Introduction to modular forms

Perspectives in Mathematical Science IV (Part II)
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Henrik Bachmann (Math. Building Room 457, henrik.bachmann@math.nagoya-u.ac.jp)
Lecture notes and exercises are available at: http://www.henrikbachmann.com/mf2018.html

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Modular forms are functions appearing in several areas of mathematics as well as mathematical physics. There are two cardinal points about them which explain why they are interesting. First of all, the space of modular forms of a given weight is finite dimensional and algorithmically computable. Secondly, modular forms occur naturally in connection with problems arising in many areas of mathematics. Together, these two facts imply that modular forms have a huge number of applications and the purpose of this lecture is to demonstrate this on examples coming from classical number theory, such as identities among divisor sums. In this course we will discuss the following topics:

- The action of the modular group on the complex upper half-plane and modular forms.
- Eisenstein series and their Fourier expansion.
- Cusp forms and Ramanujan's Delta-function.
- The space of modular and its dimension.
- Derivatives of modular forms.
- Application: Relations and congruences among Fourier coefficients of modular forms.

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1 Motivation

Modular forms have various different application in many areas. We will illustrate some applications coming from classical number theory. For this we start with the following well-known theorem:

Theorem 1.1 (Theorem of Lagrange (1770))**.** *Every positive integer is a sum of four*[1] *squares.*

For example $1 = 1^2 + 0^2 + 0^2 + 0^2$ or $30 = 1^2 + 2^2 + 3^2 + 4^2 = 0^2 + 1^2 + 2^2 + 5^2$ and

$$2018 = 3^2 + 21^2 + 28^2 + 28^2 = 12^2 + 19^2 + 27^2 + 28^2 = 17^2 + 18^2 + 27^2 + 26^2.$$

In particular these examples show that the representation as a sum of four squares is not unique.

Question: *In how many ways can a natural number n be written as a sum of four squares?*

In other words, the question asks for an explicit formula for the function

$$r_4(n) = \#\left\{(a,b,c,d) \in \mathbb{Z}^4 \mid n = a^2 + b^2 + c^2 + d^2\right\} .$$

This question was answered by Jacobi who gave the following explicit formula for $r_4(n)$.

Theorem 1.2 (Jacobi's four-square theorem (1834))**.** *For all $n \in \mathbb{Z}_{\geq 1}$ we have*

$$r_4(n) = 8 \sum_{\substack{d \mid n \\ 4 \nmid d}} d \,.$$

Here the sum runs over all positive divisors d of n, which are not divisible by 4.

Example 1.3. *i) If p is prime, then there are $8(p+1)$ ways to write p as a sum of four squares.*

ii) The divisors of 2018 are $1, 2, 1009$ and 2018, which are all not divisible by 4 and therefore we have

$$r_4(2018) = 8\,(1 + 2 + 1009 + 2018) = 24240$$

ways of writing 2018 as a sum of four squares.

To prove theorems like Theorem 1.2 it is convenient to consider the generating series of $r_4(n)$, i.e.

$$F(q) = \sum_{n \geq 0} r_4(n) q^n = 1 + 8q + 24q^2 + 32q^3 + 24q^4 + 48q^5 + 96q^6 + 64q^7 + 24q^8 + 104q^9 + \dots$$

It turns out that $F(q)$ is an example of a "modular form of weight 2 and level 4". Using the theory of modular forms, one knows that the space of modular forms of weight 2 and level 4 has dimension 2 and we can give an explicit basis for this space in terms of so-called Eisenstein series. Eisenstein series are given by q-series, whose coefficients are divisor sums, and writing F as a linear combination of them (see (1.2)) gives a proof of Theorem 1.2.

Definition 1.4. *For $l \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 1}$ the **l-th divisor sum** $\sigma_l(n)$ is defined by*

$$\sigma_l(n) = \sum_{d \mid n} d^l \,,$$

where the sum runs over all positive divisors d of n.

[1] Four is the smallest integer with this property, since 5 is not the sum of two squares and 7 is not the sum of three.

In particular $\sigma_0(n)$ counts the divisors of n and $\sigma_1(n)$ is the sum of all divisor of n. For example since the divisor of 6 are $1, 2, 3, 6$, we have $\sigma_0(6) = 4$ and $\sigma_1(6) = 1+2+3+6 = 12$. A few more examples:

n	$\sigma_1(n)$	$\sigma_3(n)$	$\sigma_5(n)$	$\sigma_7(n)$	$\sigma_9(n)$	$\sigma_{11}(n)$
1	1	1	1	1	1	1
2	3	9	33	129	513	2049
3	4	28	244	2188	19684	177148
4	7	73	1057	16513	262657	4196353
5	6	126	3126	78126	1953126	48828126
6	12	252	8052	282252	10097892	362976252
7	8	344	16808	823544	40353608	1977326744
8	15	585	33825	2113665	134480385	8594130945
9	13	757	59293	4785157	387440173	31381236757

The Eisenstein series of weight 2,4,6 and 8 are given by[2] the following q-series

$$\begin{aligned}
E_2(q) &= 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n = 1 - 24q - 72q^2 - 96q^3 - 168q^4 - 144q^5 + \dots\\
E_4(q) &= 1 + 240\sum_{n\geq 1}\sigma_3(n)q^n = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \dots\\
E_6(q) &= 1 - 504\sum_{n\geq 1}\sigma_5(n)q^n = 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 + \dots\\
E_8(q) &= 1 + 480\sum_{n\geq 1}\sigma_7(n)q^n = 1 + 480q + 61920q^2 + 1050240q^3 + 7926240q^4 + \dots .
\end{aligned} \tag{1.1}$$

The space of modular forms of weight 2 and level 4 is spanned by the two q-series $E_2(q) - 2E_2(q^2)$ and $E_2(q) - 4E_2(q^4)$. One can show, using analytic methods, that $F(q)$ is also an element in this space and therefore has to be a linear combination of these two. It turns out that

$$F(q) = -\frac{1}{3}\left(E_2(q) - 4E_2(q^4)\right), \tag{1.2}$$

which proves Theorem 1.2 (see Section 8 for a bit more details).

In this course, we will focus on modular forms of level 1. The Eisenstein series E_4, E_6 and E_8 are examples of modular forms of level 1 and weight 4, 6 and 8 respectively. The goal of this course is to prove a dimension formula for modular forms and show that every modular form is actually a polynomial in just E_4 and E_6. As one small application, we will obtain relations among divisor-sums.

One example is the following: The space of all modular forms is a graded ring and E_4^2 and E_8 are both modular forms of weight 8. As we will see, the space of modular forms of weight 8 has dimension 1 and since both E_4^2 and E_8 start with $1 + \dots$ they are equal, i.e.

$$E_8(q) = E_4(q)^2 . \tag{1.3}$$

This implies the following identity among divisor-sums by considering the coefficient of q^n in (1.3).

[2]The factors -24, 240, -504 and 480 will become clear when we give the "real definition" of the Eisenstein series in Section 3 equation (3.5).

Theorem 1.5 (Hurwitz identity)**.** *For all $n \in \mathbb{Z}_{\geq 1}$ we have*

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{j=1}^{n-1} \sigma_3(j)\sigma_3(n-j)\,.$$

For example for $n = 3$ we have $\sigma_7(3) = 1 + 3^7 = 2188$ and

$$\sigma_3(3) + 120 \sum_{j=1}^{2} \sigma_3(j)\sigma_3(3-j) = 1 + 3^3 + 120(1 \cdot (1+2^3) + (1+2^3) \cdot 1) = 28 + 120 \cdot 18 = 2188\,.$$

This identity can also be proven without using modular forms (Bonus exercise), but the proof becomes much more complicated.

2 The modular group and modular forms

The modular forms mentioned in the previous section were given by q-series. But actually modular forms are functions from the upper half plane to the complex numbers. That they can be written as q-series will follow later as a simple implication of their definition. We will start by giving the definition of the upper half plane and the action of the modular group on this space. With this, we will define modular functions and modular forms, before giving (non-trivial) examples in the next section.

The **upper half plane**, denoted $\mathbb{H}$, is the set of all complex numbers with positive imaginary part:

$$\mathbb{H} = \big\{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\big\} = \big\{x + iy \in \mathbb{C} \mid x, y \in \mathbb{R}\,, y > 0\big\}\,.$$

The **modular group** (or special linear group) $\mathrm{SL}_2(\mathbb{Z})$ is the group of 2×2-matrices with integer entries and determinant one:

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{Z}\,, ad - bc = 1 \right\}\,.$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{C}$ we define the fractional linear transformation

$$\gamma(\tau) := \frac{a\tau + b}{c\tau + d}\,.$$

This gives a left action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ (Exercise 1).
The group $\mathrm{SL}_2(\mathbb{Z})$ contains the following three matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,, \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\,,$$

which correspond to the identity and the fractional transformation $\tau \mapsto -\frac{1}{\tau}$ and $\tau \mapsto \tau + 1$. The latter two fractional transformation will play the major role in our studies, since we have following:

Proposition 2.1. *The matrices S and T generate* $\mathrm{SL}_2(\mathbb{Z})$.

Proof. Exercise 2 i). □

Remark 2.2. Some authors denote by the modular group the group of transformations generated by $\gamma(\,.\,)$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Since $(-I)(\tau) = \tau$ this group is isomorphic to $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$.

