7] or [S, p. 82, Lemma 1]).

To check that G_k is holomorphic at infinity, we will show that $G_k(\tau)$ approaches an explicit finite limit as $\tau \to i\infty$. By the uniform convergence, we can exchange summation and the limit and obtain

$$\lim_{\tau\to i\infty} G_k(\tau) = \frac{1}{2}\lim_{\tau\to i\infty} \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k} = \frac{1}{2}\sum_{0\neq n\in\mathbb{Z}} \frac{1}{n^k} = \zeta(k)\,.$$

Now we check the modularity conditions for G_k. For this, the sum must converge absolutely and therefore we are allowed to arrange the terms in any way. To show that $G_k(\tau+1) = G_k(\tau)$ we calculate

$$G_k(\tau+1) = \frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m(\tau+1)+n)^k} = \frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m\tau+(m+n))^k}\,.$$

As (m,n) runs over $\mathbb{Z}^2\backslash\{(0,0)\}$, so does $(m, m+n) = (m, n')$, so by the absolute convergence we get

$$G_k(\tau+1) = \frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m\tau+m+n)^k} = \frac{1}{2}\sum_{\substack{m,n'\in\mathbb{Z}\\(m,n')\neq(0,0)}} \frac{1}{(m\tau+n')^k} = G_k(\tau)\,.$$

Similarly, to show $G_k(-1/\tau) = \tau^k G_k(\tau)$ we derive

$$G_k(-1/\tau) = \frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(-m/\tau+n)^k} = \tau^k\frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(n\tau-m)^k} = \tau^k G_k(\tau)\,,$$

since also $(n,-m)$ runs over $\mathbb{Z}^2\backslash\{(0,0)\}$. □

We will now calculate the Fourier expansion of G_k for which we will need the following lemma.

Lemma 2.17. *(Lipschitz‘s formula) For $k \geq 2$ and $\tau \in \mathbb{H}$ we have*

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{d=1}^{\infty} d^{k-1}q^d\,. \tag{2.6}$$

Proof. (Sketch) This follows by differentiating the following two expressions of the cotangent $k-1$-times

$$\sum_{n\in\mathbb{Z}} \frac{1}{\tau+n} = \frac{\pi}{\tan(\pi\tau)} = -\pi i - 2\pi i\sum_{d=1}^{\infty} q^d\,.$$

See for example [Z, Proposition 5] for more details. □

Proposition 2.18 (Fourier expansion of G_k). *For even $k \geq 4$ the Fourier expansion of G_k is given by*

$$G_k(\tau) = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n\,. \tag{2.7}$$

Proof. Again we use absolute convergence which allows the following rearrangements

$$G_k(\tau) = \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k} \overset{k \text{ even}}{=} \frac{1}{2} \sum_{0\neq n\in\mathbb{Z}} \frac{1}{n^k} + \sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^k}$$
$$\overset{(2.6)}{=} \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} \sum_{d=1}^{\infty} d^{k-1} q^{md} = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \,.$$

□

In the definition of G_k, we needed $k > 2$ to assure absolute convergence. But the q-series in (2.7) also makes sense[3] for $k = 2$ and also defines a holomorphic function in $\tau \in \mathbb{H}$. We therefore use this equation to define the **Eisenstein series of weight** 2 by

$$G_2(\tau) := \zeta(2) + (2\pi i)^2 \sum_{n=1}^{\infty} \sigma_1(n) q^n \,. \tag{2.8}$$

This is not a modular form anymore, but plays an important role in the theory of modular forms. We have the following Proposition which gives the failure of G_2 to be a modular form.

Proposition 2.19 (Modular transformation of G_2)**.** *For $\tau \in \mathbb{H}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we have*

$$G_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 G_2(\tau) - \pi i c(c\tau+d) \,. \tag{2.9}$$

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Proof. We will just sketch the idea of this proof. For details see for example [Z, Proposition 6] or [Ko, Chapter III, Proposition 7]. The sum (2.5) does not converge absolutely for $k = 2$, but it does for $k > 2$. We now use what is called Hecke's trick and consider for $\epsilon > 0$ the function

$$G_{2,\epsilon}(\tau) = \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^2 |m\tau+n|^{2\epsilon}} \,. \tag{2.10}$$

This sum does converge absolutely and we can use the same argument as before to see that for $\epsilon > 0$

$$G_{2,\epsilon}\left(\frac{a\tau+b}{c\tau+d}\right) = (cz+d)^2 |cz+d|^{2\epsilon} G_{2,\epsilon}(\tau) \,. \tag{2.11}$$

One then shows that the limit $\epsilon \to 0$ of $G_{2,\epsilon}$ exists and equals $G_2(\tau) - \frac{\pi}{2\,\mathrm{Im}(\tau)}$. Taking the limit $\epsilon \to 0$ in (2.11) then proves the statement. □

We will later see that (2.19) means that G_2 is a **quasimodular form**.

Proposition 2.20 (L. Euler (1735))**.** *For even $k \geq 2$ we have*

$$\zeta(k) = -\frac{B_k}{2k} \frac{(2\pi i)^k}{(k-1)!} \,,$$

where B_k denotes the k-th Bernoulli number defined in (1.6)*.*

[3] The q-series (2.7) actually makes for any $k \geq 2$, but for odd k these are not modular forms.

Proof. See for example [FB, Proposition III.7.14]. □

We get the following values of $\zeta(k)$ for $k = 2, 4, 6, 8, 10, 12$:

$$\zeta(2) = \frac{\pi^2}{6}\,, \quad \zeta(4) = \frac{\pi^4}{90}\,, \quad \zeta(6) = \frac{\pi^6}{945}\,, \quad \zeta(8) = \frac{\pi^8}{9450}\,, \quad \zeta(10) = \frac{\pi^{10}}{93555}\,, \quad \zeta(12) = \frac{691\pi^{12}}{638512875}\,.$$

Using Proposition 2.20 we define for even $k \geq 2$ the **normalized Eisenstein series** by

$$E_k(\tau) = \frac{1}{\zeta(k)} G_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n\,. \tag{2.12}$$

These are the q-series which also appeared in (1.7).

2.4 Cusp forms and the discriminant function Δ

In this section, we will consider cusp forms, which are modular forms vanishing at the "cusps". By cusps, one usually denotes the classes of $\mathbb{Q} \cap \{\infty\}$ modulo the action of a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. For the level one case, where we consider the whole group $\mathrm{SL}_2(\mathbb{Z})$, we have $|\,\mathrm{SL}_2(\mathbb{Z})\backslash(\mathbb{Q} \cap \{\infty\})| = 1$, because every rational number can be send to ∞ by a linear fractional transformation. This means there is just one cusp. A cusp form of level one is, therefore, a modular form which vanishes at ∞i or, equivalently, has a vanishing constant term in its Fourier expansion.

Definition 2.21. *We define the* **discriminant function** Δ *by* ($q = e^{2\pi i \tau}$)

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}\,. \tag{2.13}$$

The function $\tau(n)$ defined by $\Delta(\tau) = \sum_{n=1}^{\infty} \tau(n) q^n$ is called **Ramanujan tau function**.

Expanding the product in the definition of Δ gives the following first values for $\tau(n)$

$$\Delta(\tau) = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + 84480q^8 - 113643q^9 + \dots\,. \tag{2.14}$$

Remark 2.22.

i) Ramanujan observed in 1915 that $\tau(n)$ is multiplicative, i.e. $\tau(m \cdot n) = \tau(m) \cdot \tau(n)$ for coprime $m, n \in \mathbb{Z}_{\geq 1}$. For example $\tau(6) = -6048 = -24 \cdot 252 = \tau(2) \cdot \tau(3)$. This was proved by Mordell the next year and later generalized by Hecke to the theory of **Hecke operators**, which we will discuss later. The function Δ is an example of a **Hecke eigenform** (meaning it is an eigenvector for all Hecke operators having 1 as the coefficient of q), which all satisfy the property that their Fourier coefficients are multiplicative.
The divisor-sums $\sigma_{k-1}(n)$ are also multiplicative and the Eisenstein series, after some normalization, are also examples of Hecke eigenforms.

ii) Lehmer (1947) conjectured that $\tau(n) \neq 0$ for all $n \geq 1$, an assertion sometimes known as Lehmer's conjecture. This conjecture is still unproven but checked for all $1 \leq n \leq 816212624008487344127999$ (due to Derickx, van Hoeij, and Zeng in 2013).

Since $|e^{2\pi i \tau}| < 1$ for $\tau \in \mathbb{H}$, the terms of the infinite product (2.13) are all non-zero and tend exponentially rapidly to 1, so Δ gives a holomorphic and everywhere non-zero function on $\mathbb{H}$. It gives the first example of a non-trivial cusp form.

Proposition 2.23. *The function $\Delta(\tau)$ is a cusp form of weight* 12.

Proof. Since $\Delta(\tau) \neq 0$, we can consider its logarithmic derivative. We find

$$\frac{1}{2\pi i}\frac{d}{d\tau}\log\Delta(\tau) = \frac{1}{2\pi i}\frac{d}{d\tau}\log\left(q\prod_{n=1}^{\infty}(1-q^n)^{24}\right) = \frac{1}{2\pi i}\frac{d}{d\tau}\left(\log(q) + 24\sum_{n=1}^{\infty}\log(1-q^n)\right).$$

Since $q = e^{2\pi i\tau}$ we have $\frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$ and therefore

$$\frac{1}{2\pi i}\frac{d}{d\tau}\log\Delta(\tau) = 1 - 24\sum_{n=1}^{\infty} n\frac{q^n}{1-q^n} = 1 - 24\sum_{n=1}^{\infty} n\sum_{d=1}^{\infty} q^{dn} = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n = E_2(\tau)\,. \tag{2.15}$$

By Proposition 2.19 and $E_2(\tau) = \frac{6}{\pi^2}G_2(\tau)$ we have

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2E_2(\tau) - \frac{6}{\pi}ic(c\tau+d)\,. \tag{2.16}$$

Combining (2.15), (2.16) and using

$$\frac{d}{d\tau}\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{ad-bc}{(c\tau+d)^2} = \frac{1}{(c\tau+d)^2}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we deduce

$$\frac{1}{2\pi i}\frac{d}{d\tau}\log\left(\frac{\Delta\left(\frac{a\tau+b}{c\tau+d}\right)}{(c\tau+d)^{12}\Delta(\tau)}\right) = \frac{1}{(c\tau+d)^2}E_2\left(\frac{a\tau+b}{c\tau+d}\right) - \frac{12}{2\pi i}\frac{c}{c\tau+d} - E_2(\tau) = 0\,.$$

