Homework 2: Eisenstein series, Derivatives, Δ and index formula

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Deadline: 4th June (23:55 JST), 2023 at TACT

Exercise 1.

(i) Express E_{18} as a linear combination of E_6^3 and $E_4^3 E_6$.

(ii) Prove the following identity among divisor sums by using the theory of modular forms: For all $n \in \mathbb{Z}_{\geq 1}$ we have

$$11\sigma_9(n) = 21\sigma_5(n) - 10\sigma_3(n) + 5040 \sum_{j=1}^{n-1} \sigma_3(j)\sigma_5(n-j)\,.$$

Exercise 2. Show that the derivative of a modular form $f \in M_k$ satisfies

$$f'\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+2} f'(\tau) + \frac{k}{2\pi i} c(c\tau+d)^{k+1} f(\tau)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Exercise 3. For a modular form $f \in M_k$, we define the **Serre derivative** by

$$\partial_k f := f' - \frac{k}{12} E_2 f\,.$$

(i) Show that for a modular form $f \in M_k$ we have $\partial_k f \in M_{k+2}$.

(ii) Show that the Serre derivative maps cusp forms to cusp forms, i.e. it gives a map $\partial_k : S_k \to S_{k+2}$.

(iii) Compute Δ' and $\partial_{12}\Delta$.

(iv) Show that for all $n \in \mathbb{Z}_{\geq 1}$ we have

$$(n-1)\tau(n) \equiv 0 \mod 24\,,$$

where $\tau(n)$ is the Ramanujan tau function.

Exercise 4. Show that for $N \geq 1$ the index of the congruence subgroup $\Gamma_0(N)$ is given by

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p \mid N} \left(1 + \tfrac{1}{p}\right),$$

where the product runs over all primes p dividing N.