Definition 2.3. *Two points $\tau_1, \tau_2 \in \mathbb{H}$ are called* $\mathrm{SL}_2(\mathbb{Z})$**-equivalent** *if there exists a $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $\gamma(\tau_1) = \tau_2$. A* **fundamental domain** *$\mathcal{F}$ for $\mathrm{SL}_2(\mathbb{Z})$ is a closed subset of $\mathbb{H}$, such that*

i) every $\tau \in \mathbb{H}$ is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to a point in $\mathcal{F}$.

ii) no two points in the interior of $\mathcal{F}$ are $\mathrm{SL}_2(\mathbb{Z})$-equivalent.

Proposition 2.4. *The following set is a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$*

$$\mathcal{F} = \left\{ \tau \in \mathbb{H} \mid |\tau| \geq 1 \text{ and } |\mathrm{Re}(\tau)| \leq \frac{1}{2} \right\}.$$

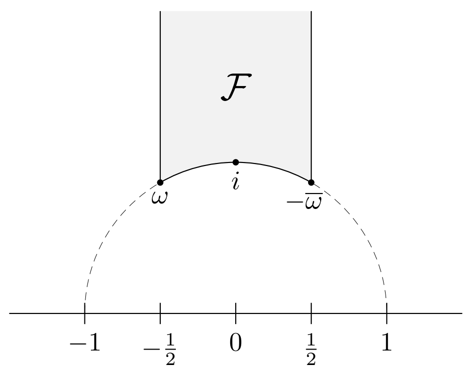


Figure 1: Fundamental domain $\mathcal{F}$ and the points $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $-\overline{\omega} = S(\omega) = \frac{1}{2} + \frac{\sqrt{3}}{2}i$.

Proof. We first show that every element $\tau \in \mathbb{H}$ is equivalent to a point in $\mathcal{F}$: For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have (see Exercise 1 i))

$$\mathrm{Im}(\gamma(\tau)) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2}\,. \tag{2.1}$$

Since c, d are integers, we can find a matrix $\gamma_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, such that $|c\tau + d|$ is minimal. In particular we get by (2.1) that

$$\mathrm{Im}(\gamma_0(\tau)) \geq \mathrm{Im}(\gamma(\tau)) \qquad \text{for all } \gamma \in \mathrm{SL}_2(\mathbb{Z})\,. \tag{2.2}$$

Since the action of T corresponds to a horizontal translation, we can find a $j \in \mathbb{Z}$, such that $\gamma_1 = T^j\gamma_0$ satisfies $-\frac{1}{2} \leq \mathrm{Re}(\gamma_1(\tau)) \leq \frac{1}{2}$. We now already have $\gamma_1(\tau) \in \mathcal{F}$ because otherwise we would have $|\gamma_1(\tau)| < 1$ and therefore

$$\mathrm{Im}(S\gamma_1(\tau)) = \frac{\mathrm{Im}(\gamma_1(\tau))}{|\gamma_1(\tau)|^2} > \mathrm{Im}(\gamma_1(\tau)) = \mathrm{Im}(\gamma_0(\tau))\,,$$

which is not possible by (2.2).

We now prove that no two points in the interior of $\mathcal{F}$ are $\mathrm{SL}_2(\mathbb{Z})$-equivalent: Let $\tau \in \mathcal{F}$ and assume we have a $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that also $\gamma(\tau) \in \mathcal{F}$. Without loss of generality we can assume that $\mathrm{Im}(\gamma(\tau)) \geq \mathrm{Im}(\tau)$ (otherwise replace γ by γ^{-1}). By (2.1) we therefore have $|c\tau + d| \leq 1$. Since $c, d \in \mathbb{Z}$ and $\tau \in \mathcal{F}$ this can just be the case if $|c| \leq 1$, which leaves us with the following cases:

i) $c = 0$, $d = \pm 1$: In this case we have $\gamma = \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}$ and therefore we either have $\gamma = I$ or $\mathrm{Re}(\tau) = \pm\frac{1}{2}$, i.e. τ is on one of the vertical boundary lines of $\mathcal{F}$.

ii) $c = \pm 1$, $d = 0$ and $|\tau| = 1$: In this case we have $\gamma = \begin{pmatrix} a & \mp 1 \\ \pm 1 & 0 \end{pmatrix} = \pm T^a S$. This gives either $a = 0$ with τ and $\gamma(\tau)$ on the unit circle (and symmetrically located with respect to the imaginary axis), $a = -1$ with $\tau = \gamma(\tau) = \omega$ or $a = 1$ with $\tau = \gamma(\tau) = -\overline{\omega}$.

iii) $c = d = \pm 1$ and $\tau = \omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$: In this case we have $\gamma = \begin{pmatrix} a & a \mp 1 \\ \pm 1 & \pm 1 \end{pmatrix} = \pm T^a S T$ which gives either $a = 0$ and $\gamma(\tau) = \omega$ or $a = 1$ and $\gamma(\tau) = -\overline{\omega}$.

iv) $c = -d = \pm 1$ and $\tau = -\overline{\omega} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$: This case is similar to case iii).

In all cases we conclude that either $\gamma = I$ or τ and $\gamma(\tau)$ are on the boundary of $\mathcal{F}$. □

Remark 2.5. The following diagram shows how the fundamental domain $\mathcal{F}$ is translated by different matrices in $\mathrm{SL}_2(\mathbb{Z})$.

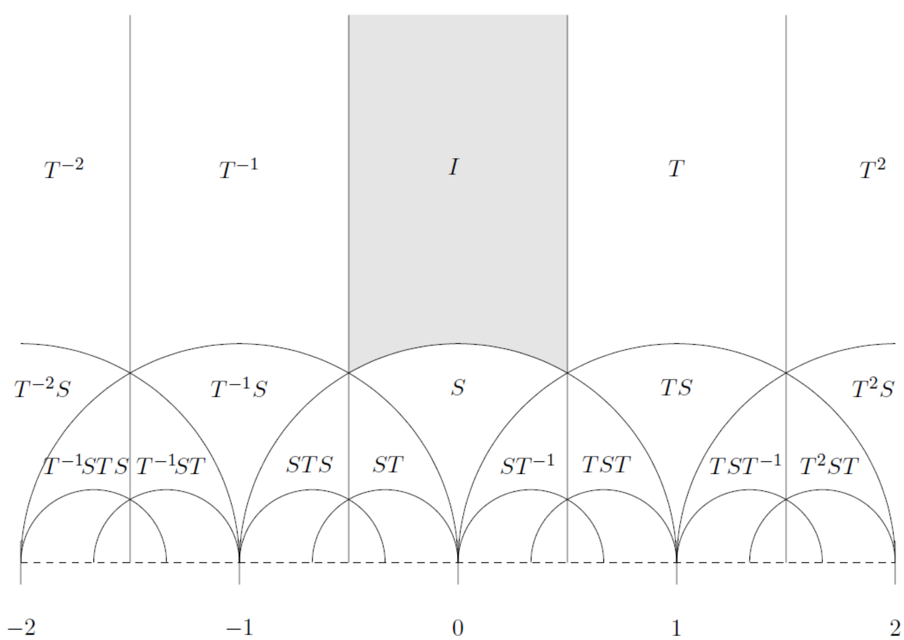


Figure 2: Translations of the fundamental domain $\mathcal{F}$.

We will now recall some basic definitions from complex analysis. For details we refer to [SS] and [FB].

Definition 2.6. *Let $U \subset \mathbb{C}$ be an open subset of the complex numbers. A function $f : U \to \mathbb{C}$ is called* **holomorphic** *on U, if for all $z_0 \in U$ the limit*

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. If it exists, it is denoted by $f'(z_0)$. By $\mathcal{O}(U)$ we denote the set of all holomorphic functions on the open set U.

A basic fact from complex analysis is that holomorphic functions are also analytic. This means that if f is holomorphic on U, then for each $z_0 \in U$ there exists a $\epsilon > 0$, such that f can be written as a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n ,$$

for all $z \in U$ with $|z - z_0| < \epsilon$ and some $a_n \in \mathbb{C}$.

Definition 2.7. *A function f is* **meromorphic** *on U, if there exists a discrete subset $P \subset U$ with*

i) f is holomorphic on $U \backslash P$.

ii) f has poles at $p_j \in P$.

If f is meromorphic on U, then for each $p \in U$, there exists a unique integer $v_p(f) \in \mathbb{Z}$, a $\epsilon > 0$ and a non-vanishing holomorphic function g on the deleted neighborhood $0 < |z - p| < \epsilon$, such that

$$f(z) = (z - p)^{v_p(f)} g(z)$$

for all $0 < |z - p| < \epsilon$. The integer $v_p(f)$ is called **the order of f at the point** $p \in U$ and we have:

i) If $v_p(f) < 0$ then f has a pole of order $|v_p(f)|$ at p.

ii) If $v_p(f) = 0$ then f has no pole and no zero at p.

iii) If $v_p(f) > 0$ then f has a zero of order $v_p(f)$ at p.

