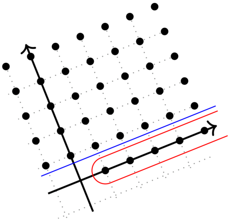
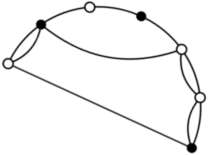
Multiple Eisenstein series and their Fourier expansion

Henrik Bachmann





(joint work with Koji Tasaka)

Analytic number theory seminar

27th May 2016

Content of this talk

- Multiple zeta values
- Multiple Eisenstein series and their Fourier expansion
- Formal iterated integrals and the Goncharov coproduct
- Shuffle regularized multiple zeta values and certain q-series
- Shuffle regularized multiple Eisenstein series
- Open questions

Multiple zeta values

Definition

For natural numbers $n_1, ..., n_{r-1} \geq 1, n_r \geq 2$, the multiple zeta value (MZV) of weight $N = n_1 + ... + n_r$ and length r is defined by

$$\zeta(n_1, ..., n_r) = \sum_{0<m_1<...<m_r} \frac{1}{m_1^{n_1} \dots m_r^{n_r}} .$$

By $\mathcal{Z}_N$ we denote the space spanned by all MZV of weight N and by $\mathcal{Z}$ the space spanned by all MZV.

- The product of two MZV can be expressed as a linear combination of MZV with the same weight (stuffle product). e.g:

$$\zeta(r) \cdot \zeta(s) = \zeta(r, s) + \zeta(s, r) + \zeta(r + s) .$$

- MZV can be expressed as iterated integrals. This gives another way (shuffle product) to express the product of two MZV as a linear combination of MZV.
- These two products give a number of $\mathbb{Q}$-relations (double shuffle relations) between MZV.

Example:

$$\zeta(3,2)+3\zeta(2,3)+6\zeta(1,4)\overset{\text{shuffle}}{=}\zeta(2)\cdot\zeta(3)\overset{\text{stuffle}}{=}\zeta(2,3)+\zeta(3,2)+\zeta(5)\,.$$

$$\Longrightarrow 2\zeta(2,3)+6\zeta(1,4)\overset{\text{double shuffle}}{=}\zeta(5)\,.$$

But there are more relations between MZV. e.g.:

$$\zeta(1,2)=\zeta(3).$$

These follow from the extended double shuffle relations.

Classical Eisenstein series

For even $k > 2$ the Eisenstein series of weight k defined by ($q = e^{2\pi i\tau}$)

$$G_k^{\spadesuit}(\tau) := \frac{1}{2} \sum_{\substack{(l,m)\in\mathbb{Z}^2 \\ (l,m)\neq(0,0)}} \frac{1}{(l\tau+m)^k} = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

are modular forms of weight k. The first sum vanishes for odd k (and there are no non trivial modular forms of odd weight) since one sums over all lattice points.

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Similarly if one would define the Riemann zeta value as a sum over all integer, i.e.

$$\zeta^{\spadesuit}(k) := \frac{1}{2} \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \frac{1}{n^k}$$

then these series would vanish for odd k and $\zeta^{\spadesuit}(k) = \zeta(k)$ for even k.

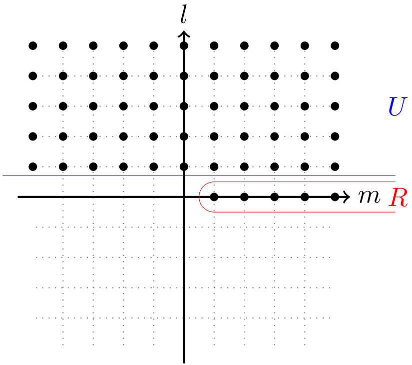
A particular order on lattices

Let $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$ be a lattice with $\tau \in \mathbb{H} := \{x + iy \in \mathbb{C} \mid y > 0\}$. We define an order $\prec$ on Λ_τ by setting

$$\lambda_1 \prec \lambda_2 :\Leftrightarrow \lambda_2 - \lambda_1 \in P$$

for $\lambda_1, \lambda_2 \in \Lambda_\tau$ and the following set which we call the set of positive lattice points

$$P := \{l\tau + m \in \Lambda_\tau \mid l > 0 \vee (l = 0 \wedge m > 0)\} = U \cup R$$



Classical Eisenstein series are ordered sums

With this order on Λ_τ one gets for even $k > 2$:

$$G_k(\tau) := \sum_{0 \prec \lambda} \frac{1}{\lambda^k} = \frac{1}{2} \sum_{\substack{(l,m)\in\mathbb{Z}^2 \\ (l,m)\neq(0,0)}} \frac{1}{(l\tau+m)^k} = G_k^{\spadesuit}(\tau).$$

Since we are not summing over all lattice points the odd Eisenstein series don't vanish anymore and we get for **all** k:

$$G_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \,.$$

This order now allows us to define a multiple version of these series in the obvious way.