In other words, $(\Delta|_{12}\gamma)(\tau) = C(\gamma)\Delta(\tau)$ for all $\tau \in \mathbb{H}$ and all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, where $C(\gamma)$ is a non-zero complex number depending only on γ. We want to show that $C(\gamma) = 1$ for all γ.
The slash operator $|_k$ gives a right action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$, i.e. for γ_1, γ_2 we get

$$C(\gamma_1)C(\gamma_2)\Delta = C(\gamma_1)\Delta|_{12}\gamma_2 = \Delta|_{12}(\gamma_1)|_{12}\gamma_2 = \Delta|_{12}(\gamma_1\gamma_2) = C(\gamma_1\gamma_2)\Delta\,.$$

Therefore $C : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}$ is a homomorphism and we just need to prove $C(T) = C(S) = 1$. By definition we have $\Delta(T(\tau)) = \Delta(\tau)$, since it is defined by a q-series, which gives $C(T) = 1$. To show $C(S) = 1$, we set $\tau = i$ in $\tau^{-12}\Delta(-\frac{1}{\tau}) = (\Delta|_{12}S)(\tau) = C(S)\Delta(\tau)$. □

Remark 2.24. Since E_4^3 and Δ are modular forms of weight 12 and $\Delta(\tau) \neq 0$ for $\tau \in \mathbb{H}$, the **modular invariant** (or j-invariant), defined by

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)}$$

is a holomorphic function in $\mathbb{H}$ satisfying $j(\gamma(\tau)) = j(\tau)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Since Δ has a zero of order 1 at ∞ and E_4 does not vanish there, the function j has a pole of order 1 at ∞. Therefore j is a modular function of weight 0, which is not a modular form. Its Fourier expansion, the Laurent expansion at $q = 0$ of $\widetilde{j}$, starts with

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \dots\,.$$

These Fourier coefficients, for the positive exponents of q, are the dimensions of the graded part of an infinite-dimensional graded algebra representation of the so called monster group.

2.5 Structure of the space of modular forms

We now come to a very important technical result about modular forms. To state and prove this result, we will use some definitions and results from complex analysis that can be found again in [FB] or [SS]. Especially the notion of contour integration will be necessary, which can be found in [SS, Section 1.3] or [FB, Chapter 2].

Proposition 2.25 (Argument principle)**.** *If f is a meromorphic function inside and on some closed contour $\mathcal{C}$ with interior $D \subset \mathbb{C}$, and f has no zeros or poles on $\mathcal{C}$, then*

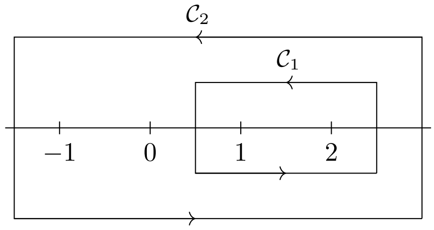
$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f'(z)}{f(z)}dz = \sum_{p\in D} v_p(f)\,.$$

Proof. See for example [FB, Proposition III.7.4]. □

Example 4. *We again consider the rational function $f(z) = \frac{z-2}{(z-1)(z+1)^2}$, which is meromorphic on $\mathbb{C}$ with $v_2(f) = 1, v_1(f) = -1$, $v_{-1}(f) = -2$ and $v_p(f) = 0$ for $p \in \mathbb{C}\backslash\{-1,1,2\}$.*

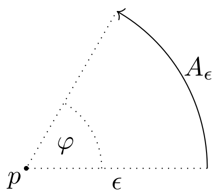
With the two contours $\mathcal{C}_1$ and $\mathcal{C}_2$ shown on the right, we get for example

$$\frac{1}{2\pi i}\int_{\mathcal{C}_1} \frac{f'(z)}{f(z)}dz = v_1(f) + v_2(f) = 0\,,$$
$$\frac{1}{2\pi i}\int_{\mathcal{C}_2} \frac{f'(z)}{f(z)}dz = v_{-1}(f) + v_1(f) + v_2(f) = -2\,.$$



Lemma 2.26. *(Integration over arcs) Let f be a meromorphic function on some open set $U \subset \mathbb{C}$. For an arc $A_\epsilon \subset U$ of radius $\epsilon > 0$, center $p \in U$, angle φ not intersecting any zeros or poles of f, we have*

$$\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{A_\epsilon} \frac{f'(z)}{f(z)}dz = \frac{\varphi}{2\pi}v_p(f)\,.$$



Proof. See for example part (4) in the proof of [FB, Theorem VI.2.3]. □

Lemma 2.27. *Let f be a modular function of weight k with no zeros or poles on a (not necessarily closed) contour $\mathcal{C} \subset \mathbb{H}$. Then*

$$\int_{\mathcal{C}} \frac{f'(\tau)}{f(\tau)}d\tau - \int_{\gamma(\mathcal{C})} \frac{f'(\tau)}{f(\tau)}d\tau = -k\int_{\mathcal{C}} \frac{c}{c\tau+d}d\tau$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Proof. Differentiating $f(\gamma(\tau)) = (c\tau+d)^k f(\tau)$ gives

$$f'(\gamma(\tau))\frac{d(\gamma(\tau))}{d\tau} = (c\tau+d)^k f'(\tau) + kc(c\tau+d)^{k-1}f(\tau)\,.$$

Dividing the left-hand side by $f(\gamma(\tau))$ and the right-hand side by $(c\tau+d)^k f(\tau)$ leads to

$$\frac{f'(\gamma(\tau))}{f(\gamma(\tau))}d(\gamma(\tau)) = \frac{f'(\tau)}{f(\tau)}d\tau + k\frac{c}{c\tau+d}d\tau$$

and therefore

$$\int_{\mathcal{C}} \left(\frac{f'(\tau)}{f(\tau)} d\tau - \frac{f'(\gamma(\tau))}{f(\gamma(\tau))} d(\gamma(\tau)) \right) = -k \int_{\mathcal{C}} \frac{c}{c\tau + d} d\tau\,.$$

$\square$

Example 5. *For $\gamma = S$ Lemma 2.27 gives for a modular function f of weight k*

$$\int_{\mathcal{C}} \frac{f'(\tau)}{f(\tau)} d\tau - \int_{S(\mathcal{C})} \frac{f'(\tau)}{f(\tau)} d\tau = -k \int_{\mathcal{C}} \frac{1}{\tau} d\tau\,. \tag{2.17}$$

Since the factor $(c\tau + d)^k$ does not vanish for $\tau \in \mathbb{H}$ and $c, d \in \mathbb{Z}$, we have $v_p(f) = v_{\gamma(p)}(f)$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and a modular function f. The following theorem gives a restriction on the orders of a modular functions, which will be crucial to describe the space M_k afterwards.

Theorem 2.28 (Valence formula)**.** *For a non-zero modular function f of weight k we have*

$$v_\infty(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_\omega(f) + \sum_{\substack{p \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \\ p \neq i, \omega}} v_p(f) = \frac{k}{12}\,. \tag{2.18}$$

Until here in lecture 4 (12th May, 2023)

Proof. The idea of the proof is to count the (order) of the zeros and poles in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ by integrating the logarithmic derivative f'/f of f around the boundary of the fundamental domain $\mathcal{F}$ and then applying the argument principle.

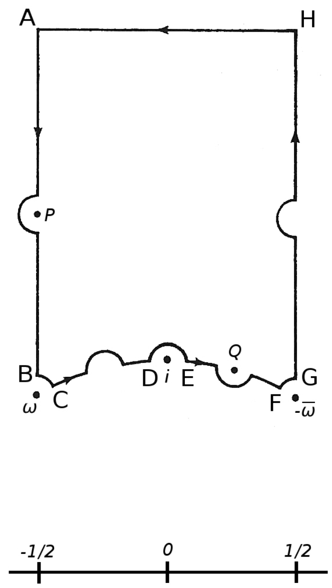


Figure 3: The contour $\mathcal{C}$.

More precisely, we need an approximation first and start with a curve as shown in Figure 3. The contour $\mathcal{C}$ is chosen in such a way that it contains exactly one represent in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ of each pole and zero, except i, ω (and $-\overline{\omega} = S(\omega)$) which are kept outside.

Since f is a modular functions, it is meromorphic at ∞. This means that for some $T \in \mathbb{R}$ the function f has no poles or zeros with imaginary part larger than T. Therefore we can choose the the top line from $H = \frac{1}{2} + iT$ to $A = -\frac{1}{2} + iT$ such that f does not have any poles or zeros on or on top of the line HA.

The rest of the contour follows the boundary of $\mathcal{F}$ with a few exceptions: For each zero or pole $P \neq i, \omega$ on the boundary, we simply circle around it with a small enough radius and the other way round for the congruent point on the other side of the boundary (this way we will only count the point once). This procedure is illustrated for two such points P and Q in Figure 3.

So far we still followed the boundary of $\mathcal{F}$ (modulo $\mathrm{SL}_2(\mathbb{Z})$) but since we don't want to include i and ω we also have to circle around those points with a small enough radius ϵ (and the same way for $-\overline{\omega} = S(\omega)$).