Equivalent to above condition is that f has a **Laurent expansion** in all $p \in U$ of the form

$$f(z) = \sum_{n=v_p(f)}^{\infty} a_n (z - p)^n ,$$

for $0 < |z - p| < \epsilon$ and $a_n \in \mathbb{C}$ with $a_{v_p(f)} \neq 0$.

Example 2.8. *i) The rational function $f(z) = \frac{z-2}{(z-1)(z+1)^2}$ is meromorphic on $\mathbb{C}$ with $v_2(f) = 1, v_1(f) = -1$ and $v_{-1}(f) = -2$. Its Laurent expansion around $z = 1$ is given by*

$$f(z) = -\frac{1}{4}(z-1)^{-1} + \frac{1}{2} - \frac{7}{16}(z-1) + \frac{5}{16}(z-1)^2 + \dots .$$

ii) The function $e^{\frac{1}{z}}$ is holomorphic on $\mathbb{C}\backslash\{0\}$, but it is not meromorphic on $\mathbb{C}$, since it has an essential singularity at $z = 0$.

In the following $k \in \mathbb{Z}$ will always denote an integer.

Definition 2.9. *A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a* **weakly modular function of weight** k*, if it satisfies*

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)\,, \tag{2.3}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ *and all* $\tau \in \mathbb{H}$.

Since $-I \in \mathrm{SL}_2(\mathbb{Z})$, a weakly modular function of weight k satisfies $f(\tau) = (-1)^k f(\tau)$. This shows that there are no non-trivial weakly modular functions of odd weight.
If f is a weakly modular function of weight k we have

$$\begin{aligned} f(\tau + 1) &= f(\tau)\,, \\ f(-1/\tau) &= \tau^k f(\tau)\,, \end{aligned} \tag{2.4}$$

by choosing the matrices T and S for (2.3). These two conditions are already sufficient for f to be weakly modular function of weight k.

Proposition 2.10. *A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ which satisfies* (2.4) *is already a weakly modular function of weight k.*

Proof. Exercise 2 □

Definition 2.11. *For a function $f : \mathbb{H} \to \mathbb{C}$ and a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we define the* **slash operator of weight** k *by*

$$(f|_k\gamma)(\tau) := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

This gives a right action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{O}(\mathbb{H})$ (Exercise 1) and the weakly modular functions of weight k are exactly the meromorphic functions on $\mathbb{H}$, which are invariant under this operator.

Now consider the following holomorphic map from the upper half plane to the punctured unit disc

$$\begin{aligned} \mathbb{H} &\longrightarrow \mathbb{D}^* := \{z \in \mathbb{C} \mid 0 < |z| < 1\}\,, \\ \tau &\longmapsto q_\tau := e^{2\pi i \tau}\,. \end{aligned}$$

First notice that this is indeed a map from $\mathbb{H}$ to $\mathbb{D}^*$, since if $\tau = x + iy$ then $q_\tau = e^{2\pi i \tau} = e^{-2\pi y} e^{2\pi x i}$, which lies in $\mathbb{D}^*$ because of $y > 0$.
The equation $f(\tau + 1) = f(\tau)$ implies that f can be written in the form

$$f(\tau) = \tilde{f}(q_\tau)\,,$$

where $\tilde{f}$ is a meromorphic function on the punctured unit disc $\mathbb{D}^*$.

Definition 2.12. *i) A weakly modular function f is called* **meromorphic (resp. holomorphic) in** ∞*, if the function $\tilde{f}$ extends to a meromorphic (resp. holomorphic) function at* 0.

ii) The **order at** ∞ *of a meromorphic weakly modular function f is defined by $v_\infty(f) := v_0(\tilde{f})$.*

Extending to a meromorphic (resp. holomorphic) function at 0, means that there exists a $N \in \mathbb{Z}$ (resp. $N \in \mathbb{Z}_{\geq 0}$) such that the Laurent expansion of $\tilde{f}$ around 0 has the form

$$\tilde{f}(q) = \sum_{n=N}^{\infty} a_n q^n \,,$$

for some $a_n \in \mathbb{C}$. The smallest such N is given by $v_0(\tilde{f})$.

Definition 2.13. *A weakly modular function (of weight k) f is called* **modular function (of weight k)** *if it is meromorphic at ∞.*

Definition 2.14. *A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a* **modular form of weight** k*, if*

i) f is a modular function of weight k,

ii) f is holomorphic at ∞.

By M_k we denote the space of all modular forms of weight k.

In other words, modular forms of weight k are holomorphic functions $f : \mathbb{H} \to \mathbb{C}$, which satisfy (2.3) and which have a **Fourier expansion** of the form

$$f(\tau) = \sum_{n=0}^{\infty} a_n q_\tau^n$$

for some $a_n \in \mathbb{C}$, which are called the **Fourier coefficients** of f. By abuse of notation we will in the following always write q instead of q_τ.

Example 2.15. *i) For all $k \in \mathbb{Z}$ the function $f(\tau) = 0$ is a modular form of weight k.*

ii) There are no non-trivial modular forms of odd weight.

iii) For all $c \in \mathbb{C}$ the constant function $f(\tau) = c$ is a modular form of weight 0.

Of course there are other non-trivial examples of modular forms, as we will see in the next section.

3 Eisenstein series

In this section we will introduce Eisenstein series, which are one of the most important examples of modular forms. These already appeared in the first section as q-series. Here we will give their "correct" definition as a function in a complex variable $\tau \in \mathbb{H}$ and calculate their Fourier expansion. For this we will also need to recall the **Riemann zeta function**

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} \,, \qquad (k \in \mathbb{C},\, \mathrm{Re}(k) > 1)$$

which will give the constant term in the Fourier expansion of the Eisenstein series.

Proposition 3.1. *For even $k \geq 4$ the* **Eisenstein series of weight** k*, defined by*

$$G_k(\tau) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m\tau + n)^k} \,,$$

is a modular form of weight k.

Proof. First one can check that for $k > 2$ the above sum is absolutely convergent and uniformly convergent on compacts subset (actually also on $\mathcal{F}$) of $\mathbb{H}$ and therefore G_k is a holomorphic function on $\mathbb{H}$. For the proof of this fact we refer to the literature (see for example [Ki, Lemma 2.7] or [S, p. 82, Lemma 1]).

To check that G_k is holomorphic at infinity, we will show that $G_k(\tau)$ approaches an explicit finite limit as $\tau \to i\infty$. By the uniformly convergence we can exchange summation and the limit and obtain

$$\lim_{\tau\to i\infty} G_k(\tau) = \frac{1}{2} \lim_{\tau\to i\infty} \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k} = \frac{1}{2}\sum_{0\neq n\in\mathbb{Z}} \frac{1}{n^k} = \zeta(k)\,.$$

Now we check the modularity conditions for G_k. For this it is important that the sum converges absolutely and therefore we are allowed to arrange the terms in any way. To show that $G_k(\tau+1) = G_k(\tau)$ we calculate

$$G_k(\tau+1) = \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m(\tau+1)+n)^k} = \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m\tau+(m+n))^k}\,.$$

As (m,n) runs over $\mathbb{Z}^2\backslash\{(0,0)\}$, so does $(m, m+n) = (m, n')$, so by the absolute convergence we get

$$G_k(\tau+1) = \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m\tau+m+n)^k} = \frac{1}{2} \sum_{\substack{m,n'\in\mathbb{Z}\\(m,n')\neq(0,0)}} \frac{1}{(m\tau+n')^k} = G_k(\tau)\,.$$

Similarly, to show $G_k(-1/\tau) = \tau^k G_k(\tau)$ we derive

$$G_k(-1/\tau) = \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(-m/\tau+n)^k} = \tau^k \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(n\tau-m)^k} = \tau^k G_k(\tau)\,,$$

since also $(n, -m)$ runs over $\mathbb{Z}^2\backslash\{(0,0)\}$. □

We will now calculate the Fourier expansion of G_k for which we will need the following lemma.

Lemma 3.2. *(Lipschitz's formula) For $k \geq 2$ and $\tau \in \mathbb{H}$ we have*

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{d=1}^{\infty} d^{k-1} q^d\,. \tag{3.1}$$

Proof. (Sketch) This follows by differentiating the following two expression of the cotangent $k-1$-times

$$\sum_{n\in\mathbb{Z}} \frac{1}{\tau+n} = \frac{\pi}{\tan(\pi\tau)} = -\pi i - 2\pi i \sum_{d=1}^{\infty} q^d\,.$$

See for example [Z, Proposition 5] for more details. □

Proposition 3.3 (Fourier expansion of G_k)**.** *For even $k \geq 4$ the Fourier expansion of G_k is given by*

$$G_k(\tau) = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n\,. \tag{3.2}$$

Proof. Again we use the absolute convergence which allows the following rearrangements

$$G_k(\tau) = \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k} \overset{k \text{ even}}{=} \frac{1}{2}\sum_{0\neq n\in\mathbb{Z}} \frac{1}{n^k} + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^k}$$

$$\overset{(3.1)}{=} \zeta(k) + \frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\sum_{d=1}^{\infty} d^{k-1}q^{md} = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\,.$$

□

In the definition of G_k we needed $k > 2$ to assure absolute convergence. But the q-series in (3.2) also makes sense for $k = 2$ and also defines a holomorphic function in $\tau \in \mathbb{H}$. We therefore use this equation to define the **Eisenstein series of weight** 2 by

$$G_2(\tau) := \zeta(2) + (2\pi i)^2 \sum_{n=1}^{\infty} \sigma_1(n)q^n\,. \tag{3.3}$$

This is not a modular form anymore, but plays an important role in the theory of modular forms. We have the following Proposition which gives the failure of G_2 to be a modular form.