Multiple Eisenstein series

Definition

For integers $n_1, \dots, n_r \geq 2$, we define the **multiple Eisenstein series** $G_{n_1,\dots,n_r}(\tau)$ on $\mathbb{H}$ by

$$G_{n_1,\dots,n_r}(\tau) = \sum_{\substack{0 \prec \lambda_1 \prec \dots \prec \lambda_r \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{n_1} \dots \lambda_r^{n_r}} \,.$$

It is easy to see that these are holomorphic functions in the upper half plane and that they fulfill the stuffle product, i.e. it is for example

$$G_3(\tau) \cdot G_4(\tau) = G_{4,3}(\tau) + G_{3,4}(\tau) + G_7(\tau) \,.$$

Remark

Use Eisenstein summation in the case $n_r = 2$.

The function g

Definition

For integers $n_1, \ldots, n_r \geq 1$ and $N = n_1 + \cdots + n_r$ define

$$g_{n_1,\ldots,n_r}(q) := (-2\pi i)^N \cdot \sum_{\substack{0<m_1<\ldots<m_r \\ 0<d_1,\ldots,d_r}} \frac{d_1^{n_1-1}}{(n_1-1)!} \cdots \frac{d_r^{n_r-1}}{(n_r-1)!} q^{m_1 d_1 + \cdots + m_r d_r} .$$

This function can be seen as a multiple version of the generating function of the divisor sums.

$$G_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n = \zeta(k) + g_k(q) .$$

Multiple Eisenstein series - Fourier expansion

Since $G_{n_1,\dots,n_r}(\tau+1) = G_{n_1,\dots,n_r}(\tau)$ we have a Fourier expansion:

Theorem (B. 2012)

For $n_1,\dots,n_r \geq 2$, $N = n_1 + \dots + n_r$ the Fourier expansion of $G_{n_1,\dots,n_r}$ is given by

$$
\begin{aligned}
G_{n_1,\dots,n_r}(\tau) &= \zeta(n_1,\dots,n_r) \\
&+ \sum_{\substack{k_1+k_2=N \\ k_1,k_2\geq 2}} \xi^{(r-1)}_{k_1} g_{k_2}(q) + \sum_{\substack{k_1+k_2+k_3=N \\ k_1,k_2,k_3\geq 2}} \xi^{(r-2)}_{k_1} g_{k_2,k_3}(q) \\
&+ \dots + \sum_{\substack{k_1+\dots+k_r=N \\ k_1,\dots,k_r\geq 2}} \xi^{(1)}_{k_1} g_{k_2,\dots,k_r}(q) + g_{n_1,\dots,n_r}(q),
\end{aligned}
$$

where $\xi^{(d)}_k \in \mathcal{Z}_k$ are $\mathbb{Q}$-linear combinations of multiple zeta values of weight k and length d.

From now on we also write $G_{n_1,\dots,n_r}(q)$ instead of $G_{n_1,\dots,n_r}(\tau)$.

Multiple Eisenstein series - Fourier expansion - Multitangent functions

For $n_1, \ldots, n_r \geq 2$ and $x \in \mathbb{H}$ define the **multitangent function** by

$$\Psi_{n_1,\ldots,n_r}(x) := \sum_{\substack{m_1 < \cdots < m_r \\ m_i \in \mathbb{Z}}} \frac{1}{(x+m_1)^{n_1} \cdots (x+m_r)^{n_r}}.$$

In the case $r = 1$ we also refer to these series as **monotangent function**.

Theorem (Bouillot 2011)

For $n_1, \ldots, n_r \geq 2$ and $N = n_1 + \cdots + n_r$ the multitangent function can be written as

$$\Psi_{n_1,\ldots,n_r}(x) = \sum_{j=2}^{N} \alpha_{N-j} \Psi_j(x)$$

with $\alpha_{N-j} \in \mathcal{Z}_{N-j}$.

Proof idea: Use partial fraction decomposition.