By the argument principle (Proposition 2.25) we obtain

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f'(\tau)}{f(\tau)} d\tau = \sum_{\substack{p \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \\ p \neq i, \omega}} v_p(f) \,. \tag{2.19}$$

On the other hand we can evaluate the contour integral over $\mathcal{C}$ on the left-hand side section by section:

i) AB and GH: The integral from A to B cancels the integral from G to H, because $f(\tau+1) = f(\tau)$, and the lines go in opposite direction, i.e.

$$\frac{1}{2\pi i} \left(\int_A^B + \int_G^H \right) \frac{f'(\tau)}{f(\tau)} d\tau = 0 \,.$$

ii) HA: By the map $q = e^{2\pi i \tau}$ the line from H to A gets send to a circle in the unit disc of radius $e^{-2\pi T}$ running clockwise around 0. Recall that $f(\tau) = \tilde{f}(q)$ and therefore we have $\frac{f'(\tau)}{f(\tau)} d\tau = \frac{\tilde{f}'(q)}{\tilde{f}(q)} dq$. By the argument principle we get

$$\frac{1}{2\pi i} \int_H^A \frac{f'(\tau)}{f(\tau)} d\tau = \frac{1}{2\pi i} \int_{|q| = e^{-2\pi T}} \frac{\tilde{f}'(q)}{\tilde{f}(q)} dq = -v_0(\tilde{f}) = -v_\infty(f) \,.$$

Here the minus sign comes from the fact that the contour integral runs clockwise around 0.

iii) BC, DE and FG: All these three sections are small arcs of a small radius ϵ which approach angles $\frac{\pi}{3}$, π and $\frac{\pi}{3}$ as $\epsilon \to 0$. Using Lemma 2.26 and noticing that all three arcs run clockwise (minus sign), we obtain

$$\begin{aligned} \lim_{\epsilon \to 0} \frac{1}{2\pi i} \left(\int_B^C + \int_D^E + \int_F^G \right) \frac{f'(\tau)}{f(\tau)} d\tau &= -\frac{1}{2\pi} \left(\frac{\pi}{3} v_\omega(f) + \pi v_i(f) + \frac{\pi}{3} v_{-\overline{\omega}}(f) \right) \\ &= -\frac{1}{2} v_i(f) - \frac{1}{3} v_\omega(f) \,, \end{aligned}$$

where we used $v_{-\overline{\omega}}(f) = v_{S(\omega)}(f) = v_\omega(f)$ in the last equation.

iv) CD and EF: First notice that the transformation $S(\tau) = -\frac{1}{\tau}$ sends the contour CD to the contour EF, but with directions reversed. By (2.17) we therefore have

$$\frac{1}{2\pi i} \left(\int_C^D \frac{f'(\tau)}{f(\tau)} d\tau + \int_E^F \frac{f'(\tau)}{f(\tau)} d\tau \right) = -\frac{k}{2\pi i} \int_C^D \frac{1}{\tau} d\tau \,.$$

Sending $\epsilon \to 0$, the integral on the right-hand side is just an integral over an arc from ω to i:

$$\lim_{\epsilon\to 0} -\frac{k}{2\pi i}\int_C^D \frac{1}{\tau}d\tau = -\frac{k}{2\pi i}\int_{\text{arc from }\omega\text{ to }i} \frac{1}{\tau}d\tau \overset{\tau=e^{\Theta i}}{=} = -\frac{k}{2\pi}\int_{\frac{2\pi}{3}}^{\frac{\pi}{2}} d\Theta = \frac{k}{2\pi}\left(\frac{2\pi}{3}-\frac{\pi}{2}\right) = \frac{k}{12}$$

and therefore we obtain

$$\frac{1}{2\pi i}\left(\int_C^D + \int_E^F\right)\frac{f'(\tau)}{f(\tau)}d\tau = \frac{k}{12}.$$

Combining the parts i) - iv) and plugging them into the left-hand side of (2.19) finishes the proof. □

Recall that the difference between a modular function and a modular form is, that a modular form is holomorphic on $\mathbb{H}$ and at ∞. This means that for $f \in M_k$ all the numbers in (2.18) are positive and therefore for a fixed k there are just finitely many solutions. This leads to the following proposition.

Proposition 2.29. *Let $k \in \mathbb{Z}$ be an integer. Then*

i) $M_0 = \mathbb{C}$,

ii) If $k = 2$, $k < 0$ *or if* k *is odd then* $M_k = 0$.

iii) If $k \in \{4, 6, 8, 10, 14\}$, *then* $M_k = \mathbb{C}E_k$.

iv) If $k < 12$ *or* $k = 14$ *then* $S_k = 0$.

v) $S_{12} = \mathbb{C}\Delta$ *and if* $k > 12$ *then* $S_k = \Delta \cdot M_{k-12}$.

vi) If $k \geq 4$ *then* $M_k = \mathbb{C}E_k \oplus S_k$.

Proof. i) We know that the constant functions are elements in M_0 and we want to show the reverse. Let $f \in M_0$ be an arbitrary modular form of weight 0 and let $c \in \mathbb{C}$ be any element in the image of f. Then $f(z) - c \in M_0$ has a zero in $\mathbb{H}$, i.e. one of the terms in (2.18) is strictly positive. Since the right-hand side is 0, this can only happen if $f(z) - c$ is the zero function, i.e. f is constant.

ii) We already saw that $M_k = 0$ if k is odd. If $k = 2$ or $k < 0$ then the right-hand side of (2.18) is negative or $\frac{1}{6}$, which has no positive solutions on the left-hand side.

iii) When $k \in \{4, 6, 8, 10, 14\}$, then there is only one possible way of choosing the $v_p(f)$, such that (2.18) holds:

$k = 4:$ $v_\omega(f) = 1$ and all other $v_p(f) = 0$.
$k = 6:$ $v_i(f) = 1$ and all other $v_p(f) = 0$.
$k = 8:$ $v_\omega(f) = 2$ and all other $v_p(f) = 0$.
$k = 10:$ $v_\omega(f) = v_i(f) = 1$ and all other $v_p(f) = 0$.
$k = 14:$ $v_\omega(f) = 2$, $v_i(f) = 1$ and all other $v_p(f) = 0$.

For such k two arbitrary modular forms $f_1, f_2 \in M_k$ have the same order at all points, i.e. $\frac{f_1}{f_2}$ is a modular form of weight 0, which by i) must be constant. Therefore f_1 and f_2 are proportional and since $E_k \in M_k$ the statement follows.

iv) If $f \in S_k$ we have $v_\infty(f) > 0$, which is not possible in (2.18) for $k < 12$ or $k = 14$.

v) We know that $v_\infty(\Delta) = 1$ and by (2.18) this can be the only zero of Δ. Therefore for any $f \in S_k$ the function $\frac{f}{\Delta}$ is a modular form of weight $k - 12$.

vi) This was part of Homework 1.

□

Theorem 2.30. *(Dimension formula) For an even positiver integer k we have*

$$\dim_{\mathbb{C}} M_k = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & , \quad k \not\equiv 2 \mod 12 \\ \lfloor \frac{k}{12} \rfloor & , \quad k \equiv 2 \mod 12 \end{cases} . \tag{2.20}$$

Proof. This will now follow by induction on k from the results in Proposition 2.29. For $k < 12$ the above dimension formula is already proven. Combing the results of Proposition 2.29 we have

$$M_{k+12} = \mathbb{C}E_{k+12} \oplus \Delta \cdot M_k$$

and since $\lfloor \frac{k}{12} \rfloor + 1 = \lfloor \frac{k+12}{12} \rfloor$ the statement follows inductively. □

k	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36
$\dim_{\mathbb{C}} M_k$	1	0	1	1	1	1	2	1	2	2	2	2	3	2	3	3	3	3	4

Figure 4: Dimension of M_k for even $0 \leq k \leq 36$.

Completion of the modular proof of Theorem 1.4. Both E_4^2 and E_8 are modular forms of weight 8. Since $\dim_{\mathbb{C}} M_8 = 1$ there must exists a $c \in \mathbb{C}$ with $E_4^2 = cE_8$. But since both have 1 as the constant term in their Fourier expansion we deduce $c = 1$. □

Both E_4^3 and E_6^2 are modular forms of weight 12 having 1 as the constant term in their Fourier expansion and therefore $E_4^3 - E_6^2 \in S_{12}$. By Proposition 2.29 v) this has to be a multiple of Δ and comparing the first few Fourier coefficients gives

$$\Delta(\tau) = \frac{E_4(\tau)^3 - E_6(\tau)^2}{1728} . \tag{2.21}$$

In general every modular form can be written (uniquely) as a polynomial in E_4 and E_6:

Proposition 2.31. *For $k \geq 0$, the set $\{E_4^a E_6^b \mid a, b \geq 0, 4a + 6b = k\}$ is a basis of the space M_k.*

Until here in lecture 5 (19th May, 2023)

Proof. We first check that the mentioned set has the correct size. Let N_k be the number of solutions to $4a + 6b = k$ in nonnegative integers a and b. For $k \leq 12$ one can check directly that $N_k = \dim_{\mathbb{C}} M_k$ (given in (2.20)) and for $k \geq 12$ one can check that $N_k = N_{k-12} + 1$. Therefore we have $N_k = \dim_{\mathbb{C}} M_k$ for all k. It remains to show that the set is linearly independent. Suppose we have a relation of the form

$$\sum_{\substack{4a+6b=k \\ a,b\geq 0}} \lambda_{a,b} E_4(\tau)^a E_6(\tau)^b = 0$$

for all $\tau \in \mathbb{H}$. If there is a pure E_4 term, say $\lambda_{a,0} E_4(\tau)^a$, then setting $\tau = i$ shows $\lambda_{a,0} E_4(i)^a = 0$ since $E_6(i) = 0$ (Homework 1). Since $E_4(i) \neq 0$ (which follows from the valence formula (2.18)) we deduce $\lambda_{a,0} = 0$. Therefore all nonzero terms in the sum have $b \geq 1$. As E_6 is not identically 0, we can divide by it and get

$$\sum_{\substack{4a+6b=k \\ a,b\geq 0}} \lambda_{a,b} E_4(\tau)^a E_6(\tau)^{b-1} = 0 ,$$

which is a linear relation in weight $k-6$. By induction we see that the remaining coefficients are 0. $\square$

Remark 2.32. Starting with a modular form $f = \sum_{n=0}^{\infty} a_n q^n \in M_k$ and choosing a and b with $4a + 6b = k$, we have $f - a_0 E_4^a E_6^b \in S_k$. By Proposition 2.29 v) we have $S_k = \Delta \cdot M_{k-12}$, i.e. we find a $g \in M_{k-12}$ with $f = a_0 E_4^a E_6^b + \Delta \cdot g$. With the explicit expression (2.21) of Δ, this gives a recursive algorithm (and in fact another way of proving Proposition 2.31) to write f as a polynomial in E_4 and E_6.

Proposition 2.33. *Modular forms with different weights are linearly independent over* $\mathbb{C}$.