Proposition 3.4 (Modular transformation of G_2)**.** *For $\tau \in \mathbb{H}$ and $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we have*

$$G_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 G_2(\tau) - \pi i c(c\tau+d)\,. \tag{3.4}$$

Proof. See for example [Z, Proposition 6] or [Ko, Chapter III, Proposition 7]. □

Proposition 3.5 (L. Euler (1735))**.** *For even $k \geq 2$ we have*

$$\zeta(k) = -\frac{B_k}{2k}\frac{(2\pi i)^k}{(k-1)!}\,,$$

*where B_k denotes the k***-th Bernoulli number** *defined by the generating function*

$$\sum_{k=0}^{\infty} \frac{B_k}{k!}X^k = \frac{X}{e^X - 1}\,.$$

Proof. See for example [FB, Proposition III.7.14]. □

A few example for the first Bernoulli numbers are given by the following table

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
B_k	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	-$\frac{1}{30}$	0	$\frac{5}{66}$	0	$-\frac{691}{2730}$	0	$\frac{7}{6}$	0	$-\frac{3617}{510}$

From this we get the following values of $\zeta(k)$ for $k = 2, 4, 6, 8, 10, 12$:

$$\zeta(2) = \frac{\pi^2}{6}\,, \quad \zeta(4) = \frac{\pi^4}{90}\,, \quad \zeta(6) = \frac{\pi^6}{945}\,, \quad \zeta(8) = \frac{\pi^8}{9450}\,, \quad \zeta(10) = \frac{\pi^{10}}{93555}\,, \quad \zeta(12) = \frac{691\pi^{12}}{638512875}\,.$$

Using Proposition 3.5 we define for even $k \geq 2$ the **normalized version the Eisenstein series** by

$$E_k(\tau) = \frac{1}{\zeta(k)} G_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \,. \tag{3.5}$$

These are the q-series which also appeared in (1.1).

4 Cusp forms and the discriminant function Δ

In this section, we will talk about a special class of modular forms, called cusp forms, which are modular forms vanishing at the "cusps". By cusps, one usually denotes the classes of $\mathbb{Q} \cap \{\infty\}$ modulo the action of a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. For the level one case, where we consider the whole group $\mathrm{SL}_2(\mathbb{Z})$, we have $|\,\mathrm{SL}_2(\mathbb{Z})\backslash(\mathbb{Q} \cap \{\infty\})| = 1$, because every rational number can be send to ∞ by a linear fractional transformation. This means there is just one cusp. A cusp form of level one is, therefore, a modular form which vanishes at ∞ or, equivalently, has a vanishing constant term in its Fourier expansion.

Definition 4.1. *i) A modular form $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$ is called a* **cusp form**, *if $a_0 = 0$.*

ii) By S_k we denote[3] *the space of all cusp forms of weight k.*

In other words, cusp forms are modular forms which vanish as $\tau \to i\infty$ or equivalently have order $v_\infty(f) > 0$ at infinity. We have the decomposition $M_k = \mathbb{C}E_k \oplus S_k$ (Exercise 3).

Definition 4.2. *We define the* **discriminant function Δ** *by ($q = e^{2\pi i \tau}$)*

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \,. \tag{4.1}$$

The function $\tau(n)$ defined by $\Delta(\tau) = \sum_{n=1}^{\infty} \tau(n) q^n$ is called **Ramanujan tau function**.

Expanding the product in the definition of Δ gives the following first values for $\tau(n)$

$$\Delta(\tau) = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + 84480q^8 - 113643q^9 + \dots \,. \tag{4.2}$$

Remark 4.3. i) Ramanujan observed in 1915 that $\tau(n)$ is multiplicative, i.e. $\tau(m \cdot n) = \tau(m) \cdot \tau(n)$ for coprime $m, n \in \mathbb{Z}_{\geq 1}$. For example $\tau(6) = -6048 = -24 \cdot 252 = \tau(2) \cdot \tau(3)$. This was proved by Mordell the next year and later generalized by Hecke to the theory of **Hecke operators**. We will not discuss Hecke operators in this lecture but they play a major role in the theory of modular forms. The function Δ is an example of a **Hecke eigenform** (meaning it is an eigenvector for all Hecke operators having 1 as the coefficient of q), which all satisfy the property that their Fourier coefficients are multiplicative.
The divisor-sums $\sigma_{k-1}(n)$ are also multiplicative and the Eisenstein series, after some normalization, are also examples of Hecke eigenforms.

ii) Lehmer (1947) conjectured that $\tau(n) \neq 0$ for all $n \geq 1$, an assertion sometimes known as Lehmer's conjecture. This conjecture is still unproven but checked for all $1 \leq n \leq 816212624008487344127999$ (due to Derickx, van Hoeij, and Zeng in 2013).

[3] "S" in S_k stands for the german word "Spitzenform" for cusp form.

Since $|e^{2\pi i\tau}| < 1$ for $\tau \in \mathbb{H}$, the terms of the infinite product (4.1) are all non-zero and tend exponentially rapidly to 1, so Δ gives a holomorphic and everywhere non-zero function on $\mathbb{H}$. It gives the first example of a non-trivial cusp form.

Proposition 4.4. *The function $\Delta(\tau)$ is a cusp form of weight* 12.

Proof. Since $\Delta(\tau) \neq 0$, we can consider its logarithmic derivative. We find

$$\frac{1}{2\pi i}\frac{d}{d\tau}\log\Delta(\tau) = \frac{1}{2\pi i}\frac{d}{d\tau}\log\left(q\prod_{n=1}^{\infty}(1-q^n)^{24}\right) = \frac{1}{2\pi i}\frac{d}{d\tau}\left(\log(q) + 24\sum_{n=1}^{\infty}\log(1-q^n)\right).$$

Since $q = e^{2\pi i\tau}$ we have $\frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$ and therefore

$$\frac{1}{2\pi i}\frac{d}{d\tau}\log\Delta(\tau) = 1 - 24\sum_{n=1}^{\infty} n\frac{q^n}{1-q^n} = 1 - 24\sum_{n=1}^{\infty} n\sum_{d=1}^{\infty} q^{dn} = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n = E_2(\tau)\,. \tag{4.3}$$

By Proposition 3.4 and $E_2(\tau) = \frac{6}{\pi^2}G_2(\tau)$ we have

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2E_2(\tau) - \frac{6}{\pi}ic(c\tau+d)\,. \tag{4.4}$$

Combining (4.3), (4.4) and using

$$\frac{d}{d\tau}\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{ad-bc}{(c\tau+d)^2} = \frac{1}{(c\tau+d)^2}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we deduce

$$\frac{1}{2\pi i}\frac{d}{d\tau}\log\left(\frac{\Delta\left(\frac{a\tau+b}{c\tau+d}\right)}{(c\tau+d)^{12}\Delta(\tau)}\right) = \frac{1}{(c\tau+d)^2}E_2\left(\frac{a\tau+b}{c\tau+d}\right) - \frac{12}{2\pi i}\frac{c}{c\tau+d} - E_2(\tau) = 0\,.$$

In other words, $(\Delta|_{12}\gamma)(\tau) = C(\gamma)\Delta(\tau)$ for all $\tau \in \mathbb{H}$ and all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, where $C(\gamma)$ is a non-zero complex number depending only on γ. We want to show that $C(\gamma) = 1$ for all γ.
The slash operator $|_k$ gives a right action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ (Exercise 1), i.e. for γ_1, γ_2 we get

$$C(\gamma_1)C(\gamma_2)\Delta = C(\gamma_1)\Delta|_{12}\gamma_2 = \Delta|_{12}(\gamma_1)|_{12}\gamma_2 = \Delta|_{12}(\gamma_1\gamma_2) = C(\gamma_1\gamma_2)\Delta\,.$$

Therefore $C : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}$ is a homomorphism and we just need to prove $C(T) = C(S) = 1$. By definition we have $\Delta(T(\tau)) = \Delta(\tau)$, since it is defined by a q-series, which gives $C(T) = 1$. To show $C(S) = 1$, we set $\tau = i$ in $\tau^{-12}\Delta(-\frac{1}{\tau}) = (\Delta|_{12}S)(\tau) = C(S)\Delta(\tau)$. ☐

Remark 4.5. Since E_4^3 and Δ are modular forms of weight 12 and $\Delta(\tau) \neq 0$ for $\tau \in \mathbb{H}$, the **modular invariant** (or j-invariant), defined by

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)}$$

is a holomorphic function in $\mathbb{H}$ satisfying $j(\gamma(\tau)) = j(\tau)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Since Δ has a zero of order 1 at ∞ and E_4 does not vanish there, the function j has a pole of order 1 at ∞. Therefore j is a modular function of weight 0, which is not a modular form. Its Fourier expansion, the Laurent expansion at $q = 0$ of $\widetilde{j}$, starts with

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \dots\,.$$

These Fourier coefficients, for the positive exponents of q, are the dimensions of the graded part of an infinite-dimensional graded algebra representation of the so called monster group.