Example:

$$\Psi_{2,3}(x) = \sum_{m_1<m_2} \frac{1}{(x+m_1)^2(x+m_2)^3}$$

$$= \sum_{m_1<m_2} \left(\frac{1}{(m_2-m_1)^3(x+m_1)^2} - \frac{3}{(m_2-m_1)^4(x+m_1)} \right) +$$

$$\sum_{m_1<m_2} \left(\frac{1}{(m_2-m_1)^2(x+m_2)^3} + \frac{2}{(m_2-m_1)^3(x+m_2)^2} + \frac{3}{(m_2-m_1)^4(x+m_2)} \right)$$

$$= 3\zeta(3)\Psi_2(x) + \zeta(2)\Psi_3(x)\,.$$

Multiple Eisenstein series - Fourier expansion - Ψ and g

Since the Lipschitz formula

$$\Psi_k(\tau) = \sum_{m \in \mathbb{Z}} \frac{1}{(x+m)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{d>0} d^{k-1} q^d$$

holds, we deduce:

Proposition

For $n_1, \dots, n_r \geq 2$ we have

$$g_{n_1,\dots,n_r}(q) = \sum_{0<l_1<\dots<l_r} \Psi_{n_1}(l_1\tau) \dots \Psi_{n_r}(l_r\tau) \, .$$

Multiple Eisenstein series - Fourier expansion

Summing over $0 \prec \lambda_1 \prec \cdots \prec \lambda_r$ is by definition equivalent to summing over all $\lambda_1, \dots, \lambda_r$ with

$$\lambda_i - \lambda_{i-1} \in P = U \cup R \qquad (\lambda_0 := 0).$$

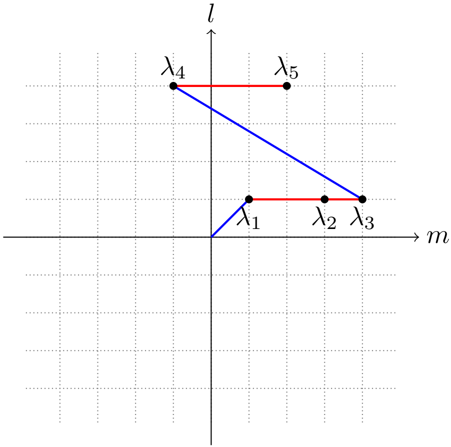
Since $\lambda_i - \lambda_{i-1}$ can be either in U or in R we can split up the sum in the definition of the MES into 2^r terms. For $w_1, \dots, w_r \in \{U, R\}$ we define

$$G^{w_1 \dots w_r}_{n_1,\dots,n_r}(\tau) = \sum_{\substack{\lambda_1,\dots,\lambda_r \in \Lambda_\tau \\ \lambda_i - \lambda_{i-1} \in w_i}} \frac{1}{\lambda_1^{n_1} \cdots \lambda_r^{n_r}}.$$

With this we get

$$G_{n_1,\dots,n_r}(\tau) = \sum_{w_1,\dots,w_r \in \{U,R\}} G^{w_1 \dots w_r}_{n_1,\dots,n_r}(\tau).$$

Example: $w_1w_2w_3w_4w_5 = URRUR$



A summand of $G^{URRUR}_{n_1,n_2,n_3,n_4,n_5}$.

By definition of the multitangent functions we can write

$$G^{URRUR}_{n_1,n_2,n_3,n_4,n_5}(\tau) = \sum_{0<l_1<l_2} \Psi_{n_1,n_2,n_3}(l_1\tau)\Psi_{n_4,n_5}(l_2\tau)\,.$$

Multiple Eisenstein series - Fourier expansion

In length $r = 2$ the $2^2 = 4$ terms are given by

$$G^{RR}_{n_1,n_2}(\tau) = \sum_{\substack{0=l_1=l_2 \\ 0<m_1<m_2}} \frac{1}{(l_1\tau + m_1)^{n_1}(l_2\tau + m_2)^{n_2}} = \zeta(n_1, n_2),$$

$$G^{UR}_{n_1,n_2}(\tau) = \sum_{\substack{0<l_1=l_2 \\ m_1<m_2}} \frac{1}{(l_1\tau + m_1)^{n_1}(l_2\tau + m_2)^{n_2}} = \sum_{0<l} \Psi_{n_1,n_2}(l\tau),$$

$$G^{RU}_{n_1,n_2}(\tau) = \sum_{\substack{0=l_1<l_2 \\ 0<m_1,m_2\in\mathbb{Z}}} \frac{1}{(l_1\tau + m_1)^{n_1}(l_2\tau + m_2)^{n_2}} = \zeta(n_1) \sum_{0<l} \Psi_{n_2}(l\tau),$$

$$G^{UU}_{n_1,n_2}(\tau) = \sum_{\substack{0<l_1<l_2 \\ m_1,m_2\in\mathbb{Z}}} \frac{1}{(l_1\tau + m_1)^{n_1}(l_2\tau + m_2)^{n_2}}$$

$$= \sum_{0<l_1<l_2} \Psi_{n_1}(l_1\tau)\Psi_{n_2}(l_2\tau).$$

Fourier expansion - example

$$G_{2,3}(\tau) = \zeta(2,3) + \sum_{0<l} \Psi_{2,3}(l\tau) + \zeta(2) \sum_{0<l} \Psi_3(l\tau) + \sum_{0<l_1<l_2} \Psi_2(l_1\tau)\Psi_3(l_2\tau)\,.$$

