Multiple Eisenstein series and their Fourier coefficients

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based on a joint work with Koji Tasaka

$$G(k_1,\dots,k_r) = \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \cdots \lambda_r^{k_r}}$$

$$= \sum_{n \geq 0} \mathcal{E}_n(k_1,\dots,k_r) q^n$$

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Overview

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Multiple zeta values & Algebraic setup

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Multiple Eisenstein series

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Shuffle regularized multiple Eisenstein series

4
Multiple Eisenstein coefficients

These slides and related papers are available on my homepage:

https://www.henrikbachmann.com/

Definition

For $k_1 \geq 2, k_2, \ldots, k_r \geq 1$ define the **multiple zeta value** (MZV)

$$\zeta(k_1, \ldots, k_r) = \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}\,.$$

By r we denote its **depth** and $k_1 + \cdots + k_r$ will be called its **weight**.

- $\mathcal{Z}$: $\mathbb{Q}$-algebra of MZVs

There are two different ways to express the product of MZV in terms of MZV.

Stuffle product (coming from the definition as iterated sums)

Example in depth two ($k_1, k_2 \geq 2$)

$$\begin{aligned}\zeta(k_1)\cdot\zeta(k_2) &= \sum_{m>0}\frac{1}{m^{k_1}}\sum_{n>0}\frac{1}{n^{k_2}}\\ &= \sum_{0<m<n}\frac{1}{m^{k_1}n^{k_2}} + \sum_{0<n<m}\frac{1}{m^{k_1}n^{k_2}} + \sum_{m=n>0}\frac{1}{m^{k_1+k_2}}\\ &= \zeta(k_1,k_2)+\zeta(k_2,k_1)+\zeta(k_1+k_2)\,.\end{aligned}$$

Shuffle product (coming from the expression as iterated integrals)

Example in depth two ($k_1, k_2 \geq 2$)

$$\zeta(k_1)\cdot\zeta(k_2) = \sum_{j=2}^{k_1+k_2-1}\left(\binom{j-1}{k_1-1}+\binom{j-1}{k_2-1}\right)\zeta(j, k_1+k_2-j)\,.$$

Define the following spaces $\mathfrak{H}^0 \subset \mathfrak{H}^1 \subset \mathfrak{H}$

- $\mathfrak{H} = \mathbb{Q}\langle x, y\rangle$ "Words in x and y"
- $\mathfrak{H}^1 = \mathbb{Q} + \mathfrak{H}y$ "Words ending in y"
- $\mathfrak{H}^0 = \mathbb{Q} + x\mathfrak{H}y$ "Words starting in x and ending in y"

For $k \geq 1$ we write

$$z_k = x^{k-1}y\,.$$

- $\mathfrak{H}^1$: span of words $z_{k_1} \dots z_{k_r}$ with $k_1, k_2, \dots k_r \geq 1$ for $r \geq 0$.
- $\mathfrak{H}^0$: span of words $z_{k_1} \dots z_{k_r}$ with $k_1 \geq 2, k_2, \dots k_r \geq 1$ for $r \geq 0$.

We can view ζ as a $\mathbb{Q}$-linear map

$$\begin{aligned} \zeta \colon \mathfrak{H}^0 &\longrightarrow \mathcal{Z} \\ z_{k_1} \dots z_{k_r} &\longmapsto \zeta(k_1, \dots, k_r)\,, \end{aligned}$$

where $\zeta(1) = 1$.

Definition (shuffle product ⧢)

Define the $\mathbb{Q}$-bilinear product ⧢ on $\mathfrak{H}$ by $1 \shuffle w = w \shuffle 1 = w$ for any word $w \in \mathfrak{H}$ and

$$a_1 w_1 \shuffle a_2 w_2 = a_1(w_1 \shuffle a_2 w_2) + a_2(a_1 w_1 \shuffle w_2)$$

for any letters $a_1, a_2 \in \{x, y\}$ and words $w_1, w_2 \in \mathfrak{H}$.

Definition (stuffle product $*$)

Define the $\mathbb{Q}$-bilinear product $*$ on $\mathfrak{H}^1$ by $1 * w = w * 1 = w$ for any word $w \in \mathfrak{H}^1$ and

$$z_i w_1 * z_j w_2 = z_i(w_1 * z_j w_2) + z_j(z_i w_1 * w_2) + z_{i+j}(w_1 * w_2)$$

for any $i, j \geq 1$ and words $w_1, w_2 \in \mathfrak{H}^1$.