5 Structure of the space of modular forms

We now come to a very important technical result about modular forms. To state and prove this result, we will use some definitions and results from complex analysis that can be found again in [FB] or [SS]. Especially the notion of contour integration will be necessary, which can be found in [SS, Section 1.3] or [FB, Chapter 2].

Proposition 5.1 (Argument principle)**.** *If f is a meromorphic function inside and on some closed contour $\mathcal{C}$ with interior $D \subset \mathbb{C}$, and f has no zeros or poles on $\mathcal{C}$, then*

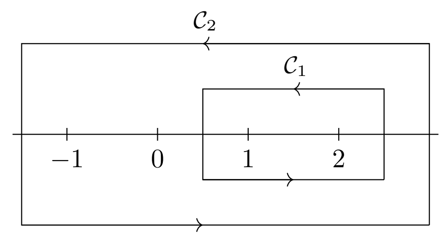
$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f'(z)}{f(z)}dz = \sum_{p\in D} v_p(f)\,.$$

Proof. See for example [FB, Proposition III.7.4]. □

Example 5.2. *We again consider the rational function $f(z) = \frac{z-2}{(z-1)(z+1)^2}$, which is meromorphic on $\mathbb{C}$ with $v_2(f) = 1, v_1(f) = -1$, $v_{-1}(f) = -2$ and $v_p(f) = 0$ for $p \in \mathbb{C}\backslash\{-1,1,2\}$.*

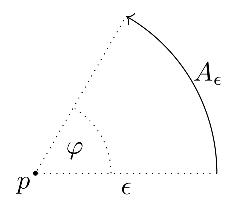
With the two contours $\mathcal{C}_1$ and $\mathcal{C}_2$ shown on the right, we get for example

$$\frac{1}{2\pi i}\int_{\mathcal{C}_1} \frac{f'(z)}{f(z)}dz = v_1(f) + v_2(f) = 0\,,$$
$$\frac{1}{2\pi i}\int_{\mathcal{C}_2} \frac{f'(z)}{f(z)}dz = v_{-1}(f) + v_1(f) + v_2(f) = -2\,.$$



Lemma 5.3. *(Integration over arcs) Let f be a meromorphic function on some open set $U \subset \mathbb{C}$. For an arc $A_\epsilon \subset U$ of radius $\epsilon > 0$, center $p \in U$, angle φ not intersecting any zeros or poles of f, we have*

$$\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{A_\epsilon} \frac{f'(z)}{f(z)}dz = \frac{\varphi}{2\pi}v_p(f)\,.$$



Proof. See for example part (4) in the proof of [FB, Theorem VI.2.3]. □

Lemma 5.4. *Let f be a modular function with no zeros or poles on a contour $\mathcal{C} \subset \mathbb{H}$. Then*

$$\int_{\mathcal{C}} \frac{f'(\tau)}{f(\tau)}d\tau - \int_{\gamma(\mathcal{C})} \frac{f'(\tau)}{f(\tau)}d\tau = -k\int_{\mathcal{C}} \frac{c}{c\tau + d}d\tau$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Proof. Differentiating $f(\gamma(\tau)) = (c\tau + d)^k f(\tau)$ gives

$$f'(\gamma(\tau))\frac{d(\gamma(\tau))}{d\tau} = (c\tau + d)^k f'(\tau) + kc(c\tau + d)^{k-1} f(\tau)\,.$$

Dividing the left-hand side by $f(\gamma(\tau))$ and the right-hand side by $(c\tau+d)^k f(\tau)$ leads to

$$\frac{f'(\gamma(\tau))}{f(\gamma(\tau))}d(\gamma(\tau)) = \frac{f'(\tau)}{f(\tau)}d\tau + k\frac{c}{c\tau+d}d\tau$$

and therefore

$$\int_{\mathcal{C}}\left(\frac{f'(\tau)}{f(\tau)}d\tau - \frac{f'(\gamma(\tau))}{f(\gamma(\tau))}d(\gamma(\tau))\right) = -k\int_{\mathcal{C}}\frac{c}{c\tau+d}d\tau\,.$$

□

Example 5.5. *For $\gamma = S$ Lemma 5.4 gives for a modular function f of weight k*

$$\int_{\mathcal{C}}\frac{f'(\tau)}{f(\tau)}d\tau - \int_{S(\mathcal{C})}\frac{f'(\tau)}{f(\tau)}d\tau = -k\int_{\mathcal{C}}\frac{1}{\tau}d\tau\,. \tag{5.1}$$

Since the factor $(c\tau+d)^k$ does not vanish for $\tau \in \mathbb{H}$ and $c,d \in \mathbb{Z}$, we have $v_p(f) = v_{\gamma(p)}(f)$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and a modular function f. The following theorem gives a restriction on the orders of a modular functions, which will be crucial to describe the space M_k afterwards.

Theorem 5.6 (Valence formula)**.** *For a non-zero modular function f of weight k we have*

$$v_\infty(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_\omega(f) + \sum_{\substack{p\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}\\ p\neq i,\omega}} v_p(f) = \frac{k}{12}\,. \tag{5.2}$$

Proof. The idea of the proof is to count the (order) of the zeros and poles in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ by integrating the logarithmic derivative f'/f of f around the boundary of the fundamental domain $\mathcal{F}$ and then applying the argument principle.

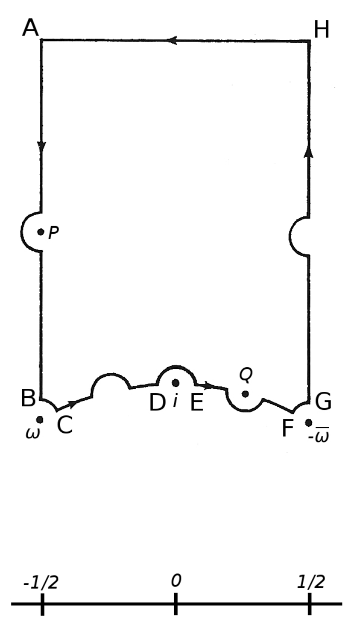


Figure 3: The contour $\mathcal{C}$.

More precisely, we need an approximation first and start with a curve as shown in Figure 3. The contour $\mathcal{C}$ is chosen in such a way that it contains exactly one represent in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ of each pole and zero, except i, ω (and $-\overline{\omega} = S(\omega)$) which are kept outside.

Since f is a modular functions, it is meromorphic at ∞. This means that for some $T \in \mathbb{R}$ the function f has no poles or zeros with imaginary part larger than T. Therefore we can choose the the top line from $H = \frac{1}{2} + iT$ to $A = -\frac{1}{2} + iT$ such that f does not have any poles or zeros on or on top of the line HA.

The rest of the contour follows the boundary of $\mathcal{F}$ with a few exceptions: For each zero or pole $P \neq i, \omega$ on the boundary, we simply circle around it with a small enough radius and the other way round for the congruent point on the other side of the boundary (this way we will only count the point once). This procedure is illustrated for two such points P and Q in Figure 3.

So far we still followed the boundary of $\mathcal{F}$ (modulo $\mathrm{SL}_2(\mathbb{Z})$) but since we dont want to include i and ω we also have to circle around those points with a small enough radius ϵ (and the same way for $-\overline{\omega} = S(\omega)$).

By the argument principle (Proposition 5.1) we obtain

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f'(\tau)}{f(\tau)} d\tau = \sum_{\substack{p \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \\ p \neq i,\omega}} v_p(f)\,. \tag{5.3}$$

On the other hand we can evaluate the contour integral over $\mathcal{C}$ on the left-hand side section by section:

i) AB and GH: The integral from A to B cancels the integral from G to H, because $f(\tau+1) = f(\tau)$, and the lines go in opposite direction, i.e.

$$\frac{1}{2\pi i}\left(\int_A^B + \int_G^H\right) \frac{f'(\tau)}{f(\tau)} d\tau = 0\,.$$

ii) HA: By the map $q = e^{2\pi i\tau}$ the line from H to A gets send to a circle in the unit disc of radius $e^{-2\pi T}$ running clockwise around 0. Recall that $f(\tau) = \tilde{f}(q)$ and therefore we have $\frac{f'(\tau)}{f(\tau)}d\tau = \frac{\tilde{f}'(q)}{\tilde{f}(q)}dq$. By the argument principle we get

$$\frac{1}{2\pi i}\int_H^A \frac{f'(\tau)}{f(\tau)} d\tau = \frac{1}{2\pi i}\int_{|q|=e^{-2\pi T}} \frac{\tilde{f}'(q)}{\tilde{f}(q)} dq = -v_0(\tilde{f}) = -v_\infty(f)\,.$$

Here the minus sign comes from the fact that the contour integral runs clockwise around 0.

iii) BC, DE and FG: All these three sections are small arcs of a small radius ϵ which approach angles $\frac{\pi}{3}$, π and $\frac{\pi}{3}$ as $\epsilon \to 0$. Using Lemma 5.3 and noticing that all three arcs run clockwise (minus sign), we obtain

