Homework 1: Modular group & basics of modular forms

Deadline: 30th April (23:55 JST), 2023 at TACT

Exercise 1. For $N \geq 1$ we define the congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \mod N \right\}.$$

(i) Show that $\Gamma_0(1) = \mathrm{SL}_2(\mathbb{Z})$ is generated by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(ii) Show that $\Gamma_0(4)$ is generated by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$.

Exercise 2. For a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, a complex number $\tau \in \mathbb{C}$ and a holomorphic function in the upper half plane $f \in \mathcal{O}(\mathbb{H})$ define

$$\gamma(\tau) := \frac{a\tau + b}{c\tau + d} \qquad \text{and} \qquad (f|_k\gamma)(\tau) := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

(i) Show that for all $\tau \in \mathbb{C}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$\mathrm{Im}(\gamma(\tau)) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2},$$

where $\mathrm{Im}(\tau)$ denotes the imaginary part of τ.

(ii) Show that $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ from the left by $\gamma(\tau)$.
(i.e. show that $\gamma(\tau) \in \mathbb{H}$, $I(\tau) = \tau$ and $\gamma'(\gamma(\tau)) = (\gamma' \cdot \gamma)(\tau)$ for all $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$.)

(iii) Show that $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{O}(\mathbb{H})$ from the right by $f|_k\gamma$.
(i.e. show that $f|_k\gamma \in \mathcal{O}(\mathbb{H})$, $f|_kI = f$ and $(f|_k\gamma')|_k\gamma = f|_k(\gamma' \cdot \gamma)$ for all $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$, $f \in \mathcal{O}(\mathbb{H})$.)

(iv) Show that if f is a meromorphic function on the upper half plan satisfying

$$f(\tau + 1) = f(\tau) \qquad \text{and} \qquad f(-1/\tau) = \tau^k f(\tau),$$

for all $\tau \in \mathbb{H}$, then f is a weakly modular function of weight k.

Exercise 3.

(i) Show that the space M_k is a $\mathbb{C}$-vector space and that for $k \geq 4$ we have $M_k = \mathbb{C}E_k \oplus S_k$.

(ii) Prove that if $f \in M_k$ and $g \in M_l$, then $f \cdot g \in M_{k+l}$.

(iii) For any $f \in M_4$ show that $f\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 0$.

(iv) For any $g \in M_6$ show that that $g(i) = 0$.