Proof. Suppose we have nonzero modular forms $f_1, f_2, \ldots, f_m$ with respective weights $k_1 < k_2 < \cdots < k_m$, such that they admit a nontrivial linear relation

$$\alpha_1 f_1(\tau) + \alpha_2 f_2(\tau) + \cdots + \alpha_m f_m(\tau) = 0 \tag{2.22}$$

for all $\tau \in \mathbb{H}$ and $\alpha_j \neq 0$ for $j = 1, \ldots, m$. Replacing τ by $S(\tau)$ and using the modularity, i.e. $f_j(S(\tau)) = \tau^{k_j} f_j(\tau)$, we obtain

$$\alpha_1 \tau^{k_1} f_1(\tau) + \alpha_2 \tau^{k_2} f_2(\tau) + \cdots + \alpha_m \tau^{k_m} f_m(\tau) = 0$$

for all $\tau \in \mathbb{H}$. With Fourier expansions $f_j(\tau) = \sum_{n=0}^{\infty} a_n^{(j)} q^n$ where $q = e^{2\pi i \tau}$, this is equivalent to

$$\sum_{n=0}^{\infty} \left(\alpha_1 \tau^{k_1} a_n^{(1)} + \alpha_2 \tau^{k_2} a_n^{(2)} + \cdots + \alpha_m \tau^{k_m} a_n^{(m)} \right) e^{2\pi i n \tau} = 0 \,.$$

Now consider the case of $\tau = iy$ $(y > 0)$ being on the positive imaginary axis, then

$$\sum_{n=0}^{\infty} \left(\alpha_1 (iy)^{k_1} a_n^{(1)} + \alpha_2 (iy)^{k_2} a_n^{(2)} + \cdots + \alpha_m (iy)^{k_m} a_n^{(m)} \right) e^{-2\pi n y} = 0 \,. \tag{2.23}$$

For $n > 0$ and any $r \geq 0$ we have $\lim_{y\to\infty} y^r e^{-2\pi n y} = 0$. Now let N be the smallest integer, such that at least for one $1 \leq j \leq m$ we have $a_N^{(j)} \neq 0$. Dividing (2.23) by $e^{-2\pi N y}$ and taking the limit $y \to \infty$ we obtain

$$\lim_{y\to\infty} \alpha_1 (iy)^{k_1} a_N^{(1)} + \alpha_2 (iy)^{k_2} a_N^{(2)} + \cdots + \alpha_m (iy)^{k_m} a_N^{(m)} = 0 \,.$$

But the left-hand side of this equation is the limit $y \to \infty$ of a non-constant polynomial in y, which can not be zero and therefore a relation of the form (2.22) can not exist. $\square$

Proposition 2.34. *The modular forms* E_4 *and* E_6 *are algebraically independent over* $\mathbb{C}$.

Proof. Let $P \in \mathbb{C}[X, Y]$ be with $P(E_4(\tau), E_6(\tau)) = 0$ for all $\tau \in \mathbb{H}$. By Proposition 2.33 we can reduce this to the case where $P(E_4, E_6)$ is a sum of modular forms of the same weight k. But by Proposition 2.31 we know that $E_4^a E_6^b$ with $4a + 6b = k$ are linearly independent and therefore we conclude $P = 0$. $\square$

Summarizing all the results we get the following description of the space of modular forms.

Corollary 2.35. *Let* M *denote the* **space of all modular forms** *(of level 1), then we have*

$$M = \bigoplus_{k=0}^{\infty} M_k = \mathbb{C}[E_4, E_6] \cong \mathbb{C}[X, Y] \,,$$

i.e. M *is a graded* $\mathbb{C}$*-algebra, which is isomorphic to the polynomial ring in two variables.*

Finally, we give another consequence of the valence formula (2.18). The following statement can be used to determine if two modular forms of a given weight k are the same if they have the same first $\frac{k}{12}$ Fourier coefficients.

Corollary 2.36 (Sturm bound)**.** *Let $f, g \in M_k$ be two modular forms of weight k with Fourier expansions $f(\tau) = \sum_{n\geq 0} a_n q^n$ and $g(\tau) = \sum_{n\geq 0} b_n q^n$. If $a_n = b_n$ for $0 \leq n < \frac{k}{12} + 1$ then $f = g$.*

Proof. If we set $h = f - g$ then $h \in M_k$ and $v_\infty(h) > \frac{k}{12}$ since $h(\tau) = \sum_{n\geq \frac{k}{12}+1}(a_n - b_n)q^n$. By the valence formula (2.18) this is just possible if h is the zero function and therefore $f = g$. □

3 Modular forms for congruence subgroups

In this course, we just considered modular forms of level 1. We want to consider higher level modular forms or more precisely modular forms for congruence subgroups. A complete discussion of modular forms for higher level can be found for example in [DS]. So far we always required that a modular form (or (weakly-)modular function) satisfies $f|_k\gamma = f$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. This condition will be weakened now and we will just require it for $\gamma \in \Gamma$, where $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ are certain subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

3.1 Congruence subgroups

Definition 3.1. *i) For $N \in \mathbb{Z}_{\geq 1}$ we define the following subgroups of* $\mathrm{SL}_2(\mathbb{Z})$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \mod N \right\},$$
$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \;\middle|\; a \equiv d \equiv 1 \mod N \right\},$$
$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) \;\middle|\; b \equiv 0 \mod N \right\}.$$

By definition we have the inclusions

$$\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z}).$$

ii) A subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is called **congruence subgroup** *if there exists a N with $\Gamma(N) \subset \Gamma$. The smallest such N is called the* **level of** Γ.

We have $\Gamma(1) = \Gamma_1(1) = \Gamma_0(1) = \mathrm{SL}_2(\mathbb{Z})$ and hence $\mathrm{SL}_2(\mathbb{Z})$ is the only congruence subgroup of level 1. The congruence subgroups all have finite index, which can be computed explicitly:

Proposition 3.2. *For $N \geq 1$ we have*

$$[\Gamma_1(N) : \Gamma(N)] = N,$$
$$[\Gamma_0(N) : \Gamma_1(N)] = N \prod_{\substack{p|N \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right),$$
$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{\substack{p|N \\ p \text{ prime}}} \left(1 + \frac{1}{p}\right).$$

Proof. The last formula is part of Homework 2. □

For example we have $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(2)] = 2\left(1 + \frac{1}{2}\right) = 3$. As before we can define weakly modular functions for Γ:

Definition 3.3. *Let Γ be a congruence subgroup. A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is a* **weakly modular function** *of weight $k \in \mathbb{Z}$ for Γ, if $f_{|\gamma} = f$ for all $\gamma \in \Gamma$.*

Again we could give the definition for a fundamental domain for Γ, which is just the same as before by replacing $\mathrm{SL}_2(\mathbb{Z})$ by Γ. These are given by $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$-many translates of $\mathcal{F}$:

Proposition 3.4. *Let Γ be a congruence subgroup written as a disjoint union of cosets as*

$$\mathrm{SL}_2(\mathbb{Z}) = \bigcup_{i=1}^{n} \alpha_i \Gamma \,.$$

Then $\mathcal{G} = \bigcup_{i=1}^{n} \alpha^{-1} \mathcal{F}$ is a fundamental domain for Γ.

Proof. We will just give the argument why any point $\tau \in \mathbb{H}$ is Γ-equivalent to a point in $\mathcal{G}$. Since $\mathcal{F}$ is a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ we can find a $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $\gamma.\tau \in \mathcal{F}$. By the above decomposition we also have $\gamma = \alpha_i \gamma'$ for some $1 \le i \le n$ and $\gamma' \in \Gamma$. Therefore we get that $\gamma'.\tau \in \alpha_i^{-1} \mathcal{F} \subset \mathcal{G}$. $\square$

For example, one can check that

$$\mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(2) \cup S\Gamma_0(2) \cup (ST)^{-1}\Gamma_0(2),$$

which gives the following fundamental domain for $\Gamma_0(2)$:

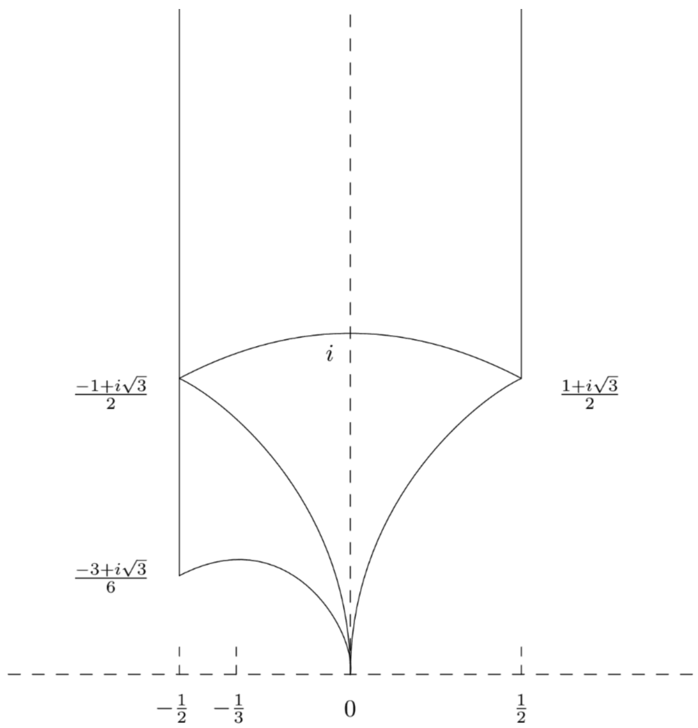


Figure 5: A fundamental domain of $\Gamma_0(2)$.

3.2 Modular forms for congruence subgroups

Definition 3.5. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup and let $k \in \mathbb{Z}$. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a* **modular form of weight** k **for** Γ *if* *i)* $f|_k\gamma = f$ *for all* $\gamma \in \Gamma$*,*

ii) $f|_k\gamma$ *is holomorphic at* ∞ *for all* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$*.*

By $M_k(\Gamma)$ *we denote the space of modular forms of weight* k *for* Γ*, i.e. with the notation used before we have* $M_k = M_k(\mathrm{SL}_2(\mathbb{Z}))$*. We say* f *is a cusp form if* $f|_k\gamma$ *vanishes at* ∞ *for all* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ *and* $S_k(\Gamma)$ *denotes the space of all cusp forms.*

Until here in lecture 6 (26th May, 2023)

Remark 3.6. Similar to the level 1 case there exist dimension formulas and valence formulas for the higher level case. For a given congruence group Γ the valence formula can be obtained similarly like we did in the classical case. But these do not suffice to obtain the dimension formulas, which need more tools from algebraic geometry. Nevertheless the valence formula also give a Sturm bound and one can show that two modular forms $f, g \in M_k(\Gamma)$ are the same if their first $(\frac{k}{12}+1)[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$ Fourier coefficients agree.