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{1}{2\pi i}\left(\int_B^C + \int_D^E + \int_F^G\right) \frac{f'(\tau)}{f(\tau)} d\tau &= -\frac{1}{2\pi}\left(\frac{\pi}{3}v_\omega(f) + \pi v_i(f) + \frac{\pi}{3}v_{-\overline{\omega}}(f)\right) \\ &= -\frac{1}{2}v_i(f) - \frac{1}{3}v_\omega(f)\,,\end{aligned}$$

where we used $v_{-\overline{\omega}}(f) = v_{S(\omega)}(f) = v_\omega(f)$ in the last equation.

iv) CD and EF: First notice that the transformation $S(\tau) = -\frac{1}{\tau}$ sends the contour CD to the contour EF, but with directions reversed. By (5.1) we therefore have

$$\frac{1}{2\pi i}\left(\int_C^D \frac{f'(\tau)}{f(\tau)}d\tau + \int_E^F \frac{f'(\tau)}{f(\tau)}d\tau\right) = -\frac{k}{2\pi i}\int_C^D \frac{1}{\tau}d\tau\,.$$

Sending $\epsilon \to 0$, the integral on the right-hand side is just an integral over an arc from ω to i:

$$\lim_{\epsilon\to 0} -\frac{k}{2\pi i}\int_C^D \frac{1}{\tau}d\tau = -\frac{k}{2\pi i}\int_{\text{arc from }\omega\text{ to }i}\frac{1}{\tau}d\tau \overset{\tau=e^{\Theta i}}{=} -\frac{k}{2\pi}\int_{\frac{2\pi}{3}}^{\frac{\pi}{2}} d\Theta = \frac{k}{2\pi}\left(\frac{2\pi}{3}-\frac{\pi}{2}\right) = \frac{k}{12}$$

and therefore we obtain

$$\frac{1}{2\pi i}\left(\int_C^D + \int_E^F\right)\frac{f'(\tau)}{f(\tau)}d\tau = \frac{k}{12}\,.$$

Combining the parts i) - iv) and plugging them into the left-hand side of (5.3) finishes the proof. □

Recall that the difference between a modular function and a modular form is, that a modular form is holomorphic on $\mathbb{H}$ and at ∞. This means that for $f \in M_k$ all the numbers in (5.2) are positive and therefore for a fixed k there are just finitely many solutions. This leads to the following proposition.

Proposition 5.7. *Let $k \in \mathbb{Z}$ be an integer. Then*

i) $M_0 = \mathbb{C}$,

ii) If $k = 2$, $k < 0$ or if k is odd then $M_k = 0$.

iii) If $k \in \{4, 6, 8, 10, 14\}$, then $M_k = \mathbb{C}E_k$.

iv) If $k < 12$ or $k = 14$ then $S_k = 0$.

v) $S_{12} = \mathbb{C}\Delta$ and if $k > 12$ then $S_k = \Delta \cdot M_{k-12}$.

vi) If $k \geq 4$ then $M_k = \mathbb{C}E_k \oplus S_k$.

Proof. i) We know that the constant functions are elements in M_0 and we want to show the reverse. Let $f \in M_0$ be an arbitrary modular form of weight 0 and let $z \in \mathbb{C}$ be any element in the image of f. Then $f(z) - c \in M_0$ has a zero in $\mathbb{H}$, i.e. one of the terms in (5.2) is strictly positive. Since the right-hand side is 0, this can only happen if $f(z) - c$ is the zero function, i.e. f is constant.

ii) We already saw that $M_k = 0$ if k is odd. If $k = 2$ or $k < 0$ then the right-hand side of (5.2) is negative or $\frac{1}{6}$, which has no positive solutions on the left-hand side.

iii) When $k \in \{4, 6, 8, 10, 14\}$, then there is only one possible way of choosing the $v_p(f)$, such that (5.2) holds:

$k = 4:$ $v_\omega(f) = 1$ and all other $v_p(f) = 0$.
$k = 6:$ $v_i(f) = 1$ and all other $v_p(f) = 0$.
$k = 8:$ $v_\omega(f) = 2$ and all other $v_p(f) = 0$.
$k = 10:$ $v_\omega(f) = v_i(f) = 1$ and all other $v_p(f) = 0$.
$k = 14:$ $v_\omega(f) = 2$, $v_i(f) = 1$ and all other $v_p(f) = 0$.

For such k two arbitrary modular forms $f_1, f_2 \in M_k$ have the same order at all points, i.e. $\frac{f_1}{f_2}$ is a modular form of weight 0, which by i) must be constant. Therefore f_1 and f_2 are proportional and since $E_k \in M_k$ the statement follows.

iv) If $f \in S_k$ we have $v_\infty(f) > 0$, which is not possible in (5.2) for $k < 12$ or $k = 14$.

v) We know that $v_\infty(\Delta) = 1$ and by (5.2) this can be the only zero of Δ. Therefore for any $f \in S_k$ the function $\frac{f}{\Delta}$ is a modular form of weight $k - 12$.

vi) This is Exercise 3.

□

Theorem 5.8. *(Dimension formula) For an even positiver integer k we have*

$$\dim_{\mathbb{C}} M_k = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & , \quad k \not\equiv 2 \mod 12 \\ \lfloor \frac{k}{12} \rfloor & , \quad k \equiv 2 \mod 12 \end{cases} . \tag{5.4}$$

Proof. This will now follow by induction on k from the results in Proposition 5.7. For $k < 12$ the above dimension formula is already proven. Combing the results of Proposition 5.7 we have

$$M_{k+12} = \mathbb{C}E_{k+12} \oplus \Delta \cdot M_k$$

and since $\lfloor \frac{k}{12} \rfloor + 1 = \lfloor \frac{k+12}{12} \rfloor$ the statement follows inductively. □

k	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36
$\dim_{\mathbb{C}} M_k$	1	0	1	1	1	1	2	1	2	2	2	2	3	2	3	3	3	3	4

Figure 4: Dimension of M_k for even $0 \leq k \leq 36$.

Proof of Theorem 1.5. Both E_4^2 and E_8 are modular forms of weight 8. Since $\dim_{\mathbb{C}} M_8 = 1$ there must exists a $c \in \mathbb{C}$ with $E_4^2 = cE_8$. But since both have 1 as the constant term in their Fourier expansion we deduce $c = 1$. □

Both E_4^3 and E_6^2 are modular forms of weight 12 having 1 as the constant term in their Fourier expansion and therefore $E_4^3 - E_6^2 \in S_{12}$. By Proposition 5.7 v) this has to be a multiple of Δ and comparing the first few Fourier coefficients gives

$$\Delta(\tau) = \frac{E_4(\tau)^3 - E_6(\tau)^2}{1728} . \tag{5.5}$$

In general every modular form can be written (uniquely) as a polynomial in E_4 and E_6:

Proposition 5.9. *For $k \geq 0$, the set $\{E_4^a E_6^b \mid a, b \geq 0, 4a + 6b = k\}$ is a basis of the space M_k.*

Proof. We first check that the mentioned set has the correct size. Let N_k be the number of solutions to $4a + 6b = k$ in nonnegative integers a and b. For $k \leq 12$ one can check directly that $N_k = \dim_{\mathbb{C}} M_k$ (given in (5.4)) and for $k \geq 12$ one can check that $N_k = N_{k-12} + 1$. Therefore we have $N_k = \dim_{\mathbb{C}} M_k$

for all k. It remains to show that the set is linearly independent. Suppose we have a relation of the form

$$\sum_{\substack{4a+6b=k\\a,b\geq 0}} \lambda_{a,b} E_4(\tau)^a E_6(\tau)^b = 0$$

for all $\tau \in \mathbb{H}$. If there is a pure E_4 term, say $\lambda_{a,0}E_4(\tau)^a$, then setting $\tau = i$ shows $\lambda_{a,0}E_4(i)^a = 0$ since $E_6(i) = 0$ (Exercise 6 ii)). Since $E_4(i) \neq 0$ (which follows from the valence formula (5.2)) we deduce $\lambda_{a,0} = 0$. Therefore all nonzero terms in the sum have $b \geq 1$. As E_6 is not identically 0, we can divide by it and get

$$\sum_{\substack{4a+6b=k\\a,b\geq 0}} \lambda_{a,b} E_4(\tau)^a E_6(\tau)^{b-1} = 0\,,$$

which is a linear relation in weight $k-6$. By induction we see that the remaining coefficients are 0. □

Remark 5.10. Starting with a modular form $f = \sum_{n=0}^{\infty} a_n q^n \in M_k$ and choosing a and b with $4a+6b = k$, we have $f - a_0 E_4^a E_6^b \in S_k$. By Proposition 5.7 v) we have $S_k = \Delta \cdot M_{k-12}$, i.e. we find a $g \in M_{k-12}$ with $f = a_0 E_4^a E_6^b + \Delta \cdot g$. With the explicit expression (5.5) of Δ, this gives a recursive algorithm (and in fact another way of proving Proposition 5.9) to write f as a polynomial in E_4 and E_6.

Proposition 5.11. *Modular forms with different weights are linearly independent over* $\mathbb{C}$.