To evaluate the second term we use $\Psi_{2,3}(x) = 3\zeta(3)\Psi_2(x) + \zeta(2)\Psi_3(x)$ and obtain

$$G_{2,3}(\tau) = \zeta(2,3) + 3\zeta(3) \sum_{0<l} \Psi_2(l\tau) + 2\zeta(2) \sum_{0<l} \Psi_3(l\tau) + \sum_{0<l_1<l_2} \Psi_2(l_1\tau)\Psi_3(l_2\tau)\,.$$

With this we get the Fourier expansion of $G_{2,3}$:

$$G_{2,3}(q) = \zeta(2,3) + 3\zeta(3)g_2(q) + 2\zeta(2)g_3(q) + g_{2,3}(q)\,.$$

Fourier expansion - general idea

The general idea to compute the Fourier expansion of $G_{n_1,\dots,n_r}(\tau)$:

- For each of the 2^r words of length r in the alphabet $\{U,R\}$, i.e a word of the form

$$w_1 \dots w_r = \underset{1}{R}\underset{2}{R}\dots R\underset{t_1}{U}R\dots R\underset{t_2}{U}\dots\underset{t_k}{U}R\dots\underset{r}{R},$$

  where $1 \leq t_1 < \dots < t_k \leq r$ are the positions of the U, we get a term of the form

$$G^{w_1\dots w_r}_{n_1,\dots,n_r} = \zeta(n_1,\dots,n_{t_1-1}) \sum_{0<l_1<\dots<l_k} \Psi_{n_{t_1},\dots,n_{t_2-1}}(l_1\tau)\dots\Psi_{n_{t_k}\dots,n_r}(l_k\tau)\,.$$

- Reduce the multitangent functions $\Psi_{n_{t_j},\dots,n_{t_i-1}}(x)$ to a linear combination of MZV and monotangent functions $\Psi_n(x)$ and then write the remaining sums of monotangent functions in terms of the q-series g.

Summary: Multiple Eisenstein series

For $n_1, \dots, n_r \geq 2$ the multiple Eisenstein series $G_{n_1,\dots,n_r}(\tau)$ are holomorphic functions having a Fourier expansion with the multiple zeta value $\zeta(n_1, \dots, n_r)$ as the constant term. By construction they fulfill the stuffle product.

This leads to the following questions:

- Is there a "good" definition of multiple Eisenstein series for $n_1, \dots, n_r \geq 1$?
- Does then these multiple Eisenstein series fulfill the shuffle and stuffle product?

The space of formal iterated integrals

Following Goncharov we consider the algebra $\mathcal{I}$ generated by the elements

$$\mathbb{I}(a_0; a_1, \dots, a_N; a_{N+1}), \quad a_i \in \{0,1\}, N \geq 0.$$

together with the following relations

(i) For any $a, b \in \{0,1\}$ the unit is given by $\mathbb{I}(a;b) := \mathbb{I}(a;\emptyset;b) = 1$.

(ii) The product is given by the shuffle product $ш$

$$\mathbb{I}(a_0; a_1, \dots, a_M; a_{M+N+1})\mathbb{I}(a_0; a_{M+1}, \dots, a_{M+N}; a_{M+N+1})$$
$$= \sum_{\sigma \in sh_{M,N}} \mathbb{I}(a_0; a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(M+N)}; a_{M+N+1}),$$

(iii) The path composition formula holds: for any $N \geq 0$ and $a_i, x \in \{0,1\}$, one has

$$\mathbb{I}(a_0; a_1, \dots, a_N; a_{N+1}) = \sum_{k=0}^{N} \mathbb{I}(a_0; a_1, \dots, a_k; x)\mathbb{I}(x; a_{k+1}, \dots, a_N; a_{N+1}).$$

(iv) For $N \geq 1$ and $a_i, a \in \{0,1\}$, $\mathbb{I}(a; a_1, \dots, a_N; a) = 0$.

