Homework 4: Period polynomials & Combinatorial proofs

Deadline: 4th August (23:55 JST), 2023 at TACT

Exercise 1. Show the following (i.e. give a proof of Lemma 6.3):

i) We have

$$W_k^+ = W_k^{\mathrm{od}}, \qquad W_k^- = W_k^{\mathrm{ev}}.$$

ii) The spaces W_k and $W_k^{\pm}$ can also be written as

$$W_k = \ker(1 - T - T'),$$

where $T' = -U^2 S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and

$$W_k^{\pm} = \ker(1 - T \mp T\epsilon).$$

Exercise 2. Define for $k \geq 2$ the function

$$p_k(m,n) = \frac{1}{mn^{k-1}} + \frac{1}{2}\sum_{r=2}^{k-2} \frac{1}{m^r n^{k-r}} + \frac{1}{m^{k-1}n}.$$

Show that for even $k \geq 2$

$$p_k(m,n) = p_k(m+n,n) + p_k(m,m+n) + \sum_{\substack{0<j<k \\ j \text{ even}}} \frac{1}{m^j n^{k-j}}.$$

Use this to show that for even $k \geq 4$

$$\sum_{\substack{0<j<k \\ j \text{ even}}} \zeta(j)\zeta(k-j) = \frac{k+1}{2}\zeta(k).$$

Exercise 3. (Bonus) Show that for $k \geq 3, n \geq 1$

$$\sigma_{k-1}(n) = \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Man}_n} (a^{k-2} - |c|^{k-2}).$$

Here the matrices Man_n are defined by

$$\mathrm{Man}_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z}, ad-bc = n, \begin{array}{c} a > |c|, d > |b|, bc \leq 0 \\ b = 0 \Rightarrow -\frac{a}{2} < c \leq \frac{a}{2}, \\ c = 0 \Rightarrow -\frac{d}{2} < b \leq \frac{d}{2} \end{array} \right\}.$$