Proof. Suppose we have nonzero modular forms $f_1, f_2, \ldots, f_m$ with respective weights $k_1 < k_2 < \cdots < k_m$, such that they admit a nontrivial linear relation

$$\alpha_1 f_1(\tau) + \alpha_2 f_2(\tau) + \cdots + \alpha_m f_m(\tau) = 0 \tag{5.6}$$

for all $\tau \in \mathbb{H}$ and $\alpha_j \neq 0$ for $j = 1, \ldots, m$. Replacing τ by $S(\tau)$ and using the modularity, i.e. $f_j(S(\tau)) = \tau^{k_j} f_j(\tau)$, we obtain

$$\alpha_1 \tau^{k_1} f_1(\tau) + \alpha_2 \tau^{k_2} f_2(\tau) + \cdots + \alpha_m \tau^{k_m} f_m(\tau) = 0$$

for all $\tau \in \mathbb{H}$. With Fourier expansions $f_j(\tau) = \sum_{n=0}^{\infty} a_n^{(j)} q^n$ where $q = e^{2\pi i \tau}$, this is equivalent to

$$\sum_{n=0}^{\infty} \left(\alpha_1 \tau^{k_1} a_n^{(1)} + \alpha_2 \tau^{k_2} a_n^{(2)} + \cdots + \alpha_m \tau^{k_m} a_n^{(m)} \right) e^{2\pi i n \tau} = 0\,.$$

Now consider the case of $\tau = iy$ $(y > 0)$ being on the positive imaginary axis, then

$$\sum_{n=0}^{\infty} \left(\alpha_1 (iy)^{k_1} a_n^{(1)} + \alpha_2 (iy)^{k_2} a_n^{(2)} + \cdots + \alpha_m (iy)^{k_m} a_n^{(m)} \right) e^{-2\pi n y} = 0\,. \tag{5.7}$$

For $n > 0$ and any $r \geq 0$ we have $\lim_{y\to\infty} y^r e^{-2\pi n y} = 0$. Now let N be the smallest integer, such that at least for one $1 \leq j \leq m$ we have $a_N^{(j)} \neq 0$. Dividing (5.7) by $e^{-2\pi N y}$ and taking the limit $y \to \infty$ we obtain

$$\lim_{y\to\infty} \alpha_1 (iy)^{k_1} a_N^{(1)} + \alpha_2 (iy)^{k_2} a_N^{(2)} + \cdots + \alpha_m (iy)^{k_m} a_N^{(m)} = 0\,.$$

But the left-hand side of this equation is the limit $y \to \infty$ of a non-constant polynomial in y, which can not be zero and therefore a relation of the form (5.6) can not exist. □

Proposition 5.12. *The modular forms* E_4 *and* E_6 *are algebraically independent over* $\mathbb{C}$.

Proof. Let $P \in \mathbb{C}[X,Y]$ be with $P(E_4(\tau), E_6(\tau)) = 0$ for all $\tau \in \mathbb{H}$. By Proposition 5.11 we can reduce this to the case where $P(E_4, E_6)$ is a sum of modular forms of the same weight k. But by Proposition 5.9 we know that $E_4^a E_6^b$ with $4a+6b = k$ are linearly independent and therefore we conclude $P = 0$. $\square$

Summarizing all the results we get the following description of the space of modular forms.

Corollary 5.13. *Let M denote the* **space of all modular forms** *(of level 1), then we have*

$$M = \bigoplus_{k=0}^{\infty} M_k = \mathbb{C}[E_4, E_6] \cong \mathbb{C}[X,Y]\,,$$

i.e. M is a graded $\mathbb{C}$-algebra, which is isomorphic to the polynomial ring in two variables.

6 Derivatives of modular forms

Modular forms are holomorphic function and therefore we can differentiate them with respect to τ. It is convenient to consider the following notation for a modular form $f = \sum_{n=0}^{\infty} a_n q^n$:

$$f' := \frac{1}{2\pi i}\frac{d}{d\tau} f = q\frac{d}{dq} f = \sum_{n=1}^{\infty} n a_n q^n\,.$$

Here the factor $2\pi i$ has been included in order to preserve the rationality properties of the Fourier coefficients. The derivative of a modular form is, in general, not a modular form anymore. The failure of modularity is given by the following proposition.

Proposition 6.1. *The derivative of a modular form $f \in M_k$ satisfies*

$$f'\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+2} f'(\tau) + \frac{k}{2\pi i} c (c\tau+d)^{k+1} f(\tau)\,.$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Proof. Exercise 6. $\square$

Definition 6.2. *For a modular form $f \in M_k$, we define the* **Serre derivative** *by*

$$\partial_k f := f' - \frac{k}{12} E_2 f\,.$$

Proposition 6.3. *For a modular form $f \in M_k$ we have $\partial_k f \in M_{k+2}$.*

Proof. We set $g(\tau) = f'(\tau) - \frac{k}{12} E_2(\tau) f(\tau)$ and by using Proposition 6.1 and the formula

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) - \frac{6}{\pi} ic(c\tau+d)\,, \tag{6.1}$$

which was a consequence of Proposition 3.4, we obtain for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$

$$\begin{aligned} g\left(\frac{a\tau+b}{c\tau+d}\right) &= f'\left(\frac{a\tau+b}{c\tau+d}\right) - \frac{k}{12}E_2\left(\frac{a\tau+b}{c\tau+d}\right) f\left(\frac{a\tau+b}{c\tau+d}\right) \\ &= (c\tau+d)^{k+2}f'(\tau) + \frac{k}{2\pi i}c(c\tau+d)^{k+1}f(\tau) \\ &\quad - \frac{k}{12}\left((c\tau+d)^2E_2(\tau) - \frac{6}{\pi}ic(c\tau+d)\right)(c\tau+d)^k f(\tau) \\ &= (c\tau+d)^{k+2}\left(f'(\tau) - \frac{k}{12}E_2(\tau)f(\tau)\right) = (c\tau+d)^{k+2}g(\tau)\,. \end{aligned}$$

Since g is also holomorphic in $\mathbb{H}$ and at ∞ we obtain $g \in M_{k+2}$. □

Definition 6.4. *The ring of* **quasimodular forms** *is defined by* $\widetilde{M} = \mathbb{C}[E_2, E_4, E_6]$.

Proposition 6.5. *The ring of quasimodular forms is closed under differentiation and we have*

$$E_2' = \frac{E_2^2 - E_4}{12}\,, \quad E_4' = \frac{E_2E_4 - E_6}{3}\,, \quad E_6' = \frac{E_2E_6 - E_4^2}{2}\,.$$

Proof. By Proposition 6.3 we have $\partial_4 E_4 = E_4' - \frac{1}{3}E_2E_4 \in M_6$ and $\partial_6 E_6 = E_6' - \frac{1}{2}E_2E_6 \in M_8$. Since both spaces are one-dimensional with basis E_6 and E_4^2 respectively we get the second and third equation after comparing the first Fourier coefficients. Using again the modularity formula (4.4) of E_2 and doing a similar calculation as in Proposition 6.3 one can also show that $E_2' - \frac{1}{12}E_2^2 \in M_4$. Therefore this is also a multiple of E_4, which turns out to be $-\frac{1}{12}$ by comparing the Fourier coefficients. □

7 Relations and congruences among Fourier coefficients

We know that for even $k_1, \ldots, k_r \geq 4$ and $a_1, \ldots, a_r \geq 1$ we have $E_{k_1}^{a_1} \ldots E_{k_r}^{a_r} \in M_{a_1k_1 + \cdots + a_rk_r}$. The possible choices of k_j and a_j are much larger than the dimension of $M_{a_1k_1+\cdots+a_rk_r}$ given by the dimension formula before. Therefore we obtain various relations among the divisor-sums, such as

$$\sigma_7(n) = \sigma_3(n) + 120\sum_{j=1}^{n-1}\sigma_3(j)\sigma_3(n-j)$$

$$11\sigma_9(n) = 21\sigma_5(n) - 10\sigma_3(n) + 5040\sum_{j=1}^{n-1}\sigma_3(j)\sigma_5(n-j)\,,$$

which are consequences of the equalities $E_8 = E_4^2$ and $E_{10} = E_4E_6$ in the one-dimensional spaces M_8 and M_{10}. These also imply non-trivial congruences, such as for example $11\sigma_9(n) \equiv 21\sigma_5(n) - 10\sigma_3(n) \mod 5040$. The results on the derivatives of modular forms, given in the section before, even give more relations. For example since $E_2'(\tau) = -24\sum_{n=1}^{\infty} n\sigma_1(n)q^n$ the first equation in Proposition 6.5 gives for all $n \in \mathbb{Z}_{\geq 1}$ the relation

$$6n\sigma_1(n) = 5\sigma_3(n) + \sigma_1(n) - 12\sum_{j=1}^{n-1}\sigma_1(j)\sigma_1(n-j)\,.$$

As a last example we give the following famous congruence for the Ramanujan tau function.

