Homework 3: $\mathfrak{sl}_2$-action & Hecke operators

Deadline: 9th July (23:55 JST), 2023 at TACT

Exercise 1. Show that (D, W, δ) is a $\mathfrak{sl}_2$-triple.

Exercise 2. Show that for $f \in M_k$, $g \in M_l$ and $n \geq 0$ we have $[f,g]_n \in M_{k+l+2n}$.

Use that $f, g \in \ker(\delta)$ and then show that the commutator relations imply for $r \geq 1$

$$[\delta, D^r] = r(W - r + 1)D^{r-1}\,.$$

Use this to show that $[f,g]_n \in \ker\delta = M$.

Exercise 3. Show hat for $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ the lattices $\Lambda' \subset \Lambda$ with $[\Lambda : \Lambda'] = n$ are in 1:1 correspondence with matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $ad = n$ and $0 \leq b \leq d-1$ via $\Lambda' = \mathbb{Z}(a\omega_1 + b\omega_2) + \mathbb{Z}d\omega_2$.

Exercise 4. (Bonus) Determine the eigenvalues of $T_n : M_k \to M_k$ for $k = 12, 24$ and $n = 2, 3$. For this choose a basis of M_k and determine the matrix of T_n with respect to this basis.

(The numbers might get big and you are allowed to use a CAS for this Exercise)