(v)*

$$\mathbb{I}(a_0; a_1, \dots, a_N; a_{N+1}) = (-1)^N \mathbb{I}(a_{N+1}; a_N, \dots, a_1; a_0)$$

Formal iterated integrals - coproduct

Goncharov defines a coproduct on $\mathcal{I}$ by

$$\Delta\left(\mathbb{I}(a_0; a_1, \ldots, a_N; a_{N+1})\right) :=$$

$$\sum \left(\prod_{p=0}^{k} \mathbb{I}(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}}) \right) \otimes \mathbb{I}(a_0; a_{i_1}, \ldots, a_{i_k}; a_{N+1}),$$

where the sum runs over all $i_0 = 0 < i_1 < \cdots < i_k < i_{k+1} = N+1$ with $0 \leq k \leq N$.

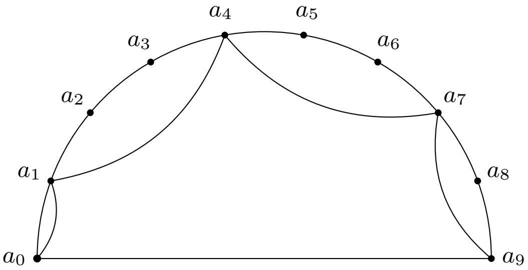
Proposition (Goncharov)

The triple $(\mathcal{I}, ш, \Delta)$ becomes a commutative graded Hopf algebra over $\mathbb{Q}$.

The calculation of Δ can be visualized by marking k of $N+2$ points on a half circle.

Coproduct - diagrams

To calculate $\Delta\left(\mathbb{I}(a_0; a_1, \ldots, a_8; a_9)\right)$ one sums over all possible diagrams of the following form.



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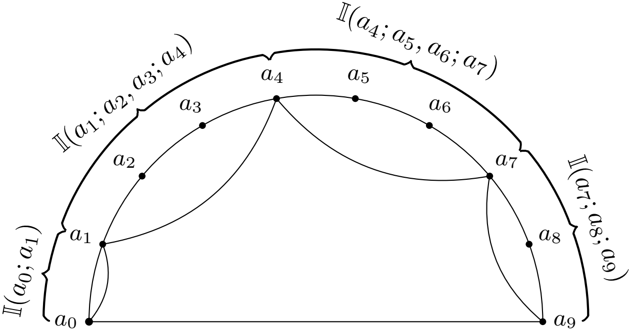


This diagram corresponds to the summand

$$\mathbb{I}(a_0; a_1)\mathbb{I}(a_1; a_2, a_3; a_4)\mathbb{I}(a_4; a_5, a_6; a_7)\mathbb{I}(a_7; a_8; a_9) \otimes \mathbb{I}(a_0; a_1, a_4, a_7; a_9) \, .$$

The space $\mathcal{I}^1$

We will consider the quotient space

$$\mathcal{I}^1 = \mathcal{I}/\mathbb{I}(0;0;1)\mathcal{I}.$$

Let us denote by

$$I(a_0; a_1, \ldots, a_N; a_{N+1})$$

the image of $\mathbb{I}(a_0; a_1, \ldots, a_N; a_{N+1})$ in $\mathcal{I}^1$. The quotient map $\mathcal{I} \to \mathcal{I}^1$ induces a Hopf algebra structure on $\mathcal{I}^1$, but for our application we just need the following statement.

Proposition

For any $w_1, w_2 \in \mathcal{I}^1$, one has $\Delta(w_1 \sqcup\!\sqcup\, w_2) = \Delta(w_1) \sqcup\!\sqcup\, \Delta(w_2)$.

The coproduct on $\mathcal{I}^1$ is given by the same formula as before by replacing $\mathbb{I}$ with I.

The space $\mathcal{I}^1$

For integers $n \geq 0, n_1, \ldots, n_r \geq 1$, we set

$$I_n(n_1, \ldots, n_r) := I(0; \underbrace{0, \ldots, 0}_{n}, \underbrace{1, 0, \ldots, 0}_{n_1}, \ldots, \underbrace{1, 0, \ldots, 0}_{n_r}; 1).$$

In particular, we write $I(n_1, \ldots, n_r)$ to denote $I_0(n_1, \ldots, n_r)$.

Proposition

- For integers $n \geq 0, n_1, \ldots, n_r \geq 1$,

$$I_n(n_1, \ldots, n_r) = (-1)^n \sum^{*} \left(\prod_{j=1}^{r} \binom{k_j - 1}{n_j - 1} \right) I(k_1, \ldots, k_r).$$

  where the sum runs over all $k_1 + \cdots + k_r = n_1 + \cdots + n_r + n$ with $k_1, \ldots, k_r \geq 1$.
- The set $\{I(n_1, \ldots, n_r) \mid r \geq 0, n_i \geq 1\}$ forms a basis of the space $\mathcal{I}^1$.