Proposition 7.1. *i) We have*

$$\Delta = \frac{691}{65520} E_{12} - \frac{691}{156}\left(\frac{E_4^3}{720} + \frac{E_6^2}{1008}\right).$$

ii) For all $n \in \mathbb{Z}_{\geq 1}$ we have

$$\tau(n) \equiv \sigma_{11}(n) \mod 691\,.$$

Proof. We know that E_4^3 and E_6^2 are basis of the space M_{12} and by comparing the first Fourier coefficients we get the equation in i). Since 691 is prime and

$$\frac{691}{65520} E_{12}(\tau) = \frac{691}{65520} + \sum_{n=1}^{\infty} \sigma_{11}(n) q^n\,,$$

ii) follows from i) by considering the coefficient of q^n. □

8 Modular forms of higher level

In this course, we just considered modular forms of level 1. We want to end this lecture notes with a few comments on higher level modular forms or more precisely modular forms for congruence subgroups. A complete discussion of modular forms for higher level can be found for example in [DS]. So far we always required that a modular form (or (weakly-)modular function) satisfies $f|_k\gamma = f$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. This condition will be weakened now and we will just require it for $\gamma \in \Gamma$, where $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ are certain subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

Definition 8.1. *i) For $N \in \mathbb{Z}_{\geq 1}$ we define the following subgroups of* $\mathrm{SL}_2(\mathbb{Z})$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \mod N \right\},$$
$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \;\middle|\; a \equiv d \equiv 1 \mod N \right\},$$
$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) \;\middle|\; b \equiv 0 \mod N \right\}.$$

By definition we have the inclusions

$$\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z})\,.$$

ii) A subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is called **congruence subgroup** *if there exists a N with $\Gamma(N) \subset \Gamma$. The smallest such N is called the* **level of** Γ.

We have $\Gamma(1) = \Gamma_1(1) = \Gamma_0(1) = \mathrm{SL}_2(\mathbb{Z})$ and hence $\mathrm{SL}_2(\mathbb{Z})$ is the only congruence subgroup of level 1.

Definition 8.2. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup and let $k \in \mathbb{Z}$. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a* **modular form of weight k for Γ** *if*

i) $f|_k\gamma = f$ for all $\gamma \in \Gamma$,

ii) $f|_k\gamma$ is holomorphic at ∞ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

By $M_k(\Gamma)$ we denote the space of modular forms of weight k for Γ, i.e. with the notation used before we have $M_k = M_k(\mathrm{SL}_2(\mathbb{Z}))$.

Similar to the level 1 case there exist dimension formulas for the higher level case. In the following, we will just mention some details for the level 4 and weight 2 example given in the introduction.

Lemma 8.3. *i) For all $N > 0$ the function*

$$G_{2,N}(\tau) = G_2(\tau) - NG_2(N\tau)$$

is an element in $M_2(\Gamma_0(N))$.

ii) The group $\Gamma_0(4)$ is generated by $\pm T$ and $\pm \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$.

iii) We have $\dim_{\mathbb{C}} M_2(\Gamma_0(4)) = 2$.

Proof. The first statement can be proven directly by using the modular transformation of G_2 given in Proposition 3.4. For ii) we refer to [DS, Exercise 1.2.4] and iii) follows from the general formula given in [DS, Theorem 3.5.1]. □

We now come back to the example from the motivation. For this we define the **theta-function** by

$$\Theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}\,.$$

Proposition 8.4. *The theta-function satisfies the two functional equations*

$$\Theta(\tau+1) = \Theta(\tau)\,, \qquad \Theta\left(-\frac{1}{4\tau}\right) = \sqrt{\frac{2\tau}{i}}\,\Theta(\tau) \qquad (\tau \in \mathbb{H})\,. \tag{8.1}$$

Proof. The first equation follows directly from definition and the second follows from the Poisson transformation formula. See [Z, Proposition 9] for details. □

Now recall that we were interested in counting the number of ways to write a positive number as the sum of for squares, i.e. we wanted to evaluate

$$r_4(n) = \#\left\{(a,b,c,d) \in \mathbb{Z}^4 \mid n = a^2+b^2+c^2+d^2\right\}\,.$$

For this we considered the generating series of $r_4(n)$, i.e.

$$F(q) = \sum_{n\geq 0} r_4(n) q^n = 1 + 8q + 24q^2 + 32q^3 + 24q^4 + 48q^5 + 96q^6 + 64q^7 + 24q^8 + 104q^9 + \dots\,.$$

By the definition of the theta-function, it is easy to see that

$$F(q) = \Theta(\tau)^4\,.$$

Corollary 8.5. *We have $\Theta^4 \in M_2(\Gamma_0(4))$.*

Proof. By Lemma 8.3 ii), we just need to check that

$$\Theta(\tau+1)^4 = \Theta(\tau)^4\,, \qquad \Theta\left(\frac{\tau}{4\tau+1}\right)^4 = (4\tau+1)^2\Theta(\tau)^4 \qquad (\tau \in \mathbb{H})\,.$$

which can be checked directly by writing $\frac{\tau}{4\tau+1} = -\frac{1}{4(\frac{-1}{4\tau}-1)}$ and using (8.1). □

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With all this we can now give a proof of Jacobi's four-square theorem:

Proof of Theorem 1.2. By Lemma 8.3 i) and iii) one can check that $G_{2,2}$ and $G_{2,4}$ are a basis of $M_2(\Gamma_0(4))$ by checking that they are linearly independent. Looking at the first two Fourier coefficients of Θ^4, we deduce $\Theta^4 = -\frac{1}{\pi^2}G_{2,4}$, which gives the formula for $r_4(n)$ given in the Theorem. □

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Introduction to modular forms
Exercises

Perspectives in Mathematical Science IV (Part II)
Nagoya University (Fall 2018)

Deadline: 24th December 2018.

Exercise 1. For a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, a complex number $\tau \in \mathbb{C}$ and a holomorphic function in the upper half plane $f \in \mathcal{O}(\mathbb{H})$, we defined

$$\gamma(\tau) := \frac{a\tau + b}{c\tau + d} \qquad \text{and} \qquad (f|_k\gamma)(\tau) := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

i) Show that for all $\tau \in \mathbb{C}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$\mathrm{Im}(\gamma(\tau)) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2},$$

where $\mathrm{Im}(\tau)$ denotes the imaginary part of τ.

ii) Show that $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ from the left by $\gamma(\tau)$.

(i.e. show that $\gamma(\tau) \in \mathbb{H}$, $I(\tau) = \tau$ and $\gamma'(\gamma(\tau)) = (\gamma' \cdot \gamma)(\tau)$ for all $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$.)

iii) Show that $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{O}(\mathbb{H})$ from the right by $f|_k\gamma$.

(i.e. show that $f|_k\gamma \in \mathcal{O}(\mathbb{H})$, $f|_kI = f$ and $(f|_k\gamma')|_k\gamma = f|_k(\gamma' \cdot \gamma)$ for all $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$, $f \in \mathcal{O}(\mathbb{H})$.)

Exercise 2.

i) Show that $\mathrm{SL}_2(\mathbb{Z})$ is generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(i.e. any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ can be written as $\gamma = S^{s_1}T^{t_1} \dots S^{s_r}T^{t_r}$ with integers $s_1, t_1, \dots, s_r, t_r \in \mathbb{Z}$.)

ii) Show that if f is a meromorphic function on the upper half plan satisfying

$$f(\tau + 1) = f(\tau),$$
$$f(-1/\tau) = \tau^k f(\tau),$$

for all $\tau \in \mathbb{H}$, then f is a weakly modular function of weight k.

Exercise 3.

i) Show that the space M_k is a $\mathbb{C}$-vector space and that for $k \geq 4$ we have $M_k = \mathbb{C}E_k \oplus S_k$.

ii) Prove that if $f \in M_k$ and $g \in M_l$, then $f \cdot g \in M_{k+l}$.

Exercise 4.

i) Let f be a modular form of weight 4. Show that $f\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=0$.

ii) Let g be a modular form of weight 6. Show that $g(i)=0$.

iii) Let h be a modular form of weight 8 with $h(i)=1$. Calculate $h\left(-\frac{2}{5}+\frac{1}{5}i\right)$.

Exercise 5. Express E_{18} as a linear combination of E_6^3 and $E_4^3E_6$.

Exercise 6. Show that the derivative of a modular form $f \in M_k$ satisfies

$$f'\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^{k+2}f'(\tau)+\frac{k}{2\pi i}c(c\tau+d)^{k+1}f(\tau)\,.$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Exercise 7.

i) Show that the Serre derivative maps cusp forms to cusp forms, i.e. it gives a map $\partial_k : S_k \to S_{k+2}$.

ii) Compute Δ' and $\partial_{12}\Delta$.

iii) Show that for all $n \in \mathbb{Z}_{\geq 1}$ we have

$$(n-1)\tau(n) \equiv 0 \mod 24\,,$$

where $\tau(n)$ is the Ramanujan tau function.

Exercise 8. Prove the following identity among divisor sums by using the theory of modular forms: For all $n \in \mathbb{Z}_{\geq 1}$ we have

$$11\sigma_9(n)=21\sigma_5(n)-10\sigma_3(n)+5040\sum_{j=1}^{n-1}\sigma_3(j)\sigma_5(n-j)\,.$$

Bonus exercise: Find an elementary proof of Theorem 1.5, i.e. show that for all $n \in \mathbb{Z}_{\geq 1}$ we have

$$\sigma_7(n)=\sigma_3(n)+120\sum_{j=1}^{n-1}\sigma_3(j)\sigma_3(n-j)\,,$$

without using the theory of modular forms.

(The Bonus exercise is just for fun and does not count for the grading, so you do not need to do it. You can find elementary proofs for this in the literature. Try to find your own proof!)