Proposition 3.7. *i) For all* $N > 0$ *the function*

$$G_{2,N}(\tau) = G_2(\tau) - NG_2(N\tau)$$

is an element in $M_2(\Gamma_0(N))$*.*

ii) The group $\Gamma_0(4)$ *is generated by* $\pm T$ *and* $\pm \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$*.*

iii) We have $\dim_{\mathbb{C}} M_2(\Gamma_0(4)) = 2$*.*

Proof. The first statement can be proven directly by using the modular transformation of G_2 given in Proposition 2.19:

$$G_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 G_2(\tau) - \pi i c(c\tau+d)\,. \tag{3.1}$$

Any element $\gamma \in \Gamma_0(N)$ can be written as $\gamma = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix}$. Now write $\eta = \begin{pmatrix} a & Nb \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and notice that $N\gamma(\tau) = \frac{aN\tau+bN}{cN\tau+d} = \eta(N\tau)$. Using this together (3.1) we get

$$\begin{aligned} (G_{2,N}|_k\gamma)(\tau) &= (Nc\tau+d)^{-2}\left(G_2(\gamma(\tau)) - NG_2(\underbrace{N\gamma(\tau)}_{\eta(N\tau)})\right) \\ &= G_2(\tau) - \frac{\pi i Nc}{Nc\tau+d} - NG_2(N\tau) + \frac{\pi i Nc}{cN\tau+d} \\ &= G_2(\tau) - NG_2(N\tau) = G_{2,N}(\tau)\,. \end{aligned}$$

This shows the correct transformation property. We will skip the proof for the holomorphicity at ∞. Part ii) is part of Homework 2 (see also [DS, Exercise 1.2.4]) and iii) follows from the general formula given in [DS, Theorem 3.5.1]. □

We now come back to the example from the motivation where we defined the **theta-function** by

$$\Theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}\,.$$

We mentioned that the theta-function satisfies the two functional equations

$$\Theta(\tau+1) = \Theta(\tau)\,, \qquad \Theta\left(-\frac{1}{4\tau}\right) = \sqrt{\frac{2\tau}{i}}\Theta(\tau) \qquad (\tau\in\mathbb{H})\,. \tag{3.2}$$

Now recall that we were interested in counting the number of ways to write a positive number as the sum of for squares, i.e. we wanted to evaluate

$$r_4(n) = \#\left\{(a,b,c,d)\in\mathbb{Z}^4 \mid n = a^2+b^2+c^2+d^2\right\}\,.$$

For this we considered the generating series of $r_4(n)$, i.e.

$$F(q) = \sum_{n\geq 0} r_4(n)q^n = 1+8q+24q^2+32q^3+24q^4+48q^5+96q^6+64q^7+24q^8+104q^9+\dots\,.$$

By the definition of the theta-function, it is easy to see that

$$F(q) = \Theta(\tau)^4\,.$$

Corollary 3.8. *We have* $\Theta^4 \in M_2(\Gamma_0(4))$.

Proof. By Lemma 3.7 ii), we just need to check that

$$\Theta(\tau+1)^4 = \Theta(\tau)^4\,, \qquad \Theta\left(\frac{\tau}{4\tau+1}\right)^4 = (4\tau+1)^2\Theta(\tau)^4 \qquad (\tau\in\mathbb{H})\,.$$

which can be checked directly by writing $\frac{\tau}{4\tau+1} = -\frac{1}{4(\frac{-1}{4\tau}-1)}$ and using (3.2). □

With all this we can now give a proof of Jacobi's four-square theorem:

Proof of Theorem 1.2. By Lemma 3.7 i) and iii) one can check that $G_{2,2}$ and $G_{2,4}$ are a basis of $M_2(\Gamma_0(4))$ by checking that they are linearly independent. Looking at the first two Fourier coefficients of Θ^4, we deduce $\Theta^4 = -\frac{1}{\pi^2}G_{2,4}$, which gives the formula for $r_4(n)$ given in the Theorem. □

4 Quasimodular forms

4.1 Derivatives of modular forms

Modular forms are holomorphic function and therefore we can differentiate them with respect to τ. It is convenient to consider the following notation for a modular form $f = \sum_{n=0}^{\infty} a_n q^n$:

$$f' := Df := \frac{1}{2\pi i}\frac{d}{d\tau}f = q\frac{d}{dq}f = \sum_{n=1}^{\infty} n a_n q^n\,.$$

Here the factor $2\pi i$ has been included in order to preserve the rationality properties of the Fourier coefficients. The derivative of a modular form is, in general, not a modular form anymore. The failure of modularity is given by the following proposition.

Proposition 4.1. *The derivative of a modular form* $f\in M_k$ *satisfies*

$$f'\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+2}f'(\tau) + \frac{k}{2\pi i}c(c\tau+d)^{k+1}f(\tau)\,.$$

for all $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Proof. Homework 2. □

Definition 4.2. *For a modular form $f \in M_k$, we define the* **Serre derivative** *by*

$$\partial_k f := f' - \frac{k}{12} E_2 f \,.$$

Proposition 4.3. *For a modular form $f \in M_k$ we have $\partial_k f \in M_{k+2}$.*

Proof. Homework 2.

□

Proposition 4.4. *The ring $\widetilde{M} = \mathbb{C}[E_2, E_4, E_6]$ is closed under differentiation and we have*

$$DE_2 = \frac{E_2^2 - E_4}{12}\,, \quad DE_4 = \frac{E_2 E_4 - E_6}{3}\,, \quad DE_6 = \frac{E_2 E_6 - E_4^2}{2}\,.$$

Proof. By Proposition 4.3 we have $\partial_4 E_4 = E_4' - \frac{1}{3} E_2 E_4 \in M_6$ and $\partial_6 E_6 = E_6' - \frac{1}{2} E_2 E_6 \in M_8$. Since both spaces are one-dimensional with basis E_6 and E_4^2 respectively we get the second and third equation after comparing the first Fourier coefficients. Using again the modularity formula (2.16) of E_2 and doing a similar calculation as in Proposition 4.3 one can also show that $E_2' - \frac{1}{12} E_2^2 \in M_4$. Therefore this is also a multiple of E_4, which turns out to be $-\frac{1}{12}$ by comparing the Fourier coefficients. □

The first equation in Proposition 4.4 gives for all $n \in \mathbb{Z}_{\geq 1}$ the relation

$$6n\sigma_1(n) = 5\sigma_3(n) + \sigma_1(n) - 12 \sum_{j=1}^{n-1} \sigma_1(j)\sigma_1(n-j)\,.$$

4.2 Quasimodular forms

The ring $\widetilde{M}$ is called the ring of quasimodular forms (for $\mathrm{SL}_2(\mathbb{Z})$). By definition it contains the ring of modular forms M as a subring. We will now give a more intrinsic/general definition of quasimodular forms for congruence subgroups $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$. Recall that the Eisenstein series G_2 was not modular, i.e. not invariant under the slash operator, but we saw that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$(G_2|_2)\gamma(\tau) = (c\tau + d)^{-2} G_2\left(\frac{a\tau + b}{c\tau + d}\right) = G_2(\tau) - \pi i \frac{c}{c\tau + d}\,. \tag{4.1}$$

We denote by $\mathrm{Hol}_0(\mathbb{H})$ the ring of holomorphic functions φ of **moderate growths** on $\mathbb{H}$, i.e. for all $C > 0$, $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $x \in \mathbb{R}$ one has $(\varphi_{|0\gamma})(x + iy) = O(e^{Cy})$ as[4] $y \to \infty$.

Definition 4.5. *A* **quasimodular form** *of weight $k \geq 0$ and depth at most $p \geq 0$ for Γ is a function $\varphi \in \mathrm{Hol}_0(\mathbb{H})$ such that there exist $\varphi_0, \dots, \varphi_p \in \mathrm{Hol}_0(\mathbb{H})$ so that for all $\tau \in \mathbb{H}$ and all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ one has*

$$(\varphi|_k\gamma)(\tau) = \varphi_0(\tau) + \varphi_1(\tau)\frac{c}{c\tau + d} + \cdots + \varphi_p(\tau)\left(\frac{c}{c\tau + d}\right)^p\,.$$

By $\widetilde{M}_k^{(\leq p)}(\Gamma)$ we denote the $\mathbb{C}$-vector space of all quasimodular forms of weight k and depth at most p for Γ. We set $\widetilde{M}_k(\Gamma) = \bigcup_{p \geq 0} \widetilde{M}_k^{(\leq p)}(\Gamma)$ and $\widetilde{M}(\Gamma) = \bigoplus_{k \geq 0} \widetilde{M}_k(\Gamma)$.
Notice that by (4.1) we have $G_2 \in \widetilde{M}_2^{\leq 1}(\mathrm{SL}_2(\mathbb{Z}))$.

[4] Here $f(y) = O(g(y))$ as $y \to \infty$ means that there exist real numbers M and y_0 such that $|f(y)| \leq Mg(y)$ for all $y \geq y_0$.

Proposition 4.6. *(i) The space $\widetilde{M}(\Gamma)$ is closed under differentiation and $D(\widetilde{M}_k^{(\leq p)}(\Gamma)) \subset \widetilde{M}_{k+2}^{(\leq p+1)}(\Gamma)$ for all $k, p \geq 0$.*

(ii) Every quasimodular form on Γ is a polynomial in G_2 with modular coefficients. More precisely, we have

$$\widetilde{M}_k^{(\leq p)}(\Gamma) = \bigoplus_{r=0}^{p} M_{k-2r}(\Gamma) \cdot G_2^r$$

for all $k, p \geq 0$.

Proof. See [Z, Proposition 20]. □

In the case $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ we see that $\widetilde{M}(\mathrm{SL}_2(\mathbb{Z})) = \widetilde{M} = \mathbb{C}[E_2, E_4, E_6]$.