We get $\mathbb{Q}$-(sub)algebras

$$\mathfrak{H}^0_{\shuffle} \subset \mathfrak{H}^1_{\shuffle} \subset \mathfrak{H}_{\shuffle} \qquad \text{and} \qquad \mathfrak{H}^0_* \subset \mathfrak{H}^1_* .$$

Stuffle product Example in depth two ($k_1, k_2 \geq 1$)

$$z_{k_1} * z_{k_2} = z_{k_1} z_{k_2} + z_{k_2} z_{k_1} + z_{k_1+k_2}\,.$$

Shuffle product Example in depth two ($k_1, k_2 \geq 1$)

$$z_{k_1} \sqcup\!\sqcup\, z_{k_2} = \sum_{j=1}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) z_j z_{k_1+k_2-j}\,.$$

The map $\zeta : \mathfrak{H}^0_\bullet \to \mathcal{Z}$ is an algebra homomorphism for $\bullet \in \{*, \sqcup\!\sqcup\}$.

Finite double shuffle relations (FDSR)

For $w, v \in \mathfrak{H}^0$ we have

$$\zeta(w \sqcup\!\sqcup\, v - w * v) = 0\,.$$

We can extend the map $\zeta : \mathfrak{H}^0 \longrightarrow \mathcal{Z}$ in two ways to obtain algebra homomorphisms

$$\zeta^\bullet : \mathfrak{H}^1_\bullet \longrightarrow \mathcal{Z}$$
$$z_{k_1} \dots z_{k_r} \longmapsto \zeta^\bullet(k_1, \dots, k_r)\,,$$

for $\bullet \in \{*, ш\}$, which are both uniquely determined by $\zeta^\bullet(z_1) = 0$ and $\zeta^\bullet_{|\mathfrak{H}^0} = \zeta$.

- $\zeta^{ш}(k_1, \dots, k_r)$: shuffle regularized multiple zeta values.
- $\zeta^{*}(k_1, \dots, k_r)$: stuffle regularized multiple zeta values.

Extended double shuffle relations (EDSR)

For $w \in \mathfrak{H}^0$, $v \in \mathfrak{H}^1$ and $\bullet \in \{*, ш\}$ we have

$$\zeta^\bullet(w \, ш \, v - w * v) = 0\,.$$

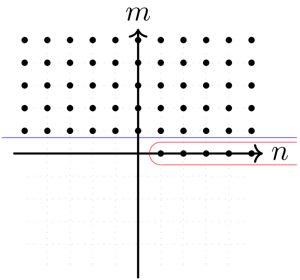
MZV-holy grail conjecture: All relations among MZV can be obtained from the EDSR.

Let $\tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$. Define an order $\succ$ on the lattice $\mathbb{Z}\tau + \mathbb{Z}$ by

$$\lambda_1 \succ \lambda_2 :\Leftrightarrow \lambda_1 - \lambda_2 \in P$$

for $\lambda_1, \lambda_2 \in \mathbb{Z}\tau + \mathbb{Z}$ and the following set of positive lattice points

$$P := \{m\tau + n \in \mathbb{Z}\tau + \mathbb{Z} \mid m > 0 \vee (m = 0 \wedge n > 0)\} \ .$$



In other words: $\lambda_1 \succ \lambda_2$ **iff** λ_1 **is above or on the right of** λ_2.

Definition

For integers $k_1, k_2, \ldots, k_r \geq 2$, we define the **multiple Eisenstein series** by

$$G(k_1, \ldots, k_r) := G(k_1, \ldots, k_r; \tau) = \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \ldots \lambda_r^{k_r}} \,.$$

These are holomorphic functions in the upper half plane and they satisfy the **stuffle product** formula, i.e. we have for example

$$G(2) \cdot G(3) = G(2,3) + G(3,2) + G(5)\,.$$

Remark

Use Eisenstein summation in the case $k_1 = 2$.

- $\mathfrak{H}^{\geq 2}$: span of words $z_{k_1} \dots z_{k_r}$ with $k_1, k_2, \dots k_r \geq 2$ for $r \geq 0$.