Example : Write I_n as a linear combination in I's

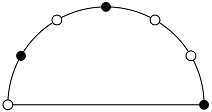
In $\mathcal{I}^1$ it is $I(0;0;1) = 0$ and therefore

$$\begin{aligned} 0 &= I(0;0;1)I(0;1,0;1) \\ &= I(0;0,1,0;1) + I(0;1,0,0;1) + I(0;1,0,0;1) \\ &= I_1(2) + 2I(3) \end{aligned}$$

which gives $I_1(2) = -2I(3) = (-1)^1 \binom{2}{1} I(3)$.

Coproduct - example

In the following we are going to calculate $\Delta(I(2,3)) = \Delta(I(0;1,0,1,0,0;1))$ and therefore we have to determine all possible markings of the diagram



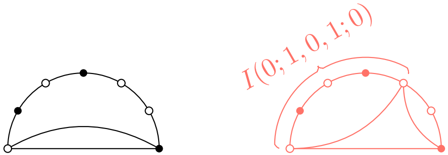
,

where the corresponding summand in the coproduct does not vanish. For simplicity we draw ○ to denote a 0 and ● to denote a 1.

We will consider the $4 = 2^2$ ways of marking the two ● in the top part of the circle separately .

Calculation of $\Delta(I(2,3))$

Diagrams with no marked ●:

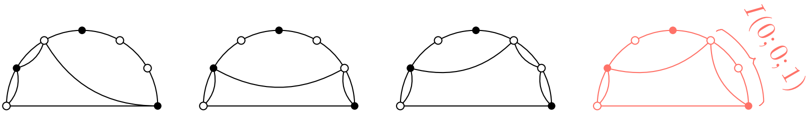


Corresponding sum in the coproduct:

$$I(0;1,0,1,0,0;1) \otimes I(0;\emptyset;1) = I(2,3) \otimes 1\,.$$

Calculation of $\Delta(I(2,3))$

Diagrams with the first ● marked:



Corresponding sum in the coproduct:

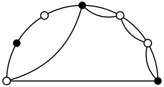
$$
\begin{aligned}
&I(0;1)\cdot I(1;0)\cdot I(0;1,0,0;1)\otimes I(0;1,0;1)\\
&+I(0;1)\cdot I(1;0,1,0;0)\cdot I(0;1)\otimes I(0;1,0;1)\\
&+I(0;1)\cdot I(1;0,1;0)\cdot I(0;0)\cdot I(0;1)\otimes I(0;1,0,0;1)\\
&=I(3)\otimes I(2)-I_1(2)\otimes I(2)+I(2)\otimes I(3)\,.
\end{aligned}
$$

Together with $I_1(2)=-2I(3)$ this gives

$$3I(3)\otimes I(2)+I(2)\otimes I(3)\,.$$

Calculation of $\Delta(I(2,3))$

Diagrams with the second ● marked:

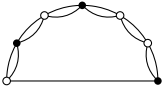


Corresponding sum in the coproduct:

$$I(0;1,0;1)\cdot I(1;0)\cdot I(0;0)\cdot I(0;1)\otimes I(0;1,0,0;1)=I(2)\otimes I(3)\,.$$

Calculation of $\Delta(I(2,3))$

Diagrams with both $\bullet$ marked:



Corresponding sum in the coproduct:

$$1 \otimes I(2,3)\,.$$

Comparison of $\Delta(I(2,3))$ and $G_{2,3}(\tau)$

Summing all 4 parts together we obtain

$$\Delta(I(2,3)) = I(2,3) \otimes 1 + 3I(3) \otimes I(2) + 2I(2) \otimes I(3) + 1 \otimes I(2,3)\,.$$

Compare this to the Fourier expansion of $G_{2,3}(\tau)$:

$$G_{2,3}(\tau) = \zeta(2,3) + 3\zeta(3)g_2(q) + 2\zeta(2)g_3(q) + g_{2,3}(q)\,.$$

Since $\Delta(I(n_1,\ldots,n_r)) \in \mathcal{I}^1 \otimes \mathcal{I}^1$ exists for all $n_1,\ldots,n_r \geq 1$ this comparison suggests, that there might be a extended definition of $G_{n_1,\ldots,n_r}$ by defining a map

$$\mathcal{I}^1 \otimes \mathcal{I}^1 \to \mathbb{C}[[q]]$$

which sends the first component to the corresponding zeta values and the second component to the functions g.