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4.3 Derivations and $\mathfrak{sl}_2$-action

In this section, we will be interested in more derivations on $\widetilde{M}$ besides $D = q\frac{d}{dq}$. For this, we first recall a few notations from algebra. For a field K and K-algebra A a **derivation** is a K-linear map $d : A \to A$ satisfying the Leibniz rule $d(ab) = d(a)b + ad(b)$ for all $a, b \in A$. The set of all derivations on A is denoted by $\mathrm{Der}(A)$. A **Lie algebra** is a K-vector space V with a K-bilinear map $[\cdot,\cdot] : V \times V \to V$, the **Lie bracket**, which satisfies $[x, x] = 0$ for all $x \in V$ and the Jacobi identity $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ for all $x, y, z \in V$. One can check by direct calculation that $\mathrm{Der}(A)$ is a Lie algebra with Lie bracket $[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$. Notice that by the Leibniz rule a derivation is already uniquely determined by its image on the algebra generators. In particular, for $\widetilde{M} = \mathbb{C}[E_2, E_4, E_6]$ a derivation $d : \widetilde{M} \to \widetilde{M}$ is uniquely determined by $d(E_2), d(E_4)$ and $d(E_6)$, since we can then, for example, evaluate

$$d(E_2^2 E_4) = d(E_2^2)E_4 + E_2^2 d(E_4) = 2E_2 d(E_2)E_4 + E_2^2 d(E_4)\,.$$

Moreover, E_2, E_4, E_6 are algebraically independent over $\mathbb{C}$ (we just proved it for E_4, E_6, but one can adapt the proof of Proposition 2.33 for quasimodular forms). Therefore, one can define a derivation d by choosing arbitrary images for $d(E_k)$ for $k = 2, 4, 6$. We define the derivations $W, \delta : \widetilde{M} \to \widetilde{M}$ by

$$\begin{aligned} W(E_2) = 2E_2, \quad W(E_4) = 4E_4, \quad W(E_6) = 6E_6,\\ \delta(E_2) = 12, \quad \delta(E_4) = \delta(E_6) = 0\,. \end{aligned}$$

Notice that for $f \in \widetilde{M}_k$ we have $W(f) = kf$, which is called the **weight operator**. The derivation δ can be seen as the derivative with respect to $\frac{E_2}{12}$. We will now see that the three derivations D, W, δ make $\widetilde{M}$ into a so-called $\mathfrak{sl}_2$-algebra. Notice that the algebra of modular forms is given by $M = \ker \delta$.

The **Lie algebra** $\mathfrak{sl}_2$ is a central object in the field of mathematics, more specifically in the study of algebraic structures known as Lie algebras. The Lie algebra $\mathfrak{sl}_2$ is associated with the special linear group $\mathrm{SL}(2, \mathbb{C})$. It comprises all 2×2 complex matrices that have trace equal to zero. $\mathfrak{sl}_2$ is three-dimensional, spanned by three elements usually denoted by X, Y, and H, with

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

These fulfill the commutation relations

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [Y, X] = H. \tag{4.2}$$

The structures of Lie algebras and their representations play pivotal roles in various mathematical and physical theories, such as quantum mechanics and representation theory. The algebra $\mathfrak{sl}_2$ has a particularly special role in these fields due to its fundamental and prototypical nature.

Definition 4.7. *A triple of operators (X, H, Y) on an algebra A is called* $\mathfrak{sl}_2$**-triple** *if they satisfy the commutator relations* (4.2)*. If furthermore the X, H, Y are derivations on A then A is called a* $\mathfrak{sl}_2$**-algebra**.

Proposition 4.8. *(D, W, δ) is a $\mathfrak{sl}_2$-triple and thus $\widetilde{M}$ is a $\mathfrak{sl}_2$-algebra.*

Proof. This is part of Homework 3. By the above explanation one just needs to check that (4.2) is satisfied by acting on E_2, E_4 and E_6, which can be done by direct calculation using the explicit formulas for the action of D given in Proposition 4.4. $\square$

Definition 4.9. *For $f \in \widetilde{M}_k, g \in \widetilde{M}_l$ and $n \geq 0$ we define the* n**-th Rankin-Cohen bracket** *by*

$$[f, g]_n = \sum_{\substack{r,s \geq 0 \\ r+s=n}} (-1)^r \binom{k+n-1}{s} \binom{l+n-1}{r} D^r(f) D^s(g)\,.$$

Proposition 4.10. *If $f, g \in \ker\delta$, then $[f, g]_n \in \ker\delta$*

Proof. This is part of Homework 3. The idea is to use the commutator relations among D, W and δ to show that for $r \geq 1$

$$[\delta, D^r] = r(W - r + 1)D^{r-1}$$

and then use this to show that $\delta[f, g]_n = 0$ if $f, g \in \ker\delta$. $\square$

Corollary 4.11. *If $f \in M_k$, $g \in M_l$ then $[f, g]_n \in M_{k+l+2n}$.*

Remark 4.12. For a general $\mathfrak{sl}_2$-algebra A with $\mathfrak{sl}_2$-triple (D, W, δ) we denote for $k \in \mathbb{C}$ the eigenspace of W for the eigenvalue k by

$$A_k = \{a \in A \mid W(a) = ka\}.$$

Then one can define for $f \in A_k, g \in A_l$ the Rankin-Cohen brackets as above (by using the general definition of the binomial coefficient $\binom{\alpha}{s}$ for $\alpha \in \mathbb{C}$) and Proposition 4.10 is still true, since just the commutator relations are used in the proof.

5 Hecke theory

5.1 Functions on lattices & Hecke operators

Let $\mathcal{L}$ be the set of all lattices in $\mathbb{C}$, i.e. lattices of the form $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\omega_1, \omega_2 \in \mathbb{C}$ which are linearly independent over $\mathbb{R}$. We can assume that $\frac{\omega_1}{\omega_2} \in \mathbb{H}$. A function $F : \mathcal{L} \to \mathbb{C}$ is called **homogeneous of degree** $-k$ if $F(\lambda\Lambda) = \lambda^{-k}F(\Lambda)$ for all $\lambda \in \mathbb{C}^\times$ and $\Lambda \in \mathcal{L}$. Notice that for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we have $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \mathbb{Z}(a\omega_1 + b\omega_2) + \mathbb{Z}(c\omega_1 + d\omega_2)$ since matrices in $\mathrm{SL}_2(\mathbb{Z})$ just make a change of basis. If F is homogeneous of degree $-k$ we get

$$\begin{aligned} F(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) &= F(\mathbb{Z}(a\omega_1 + b\omega_2) + \mathbb{Z}(c\omega_1 + d\omega_2)) \\ &= F\left((c\omega_1 + d\omega_2)\left(\mathbb{Z}\frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2} + \mathbb{Z}\right)\right) = (c\omega_1 + d\omega_2)^{-k} F\left(\mathbb{Z}\frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2} + \mathbb{Z}\right). \end{aligned}$$

Moreover, the values of a homogeneous F are determined by $F(\mathbb{Z}\tau + \mathbb{Z})$ with $\tau \in \mathbb{H}$. If we define the function $f : \mathbb{H} \to \mathbb{C}$ by $f(\tau) := F(\mathbb{Z}\tau + \mathbb{Z})$ we get

$$f(\tau) = (c\tau + d)^{-k} f\left(\frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2}\right) = (f_{|k}\gamma)(\tau),$$

i.e. homogeneous F of degree $-k$ are in correspondence with weakly modular functions of weight k. This interpretation of modular forms will be used to give a natural definition of Hecke operators, which we first define on functions on lattices as follows.

Definition 5.1. *For a homogeneous function $F : \mathcal{L} \to \mathbb{C}$ of degree $-k$ and $n \geq 1$ we define the n-th* **Hecke operator** *by*

$$T_n F(\Lambda) = n^{k-1} \sum_{\substack{\lambda' \subset \Lambda \\ [\Lambda:\Lambda'] = n}} F(\Lambda'),$$

where the sum runs over all sublattices of index n.

Notice that by definition we immediately get that $T_n F$ is also homogeneous of degree $-k$.

Lemma 5.2. *For $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ the lattices $\Lambda' \subset \Lambda$ with $[\Lambda : \Lambda'] = n$ are in 1:1 correspondence with matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $ad = n$ and $0 \leq b \leq d - 1$ via $\Lambda' = \mathbb{Z}(a\omega_1 + b\omega_2) + \mathbb{Z}d\omega_2$.*

Proof. Homework 3. □

The correspondence of lattice functions and modular forms together with above Lemma leads to the following definition of Hecke operators for modular forms.

Definition 5.3. *For a modular form $f \in M_k$ and $n \geq 1$ we define the n-th Hecke operator by*

$$T_n f(\tau) = n^{k-1} \sum_{\substack{a,d>0 \\ ad=n}} \sum_{b=0}^{d-1} d^{-k} f\left(\frac{a\tau + b}{d}\right).$$

Notice that we again get, by the correspondence with the lattice functions above, $T_n f_{|k}\gamma = T_n f$ for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Proposition 5.4. *(i) For a modular form $f \in M_k$ with $f(\tau) = \sum_{m \geq 0} c_m q^m$ the Fourier expansion of $T_n f$ is given by*

$$T_n f(\tau) = \sum_{m \geq 0} \left(\sum_{\substack{d|m \\ d|n}} d^{k-1} c_{\frac{nm}{d^2}} \right) q^m.$$

In particular, $T_n M_k \subset M_k$ and $T_n S_k \subset S_k$.

(ii) For $m, n \geq 1$ we have

$$T_n T_m = T_m T_n = \sum_{\substack{d|m \\ d|m}} d^{k-1} T_{\frac{nm}{d^2}}.$$

In particular, if $\gcd(m, n) = 1$ *then $T_n T_m = T_m T_n = T_{mn}$.*

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Proof. For (i) we use the definition of T_n to get

$$\begin{aligned} T_n f(\tau) &= n^{k-1} \sum_{\substack{a,d>0\\ ad=n}} \sum_{b=0}^{d-1} d^{-k} f\left(\frac{a\tau+b}{d}\right) \\ &= n^{k-1} \sum_{m\geq 0} \sum_{\substack{a,d>0\\ ad=n}} \sum_{b=0}^{d-1} d^{-k} c_m e^{2\pi i \frac{a\tau+b}{d} m} \\ &= n^{k-1} \sum_{m\geq 0} \sum_{\substack{a,d>0\\ ad=n}} d^{1-k} c_m e^{\frac{2\pi i a m \tau}{d}} \frac{1}{d} \sum_{b=0}^{d-1} \left(e^{\frac{2\pi i m}{d}}\right)^b . \end{aligned}$$

Now we use

$$\frac{1}{d} \sum_{b=0}^{d-1} \left(e^{\frac{2\pi i m}{d}}\right)^b = \begin{cases} 0, & d \not| m, \\ 1, & d|m \end{cases}$$

together with $n^{k-1} d^{1-k} = a^{k-1}$ to obtain (by setting $m = dm'$)

$$T_n f(\tau) = \sum_{\substack{a,d>0\\ ad=n\\ m'\geq 0}} a^{k-1} c_{dm'} e^{2\pi i a m' \tau} = \sum_{\substack{a,d>0\\ ad=n\\ m'\geq 0}} a^{k-1} c_{dm'} q^{am'} = \sum_{m\geq 0} \left(\sum_{\substack{a|m\\ a|n}} a^{k-1} c_{\frac{mn}{a^2}} \right) q^m ,$$

where in the last step we set $m = am'$, which together with $ad = n$ gives $dm' = \frac{mn}{a^2}$. From this formula, we immediately see that $T_n f$ is holomorphic at ∞ and 0 at infinity if $a_0 = 0$. Therefore, $T_n M_k \subset M_k$ and $T_n S_k \subset S_k$, since we already saw that $T_n f$ is invariant under the slash operator. For (ii), we just apply the formula from (i) twice to calculate the coefficients of $T_m T_n f$ and notice that they coincide with the formula for the coefficients of $T_{mn} f$. □

5.2 Hecke Eigenforms

For each $n \geq 1$ we defined the Hecke operator $T_n : M_k \to M_k$. A natural question is to ask for its eigenvalues and eigenvectors. Even more, one can ask if there are modular forms f which are eigenvectors for all T_n, i.e. there exist numbers $\lambda_n \in \mathbb{C}$ with

$$T_n f = \lambda_n f \qquad \text{(for all } n \geq 1). \tag{5.1}$$

If the space M_k is one-dimensional ($k = 4, 6, 8, 10, 14$) this is clearly the case. Since S_k is closed under T_n the same statement holds when S_k has dimension one. In particular, Δ is an eigenvector for all T_n since $\dim S_{12} = 1$. By Proposition 5.4 we then get that for modular forms $f(\tau) = \sum_{n\geq 0} a_n q^n$ with (5.1) the eigenvalues λ_n satisfy

$$\lambda_n a_m = \sum_{\substack{d|m\\ d|n}} d^{k-1} a_{\frac{nm}{d^2}} .$$

In particular, in the case $m = 1$ we get $\lambda_n a_1 = a_n$, i.e. it makes sense to normalize these eigenvectors such that $a_1 = 1$. This leads to the following definition.

