Introduction to modular forms
Exercises

Perspectives in Mathematical Science IV (Part II)
Nagoya University (Fall 2018)

Deadline: 24th December 2018.

Exercise 1. For a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, a complex number $\tau \in \mathbb{C}$ and a holomorphic function in the upper half plane $f \in \mathcal{O}(\mathbb{H})$, we defined

$$\gamma(\tau) := \frac{a\tau + b}{c\tau + d} \qquad \text{and} \qquad (f|_k\gamma)(\tau) := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

i) Show that for all $\tau \in \mathbb{C}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$\mathrm{Im}(\gamma(\tau)) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2},$$

where $\mathrm{Im}(\tau)$ denotes the imaginary part of τ.

ii) Show that $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ from the left by $\gamma(\tau)$.

(i.e. show that $\gamma(\tau) \in \mathbb{H}$, $I(\tau) = \tau$ and $\gamma'(\gamma(\tau)) = (\gamma' \cdot \gamma)(\tau)$ for all $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$.)

iii) Show that $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{O}(\mathbb{H})$ from the right by $f|_k\gamma$.

(i.e. show that $f|_k\gamma \in \mathcal{O}(\mathbb{H})$, $f|_kI = f$ and $(f|_k\gamma')|_k\gamma = f|_k(\gamma' \cdot \gamma)$ for all $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$, $f \in \mathcal{O}(\mathbb{H})$.)

Exercise 2.

i) Show that $\mathrm{SL}_2(\mathbb{Z})$ is generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(i.e. any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ can be written as $\gamma = S^{s_1}T^{t_1} \dots S^{s_r}T^{t_r}$ with integers $s_1, t_1, \dots, s_r, t_r \in \mathbb{Z}$.)

ii) Show that if f is a meromorphic function on the upper half plan satisfying

$$f(\tau + 1) = f(\tau),$$
$$f(-1/\tau) = \tau^k f(\tau),$$

for all $\tau \in \mathbb{H}$, then f is a weakly modular function of weight k.

Exercise 3.

i) Show that the space M_k is a $\mathbb{C}$-vector space and that for $k \geq 4$ we have $M_k = \mathbb{C}E_k \oplus S_k$.

ii) Prove that if $f \in M_k$ and $g \in M_l$, then $f \cdot g \in M_{k+l}$.

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Exercise 4.

i) Let f be a modular form of weight 4. Show that $f\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=0$.

ii) Let g be a modular form of weight 6. Show that $g(i)=0$.

iii) Let h be a modular form of weight 8 with $h(i)=1$. Calculate $h\left(-\frac{2}{5}+\frac{1}{5}i\right)$.

Exercise 5. Express E_{18} as a linear combination of E_6^3 and $E_4^3E_6$.

Exercise 6. Show that the derivative of a modular form $f \in M_k$ satisfies

$$f'\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^{k+2}f'(\tau)+\frac{k}{2\pi i}c(c\tau+d)^{k+1}f(\tau)\,.$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Exercise 7.

i) Show that the Serre derivative maps cusp forms to cusp forms, i.e. it gives a map $\partial_k : S_k \to S_{k+2}$.

ii) Compute Δ' and $\partial_{12}\Delta$.

iii) Show that for all $n \in \mathbb{Z}_{\geq 1}$ we have

$$(n-1)\tau(n) \equiv 0 \mod 24\,,$$

where $\tau(n)$ is the Ramanujan tau function.

Exercise 8. Prove the following identity among divisor sums by using the theory of modular forms: For all $n \in \mathbb{Z}_{\geq 1}$ we have

$$11\sigma_9(n)=21\sigma_5(n)-10\sigma_3(n)+5040\sum_{j=1}^{n-1}\sigma_3(j)\sigma_5(n-j)\,.$$

Bonus exercise: Find an elementary proof of Theorem 1.5, i.e. show that for all $n \in \mathbb{Z}_{\geq 1}$ we have

$$\sigma_7(n)=\sigma_3(n)+120\sum_{j=1}^{n-1}\sigma_3(j)\sigma_3(n-j)\,,$$

<u>without</u> using the theory of modular forms.

(The Bonus exercise is just for fun and does not count for the grading, so you do not need to do it. You can find elementary proofs for this in the literature. Try to find your own proof!)