Shuffle regularized zeta values and $g^{ш}$

Theorem (Ihara, Kaneko, Zagier)

There exist an algebra homomorphism $Z^{ш} : \mathcal{I}^1 \to \mathcal{Z}$ with $\zeta^{ш}(n_1, \dots, n_r) = Z^{ш}(I(n_1, \dots, n_r))$ such that

$$\zeta^{ш}(n_1, \dots, n_r) = \zeta(n_1, \dots, n_r)$$

for $n_1, \dots, n_{r-1} \geq 1$ and $n_r \geq 2$. It is uniquely determined by $Z^{ш}(I(1)) = 0$.

Theorem (B., K. Tasaka)

There exist an algebra homomorphism $\mathfrak{g}^{ш} : \mathcal{I}^1 \to \mathbb{C}[[q]]$ with $g^{ш}_{n_1,\dots,n_r}(q) := \mathfrak{g}^{ш}(I(n_1, \dots, n_r))$ such that

$$g^{ш}_{n_1,\dots,n_r}(q) = g_{n_1,\dots,n_r}(q)$$

for $n_1, \dots, n_r \geq 2$.

Proof sketch: We use generating functions and give an explicit form of $\mathfrak{g}^{ш}$.

Shuffle regularized MES

Definition

For integers $n_1, \dots, n_r \geq 1$, we define the q-series $G^{ш}_{n_1,\dots,n_r}(q) \in \mathbb{C}[[q]]$, which we call **shuffle regularized multiple Eisenstein series**, as the image of the generator $I(n_1, \dots, n_r)$ in $\mathcal{I}^1$ under the algebra homomorphism $(Z^{ш} \otimes \mathfrak{g}^{ш}) \circ \Delta$:

$$G^{ш}_{n_1,\dots,n_r}(q) := (Z^{ш} \otimes \mathfrak{g}^{ш}) \circ \Delta\big(I(n_1, \dots, n_r)\big).$$

We denote the space spanned by all shuffle regularized multiple Eisenstein series where the corresponding MZV exists by

$$\mathcal{E}_N = \langle G^{ш}_{n_1,\dots,n_r}(q) \mid N = n_1 + \dots + n_r,\ r \geq 0, n_i \geq 1, n_r \geq 2 \rangle_{\mathbb{Q}}.$$

In the definition we identify $\mathcal{Z} \otimes \mathbb{C}[[q]]$ with $\mathbb{C}[[q]]$ in the obvious way.

Shuffle regularized multiple Eisenstein series

Theorem (B., K. Tasaka 2014)

For all $n_1, \dots, n_r \geq 1$ the shuffle regularized multiple Eisenstein series $G^{ш}_{n_1,\dots,n_r}$ have the following properties:

- Setting $q = e^{2\pi i \tau}$ they are holomorphic functions on the upper half plane having a Fourier expansion with the shuffle regularized multiple zeta values as the constant term.
- They fulfill the shuffle product, i.e. we have an algebra homomorphism $\mathcal{I}^1 \to \mathbb{C}[[q]]$ by sending the generators $I(n_1, \dots, n_r)$ to $G^{ш}_{n_1,\dots,n_r}(q)$.
- For integers $n_1, \dots, n_r \geq 2$ they equal the multiple Eisenstein series

$$G^{ш}_{n_1,\dots,n_r}(q) = G_{n_1,\dots,n_r}(q)$$

  and therefore they fulfill the stuffle product in these cases.

Proof sketch: The first statement follows directly by definition. The second statement follows from the fact that Δ, $Z^{ш}$ and $\mathfrak{g}^{ш}$ are algebra homomorphism and hence $(Z^{ш} \otimes \mathfrak{g}^{ш}) \circ \Delta$ is also an algebra homomorphism.

Coproduct $\leftrightarrow$ multiple Eisenstein series

Proof sketch continued:

- To show that the $G^{\sqcup\!\sqcup}$ coincide with the G in the case $n_1, \dots, n_r \geq 2$ we give an explicit formula for the coproduct.
- For this we also split up the possible diagrams into $2^r = \sum_{k=0}^{r} \binom{r}{k}$ groups, where $\binom{r}{k}$ gives the number of ways marking k of the r $\bullet$.
- We show that the term

$$G_{n_1,\dots,n_r}^{w_1 \dots w_r}$$

  in the calculation of the Fourier expansion corresponds to the diagrams where the positions of the U in the word $w_1 \dots w_r$ coincide with the positions of the marked $\bullet$ by giving explicit formulas for both terms.
- The reduction of multitangent to monotangent functions (i.e. partial fraction expansion) in some sense then corresponds to the reduction of the I_n into linear combinations of the I's.