Definition 5.5. *(i) A modular form $f \in M_k$ is called* **Hecke eigenform** *if it is an eigenvector for all T_n with $n \geq 1$.*

(ii) A Hecke eigenform $f(\tau) = \sum_{n\geq 0} a_n q^n$ with $a_1 = 1$ is called **Hecke form** *(or often normalized Hecke eigenform).*

Hecke eigenforms have the Fouriercoefficient a_n as the eigenvalues for the operator T_n. As a direct consequence of Proposition 5.4 we obtain the following.

Proposition 5.6. *If $f(\tau) = \sum_{n\geq 0} a(n)q^n \in M_k$ is a Hecke form then for all $m, n \geq 1$*

$$a(n)a(m) = \sum_{\substack{d|m \\ d|n}} d^{k-1} a(\frac{nm}{d^2}). \tag{5.2}$$

In the special case $\gcd(m,n) = 1$ we get $a(n)a(m) = a(mn)$. In particular, the coefficients $a(n)$ are uniquely determined by $a(p)$ for p prime, since for $r \geq 1$

$$a(p^{r+1}) = a(p)a(p^r) - p^{k-1}a(p^{r-1}) \tag{5.3}$$

and $a(p_1^{r_1} \cdots p_l^{r_l}) = a(p_1^{r_1}) \cdots a(p_l^{r_l})$ for distinct primes $p_1, \dots, p_l$.

Proof. This is an immediate consequence of above discussion and Proposition 5.4. The recursion (5.3) follows from (5.2) by taking $m = p$ and $n = p^r$. □

Proposition 5.7. *(i) For even $k \geq 4$ the Eisenstein series*

$$\mathbb{G}_k = \frac{(k-1)!}{(2\pi i)^k} G_k = -\frac{B_k}{2k} + \sum_{n\geq 1} \sigma_{k-1}(n)q^n$$

are Hecke forms.

(ii) Δ is a Hecke form, i.e. $\tau(n)$ is multiplicative.

Proof. For (i) it suffices to check that the divisor sums σ_{k-1} satisfy (5.3) and are multiplicative. Then $\mathbb{G}_k$ satisfy (5.1) with $\lambda_n = \sigma_{k-1}(n)$. For p prime we have $\sigma_{k-1}(p^r) = 1 + p^{k-1} + \dots + p^{r(k-1)}$ from which we obtain by direct calculation $\sigma_{k-1}(p^{r+1}) = \sigma_{k-1}(p)\sigma_{k-1}(p^r) - p^{k-1}\sigma_{k-1}(p^{r-1})$ for $r \geq 1$. That the σ_{k-1} are multiplicative follows directly from the definition since for coprime m, n the divisors $d|m \cdot n$ are in 1:1 correspondence with divisors $d_1|m$ and $d_2|n$ via $d = d_1 d_2$,i.e.

$$\sigma_{k-1}(mn) = \sum_{d|mn} d^{k-1} = \sum_{d_1|m} d_1^{k-1} \sum_{d_2|n} d_2^{k-1} = \sigma_{k-1}(m)\sigma_{k-1}(n).$$

Statement (ii) follows from the discussion above. □

Theorem 5.8. *The Hecke forms in M_k form a basis of M_k for every $k \geq 1$.*

Proof. See, for example, [Z2, Chapter 2, Theorem 2]. □

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5.3 L-series

For a $f(\tau) = \sum_{n\geq 0} a(n)q^n$ we define its **L-series** by

$$L(f,s) = \sum_{m\geq 1} \frac{a(m)}{m^s}.$$

Here $s \in \mathbb{C}$ is a complex number such that the above sum converges. The region of convergence, i.e., admissible s depends on the growth of $a(n)$. For example, for a modular form $f \in M_k$ we will see that the L-series converges for $\mathrm{Re}(s) > k$. If the $a(n)$ are multiplicative, i.e. $a(m)a(n) = a(mn)$ for coprime m, n, then $a(p_1^{r_1} \cdots p_l^{r_l}) = a(p_1^{r_1}) \cdots a(p_l^{r_l})$ and the L-series of f has an **Euler product**

$$L(f,s) = \sum_{m\geq 1} \frac{a(m)}{m^s} = \prod_{p \text{ prime}} \sum_{r\geq 0} \frac{a(p^r)}{p^{rs}}.$$

Example 6. *If* $f(\tau) = \frac{q}{1-q} = \sum_{n\geq 1} q^n$*, i.e.* $a(n) = 1$*, the L-series of* f *is the Riemann zeta function*

$$L(f,s) = \zeta(s) = \sum_{m\geq 1} \frac{1}{n^s},$$

which converges absolutely for $\mathrm{Re}(s) > 1$*. Since* $\sum_{r\geq 0} \frac{1}{p^{rs}} = \frac{1}{1-p^{-s}}$ *the Euler product is given by*

$$\zeta(s) = \prod_{p \ prime} \frac{1}{1-p^{-s}}. \tag{5.4}$$

The Euler product for Hecke forms can also be written down explicitly:

Proposition 5.9. *If* $f(\tau) = \sum_{n\geq 0} a(n)q^n \in M_k$ *is a Hecke form, then*

$$L(f,s) = \prod_{p \ prime} \frac{1}{1 - a(p)p^{-s} + p^{k-1-2s}}.$$

Proof. Set $A_p(X) = \sum_{r\geq 0} a(p^r)X^r$. By using (5.3) we obtain

$$\begin{aligned} A_p(X) &= 1 + \sum_{r\geq 0} a(p^{r+1})X^{r+1} = 1 + \sum_{r\geq 0} a(p)a(p^r)X^{r+1} - \sum_{r\geq 1} p^{k-1}a(p^{r-1})X^{r+1} \\ &= 1 + a(p)XA_p(X) - p^{k-1}X^2A_p(X) \end{aligned}$$

and therefore

$$A_p(X) = \frac{1}{1 - a(p)X + p^{k-1}X^2}.$$

From this the result follows since $L(f,s) = \prod_{p \text{ prime}} A_p(p^{-s})$. $\square$

Example 7. *For* $f = \mathbb{G}_k$ *we have*

$$\begin{aligned} a(p^r) &= \sigma_{k-1}(p^r) = 1 + p^{k-1} + \cdots + p^{r(k-)1} = \frac{p^{(r+1)(k-1)} - 1}{p^{k-1} - 1}, \\ A_p(X) &= \sum_{r\geq 0} \frac{p^{(r+1)(k-1)} - 1}{p^{k-1} - 1} X^r = \frac{1}{(1 - p^{k-1}X)(1 - X)}, \\ L(\mathbb{G}_k, s) &= \prod_{p \ prime} \frac{1}{(1 - p^{k-1-s})(1 - p^{-s})} = \zeta(s-k+1)\zeta(s). \end{aligned}$$

Here we see that the L-series converges for $\mathrm{Re}(s) > k$*.*

Theorem 5.10. *Let* $f \in M_k$ *and* $f(\tau) = \sum_{n\geq 0} a(n)q^n$*.*

(i) We have the growth estimates

$$a(n) = O(n^{k-1}) \quad (f \in M_k), \qquad a(n) = O(n^{\frac{k}{2}}) \quad (f \in S_k).$$

Hence $L(f,s)$ *converges absolutely and locally unifrom,y in the half-plane* $\mathrm{Re}(s) > k$ *for any* $f \in M_k$ *and in the larger half-plane* $\mathrm{Re}(s) > \frac{k}{2} + 1$ *if* f *is a cusp form.*

(ii) $L(f,s)$ *has a meromorphic continuation to the whole complex plane. If* f *is a cusp form then this continuation is holomorphic everywhere. Otherwise, it has a simple pole of residue* $\frac{(2\pi i)^k}{(k-1)!}a(0)$ *at* $s = k$*. The completed L-series* $L^*(f,s) = (2\pi)^{-s}\Gamma(s)L(f,s)$ *(Here* L *is the meromophic continuation) satisfies the functional equation*

$$L^*(f,s) = (-1)^{\frac{k}{2}} L^*(f, k-s).$$

Proof. See, for example, [Z2, Chapter 2, Theorem 4]. $\square$

6 Period polynomials of modular forms

6.1 Definition & Eichler-Shimura isomorphism

Let $V_k \subset \mathbb{Q}[X,Y]$ denote the space of all homogeneous polynomials of degree $k-2$. On V_k we define a right-action of $\mathrm{GL}_2(\mathbb{Z})$ for a $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ and $F \in V_k$ by

$$(F|\gamma)(X,Y) = F(aX+bY, cX+dY)\,. \tag{6.1}$$

This action can then extended linearly to an action of the group ring $\mathbb{Z}[\mathrm{GL}_2(\mathbb{Z})]$ on V_k. The following elements in $\mathrm{GL}_2(\mathbb{Z})$ will be of importance when working with the above group action.