□

Shuffle regularized MES - double shuffle relations

Since the shuffle regularized Eisenstein series fulfill the shuffle product we have

$$G_2^{⧢}(q) \cdot G_3^{⧢}(q) \overset{\text{shuffle}}{=} G_{3,2}^{⧢}(q) + 3G_{2,3}^{⧢}(q) + 6G_{1,4}^{⧢}(q)$$

We also have the stuffle product whenever the indices are greater equal to 2:

$$G_2^{⧢}(q) \cdot G_3^{⧢}(q) \overset{\text{stuffle}}{=} G_{3,2}^{⧢}(q) + G_{2,3}^{⧢}(q) + G_5^{⧢}(q)\,.$$

This gives the same relation between MES as we had before for MZV:

$$2G_{2,3}^{⧢}(q) + 6G_{1,4}^{⧢}(q) \overset{\text{double shuffle}}{=} G_5^{⧢}(q)\,.$$

But we don't have all relations of MZV since the stuffle product of MES fails when at least one $n_j = 1$. While Euler has shown that $\zeta(3) - \zeta(1,2) = 0$ we get

$$G_3^{⧢}(q) - G_{1,2}^{⧢}(q) = \frac{1}{2} q \frac{d}{dq} G_1^{⧢}(q) \neq 0\,.$$

Shuffle regularized MES - double shuffle relations

Euler also showed that

$$\zeta(6)^2 = \frac{715}{691}\zeta(12)$$

and this relation can also be proven by using the extended double shuffle relations of multiple zeta values.

For multiple Eisenstein series this relation does not hold since there are cusp forms in weight 12 and it is

$$G_6(\tau)^2 = \frac{715}{691}G_{12}(\tau) + \alpha\Delta(q)$$

with some $\alpha \in \mathbb{C}\backslash\{0\}$ and $\Delta(q) = q\prod_{n>0}(1-q^n)^{24}$.

Shuffle regularized MES - Current project

Currently we are interested in the following question:

Question

Is the $\mathbb{C}$-vector space spanned by all $G^{ш}$ (or equivalently by all $g^{ш}$) closed under $\mathrm{d} = q\frac{d}{dq}$?

Shuffle regularized MES - Current project

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Theorem

For $k \geq 1$ we have

$$\frac{(-2\pi i)^2}{k}\,\mathrm{d}\,g_k^{ш}(q) = (k+1)g_{k+2}^{ш}(q) - \sum_{n=2}^{k+1}(2^n-2)g_{k+2-n,n}^{ш}(q)\,.$$

There is a similar formula for $\mathrm{d}\,G_k^{ш}$ proven by Kaneko.

Shuffle regularized MES - Open questions

There are a lot of other open questions for multiple Eisenstein series:

- What is exactly the failure of the stuffle product of shuffle regularized multiple Eisenstein series?
- What is the dimension of the space $\mathcal{E}_N$?
- Consider the projection $\pi : \mathcal{E}_N \longrightarrow \mathcal{Z}_N$ to the constant term, i.e

  $$\pi(G^{ш}_{n_1,\dots,n_r}(q)) = \zeta(n_1,\dots,n_r)\,.$$

  What is the kernel of π and are there elements in the kernel which are not derivatives of MES or cusp forms?
- Which linear combinations of MES are modular forms?
- Is there an iterated integral expression for $g^{ш}$ or $G^{ш}$?
- Functional equations and special values of the L-series of g, $g^{ш}$ and $G^{ш}$?

Summary

- Multiple Eisenstein series $G_{n_1,\dots,n_r}$ which are defined for $n_1,\dots,n_r \geq 2$ are multiple versions of the classical Eisenstein series and they fulfill the stuffle product.
- Their Fourier expansions are similar to the coproduct Δ on the space $\mathcal{I}^1$ of formal iterated integrals.
- This connections enables one to define shuffle regularized multiple Eisenstein series $G^{\sqcup\!\sqcup}_{n_1,\dots,n_r}$ for all $n_1,\dots,n_r \geq 1$.
- They fulfill the shuffle product and for $n_1,\dots,n_r \geq 2$ the stuffle product since in these cases they are equal to the multiple Eisenstein series.
- Since the algebra of shuffle regularized Eisenstein series contains all modular forms this setup gives a framework to study the connection of multiple zeta values and modular forms. Yet there are a lot of open and interesting problems to be solved.