$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \delta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Definition 6.1. *Let $f \in S_k$ be a cusp form of weight k.*

(i) For $n = 0, 1, \ldots, k-2$ we define the n-th **period** *of f by*

$$r_n(f) = \int_0^{i\infty} f(\tau)\tau^n d\tau.$$

(ii) Define the **period polynomial** *of f as following polynomial in $\mathbb{C} \otimes V_k$*

$$P_f(X,Y) = \int_0^{i\infty} (X - Y\tau)^{k-2} f(\tau)\, d\tau\,. \tag{6.2}$$

Notice that

$$P_f(X,Y) = \sum_{n=0}^{k-2} (-1)^n \binom{k-2}{n} r_n(f) X^{k-2-n} Y^n\,.$$

Moreover, one can show, using the Mellin transform, that the periods of f are given by the values of its completed L-series. More precisely, we have

$$r_n(f) = i^{n+1} L^*(f, n+1).$$

Proposition 6.2. *For a cusp form $f \in S_k$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we have*

$$(P_f\,|\gamma)(X,Y) = \int_{\gamma^{-1}(0)}^{\gamma^{-1}(i\infty)} (X - Y\tau)^{k-2} f(\tau)\, d\tau\,, \tag{6.3}$$

where $\gamma(z) = \frac{az+b}{cz+d}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as before.

Proof. – will be added later –. □

In particular we see that

$$\begin{aligned} (P_f\,|(1+S))(X,Y) &= \left(\int_0^{\infty i} + \int_{\infty i}^0\right)(X - Y\tau)^{k-2} f(\tau)\, d\tau = 0\,, \\ (P_f\,|(1+U+U^2))(X,Y) &= 0\,. \end{aligned} \tag{6.4}$$

Motivated by (6.4) we define the following subspace of V_k for even $k \geq 2$

$$W_k = \ker(1+S) \cap \ker(1+U+U^2) \subset V_k\,.$$

By Proposition (6.2) we get $P_f \in \mathbb{C} \otimes W_k$ for $f \in S_k$.

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Let $V_k^{\pm}$ denote the ± 1-eigenspaces under the action of ϵ, i.e. V_k^+ are the symmetric and V_k^- are the antisymmetric polynomials. By V^{ev} and V^{od} we denote the ± 1-eigenspaces of δ, i.e. the even and odd polynomials. With this we define the symmetric (+), antisymmetric (−), even (ev) and odd (od) parts of W_k for $\bullet \in \{+,-,\mathrm{ev},\mathrm{od}\}$ by

$$W_k^{\bullet} = W_k \cap V_k^{\bullet} \,.$$

Lemma 6.3. *i) We have*

$$W_k^+ = W_k^{od}\,, \qquad W_k^- = W_k^{ev}\,.$$

ii) The spaces W_k and $W_k^{\pm}$ can also be written as

$$W_k = \ker(1 - T - T')\,, \tag{6.5}$$

where $T' = -U^2 S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and

$$W_k^{\pm} = \ker(1 - T \mp T\epsilon)\,.$$

Proof. Homework 4. □

The equation (6.5) is also called **Lewis equation**. The period polynomials of cusp forms satisfy the Lewis equation and the following theorem states, that there are no more relations. More precisely the statement of the Eichler-Shimura theorem is that we get isomorphisms of S_k to $W_k^{\pm}$ (modulo a subspace of dimension one in the even case) by sending a cusp form f to the odd P_f^+ and even part P_f^- of its period polynomial P_f. Notice that $P_f^{\pm} = \frac{1}{2} P_f|(1 \pm \epsilon)$.

Theorem 6.4. *(Eichler-Shimura Isomorphism) The map $p^{\pm} : f \mapsto P_f^{\pm}$ induces isomorphisms*

$$p^+ : S_k \xrightarrow{\sim} \mathbb{C} \otimes W_k^+\,, \qquad\qquad p^- : S_k \xrightarrow{\sim} \mathbb{C} \otimes W_k^- \big/ \mathbb{Q}(X^{k-2} - Y^{k-2})\,.$$

Proof. See for example [L]. □

Theorem 6.5. *For a Hecke eigenform $f \in S_k$ there exist two complex numbers $\omega^{\pm}(f)$ such that the polynomials $P_f^{\pm}$ have coefficients in the field generated by the Fourier coefficients of f.*

Proof. See [Z2]. □

For a modular form $f = \sum_{n \geq 0} a_n q^n \in M_k$, which is not a cusp form, i.e. $a_0 \neq 0$, the integral (6.2) does not converge. In [Z4] Zagier introduces the **extended period polynomial** for any $f = \sum_{n\geq 0} a_n q^n \in M_k$, by

$$\begin{aligned} \widehat{P}_f(X,Y) = &\int_{\tau_0}^{i\infty} (X - Y\tau)^{k-2}(f(\tau) - a_0)d\tau + \int_0^{\tau_0} (X - Y\tau)^{k-2}\left(f(\tau) - \frac{a_0}{\tau^k}\right) d\tau \\ &+ \frac{a_0}{(k-1)}\left(\frac{1}{Y} - \frac{\tau_0^{1-k}}{X}\right)(X - Y\tau_0)^{k-1}\,. \end{aligned} \tag{6.6}$$

Here $\tau_0 \in \mathbb{H}$ is arbitrary, and one can check that the definition of $\widehat{P}_f(X,Y)$ is independent of τ_0 since the derivative of the right-hand side with respect to τ_0 vanishes.

For example, the extended period polynomial of the Eisenstein series G_k is

$$\widehat{P}_{G_k}(X,Y) = \frac{2\pi i \zeta(k-1)}{2(k-1)}(X^{k-2}-Y^{k-2}) - \frac{(2\pi i)^k}{2(k-1)} \sum_{\substack{k_1+k_2=k\\ k_1,k_2\geq 0}} \frac{B_{k_1}}{k_1!}\frac{B_{k_2}}{k_2!} X^{k_1-1}Y^{k_2-1}\,.$$

In general $\widehat{P}_f(X,Y)$ is not an element in $\mathbb{C}\otimes V_k$ anymore, since we can get poles in X and Y. We therefore define the space

$$\widehat{V}_k = \bigoplus_{\substack{k_1+k_2=k\\ k_1,k_2\geq 0}} \mathbb{Q}X^{k_1-1}Y^{k_2-1}\,.$$

With this we have for any $f \in M_k$, that $\widehat{P}_f(X,Y) \in \mathbb{C}\otimes \widehat{V}_k$. The group Γ does not act on $\widehat{V}_k$ anymore, but still it makes sense to define the following subspace of $\widehat{V}_k$

$$\widehat{W}_k := \ker(1+U+U^2)\cap \ker(1+S) = \ker(1-T-T')\,,$$

which contains all elements in $\widehat{V}_k$ which vanish under the actions $1+U+U^2$ and $1+S$ (defined in the same way as before). One can then also check that $\widehat{P}_f(X,Y)\in\mathbb{C}\otimes\widehat{W}_k$ and we have the following extended version of the Theorem of Eichler-Shimura, where we define $\widehat{W}_k^{\pm}$ again by the symmetric and antisymmetric parts of $\widehat{W}_k$.

Theorem 6.6. *(Eichler-Shimura, Zagier [Z4]) The map $\widehat{p}_f^{\pm} : f \mapsto \widehat{P}_f^{\pm}$ induces isomorphism*

$$\widehat{p}_f^{+} : M_k \xrightarrow{\sim} \mathbb{C}\otimes \widehat{W}_k^{+}\,, \qquad \widehat{p}_f^{-} : M_k \xrightarrow{\sim} \mathbb{C}\otimes \widehat{W}_k^{-} = \mathbb{C}\otimes W_k^{-}\,.$$

6.2 Hecke operators

In this section we want to shortly mention how Hecke operators can be defined on period polynomials. For details we refer to [Z5]. For $n\geq 1$ we define the **Manin matrices** by

$$\mathrm{Man}_n = \left\{ \begin{pmatrix} a & b\\ c & d\end{pmatrix} \mid a,b,c,d\in\mathbb{Z}, ad-bc=n,\; \begin{matrix} a>|c|, d>|b|, bc\leq 0\\ b=0 \Rightarrow -\frac{a}{2}<c\leq\frac{a}{2},\\ c=0\Rightarrow -\frac{d}{2}<b\leq \frac{d}{2}\end{matrix} \right\}.$$

For example, for $n=2$ we have

$$\mathrm{Man}_2 = \left\{ \begin{pmatrix} 2 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix} 1 & 0\\ 0 & 2\end{pmatrix}, \begin{pmatrix} 2 & 0\\ 1 & 1\end{pmatrix}, \begin{pmatrix} 1 & 1\\ 0 & 2\end{pmatrix} \right\}.$$

Denote by $\mathcal{M}_n$ the set of 2×2-matrices with integer entries and determinant n. Then $\mathrm{Man}_n\subset\mathcal{M}_n$ and we extend the action (6.1) to an action of $\mathbb{Z}[\mathcal{M}_n]$ on V_k. Further, we define for $n\geq 1$ the element $\widetilde{T}_n\in\mathbb{Z}[\mathcal{M}_n]$ by $\tilde{T}_n = \sum_{m\in \mathrm{Man}_2} m$.

Theorem 6.7 ([Z5])**.** *The action of $\widetilde{T}_n$ on $W_k^{\pm}$ corresponds to the action of the Hecke operator T_n on S_k, i.e., for any $f\in S_k$ and $n\geq 1$ we have*

$$P^{\pm}_{T_n(f)} = P_f^{\pm} \mid \widetilde{T}_n\,.$$

As shown in [Z4] this theorem also extends to the extended periods polynomials. In particular, for even $k\geq 4$ we have

$$(X^{k-2}-Y^{k-2})|\widetilde{T}_n = \sigma_{k-1}(n)(X^{k-2}-Y^{k-2}).$$

Setting $(X, Y) = (1, 0)$ gives the following formula for the divisor sums

$$\sigma_{k-1}(n) = \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Man}_n} (a^{k-2} - c^{k-2}).$$

This formula can probably be proven purely combinatorially, since it also seems to hold for all $k \geq 2$ (also odd), when replacing c^{k-2} by $|c|^{k-2}$. (But there is no proof known to the lecturer). In general, if $f = \sum_{n \geq 0} a(n) q^n$ is a Hecke form with $P_f^{\pm}(1, 0) \neq 0$ we have

$$a(n) = \frac{1}{P_f^{\pm}(1, 0)} \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Man}_n} P_f^{\pm}(a, c).$$

For the coefficients of Δ we get

$$\tau(n) = \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Man}_n} \left(a^{10} - c^{10} - \frac{691}{36} a^2 c^2 (a^2 - c^2)^3 \right).$$

7 Combinatorial modular forms

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