We view the multiple Eisenstein series as a $\mathbb{Q}$-linear map

$$\begin{aligned} G\colon \mathfrak{H}^{\geq 2} &\longrightarrow \mathcal{O}(\mathbb{H}) \\ z_{k_1} \dots z_{k_r} &\longmapsto G(k_1, \dots, k_r)\,, \end{aligned}$$

with $G(1) = 1$.

Facts

- $\mathfrak{H}^{\geq 2}$ is closed under the stuffle product, i.e. we have $\mathbb{Q}$-algebras

$$\mathfrak{H}^{\geq 2}_{*} \subset \mathfrak{H}^{0}_{*} \subset \mathfrak{H}^{1}_{*}\,.$$

- The map G is a $\mathbb{Q}$-algebra homomorphism from $\mathfrak{H}^{\geq 2}_{*}$ to $\mathcal{O}(\mathbb{H})$.

In depth $r = 1$ we have for $k \geq 2$ and $q = e^{2\pi i\tau}$

$$G(k) = G(k;\tau) = \sum_{\substack{\lambda \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda^k} = \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m=0 \wedge n>0) \vee m>0}} \frac{1}{(m\tau + n)^k}$$

$$= \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{\substack{m>0 \\ d>0}} d^{k-1} q^{md} =: \zeta(k) + (-2\pi i)^k \, \mathrm{g}(k)\,.$$

For even $k \geq 4$ the $G(k)$ are modular forms of weight k.

Definition

For $k_1, \dots, k_r \geq 1$ we define the q-series

$$\mathrm{g}(k_1, \dots, k_r) = \sum_{\substack{m_1 > \dots > m_r > 0 \\ d_1, \dots, d_r > 0}} \frac{d_1^{k_1 - 1}}{(k_1 - 1)!} \cdots \frac{d_r^{k_r - 1}}{(k_r - 1)!} q^{m_1 d_1 + \dots + m_r d_r} \in \mathbb{Q}[[q]]\,.$$

Theorem (Gangl-Kaneko-Zagier 2006 ($r=2$), B. 2012)

For $k_1, \dots, k_r \geq 2$ the $G(k_1, \dots, k_r)$ have a Fourier expansion of the form

$$G(k_1, \dots, k_r) = \zeta(k_1, \dots, k_r) + \sum_{n>0} a_n q^n$$

and they can be written as a $\mathcal{Z}[2\pi i]$-linear combination of the q-series g.

Examples

$$G(k) = \zeta(k) + (-2\pi i)^k \, \mathrm{g}(k) \,,$$

$$G(3,2) = \zeta(3,2) + 3\zeta(3)(-2\pi i)^2 \, \mathrm{g}(2) + 2\zeta(2)(-2\pi i)^3 \, \mathrm{g}(3) + (-2\pi i)^5 \, \mathrm{g}(3,2) \,.$$

Question

Can we extend the definition of $G(k_1, \dots, k_r)$ for $k_1, \dots, k_r \geq 1$, such that we have $\zeta^{\bullet}(k_1, \dots, k_r)$ as a constant term?

③ Shuffle regularized MES - Goncharov Coproduct

On the $\mathbb{Q}$-algebra $\mathfrak{H}^1_{\sqcup\!\sqcup}$ one can define the **Goncharov coproduct** Δ, which gives $\mathfrak{H}^1_{\sqcup\!\sqcup}$ the structure of a Hopf algebra.
There exist explicit formulas for Δ and we have for example for $k \geq 1$

$$\Delta(z_k) = z_k \otimes 1 + 1 \otimes z_k \,,$$
$$\Delta(z_3 z_2) = z_3 z_2 \otimes 1 + 3 z_3 \otimes z_2 + 2 z_2 \otimes z_3 + 1 \otimes z_3 z_2 \,.$$

Compare this to the Fourier expansion of $G(3,2)$:

$$G(k) = \zeta(k) + (-2\pi i)^k \,\mathrm{g}(k)\,, \qquad (k \geq 2)$$

$$G(3,2) = \zeta(3,2) + 3\zeta(3)(-2\pi i)^2 \,\mathrm{g}(2) + 2\zeta(2)(-2\pi i)^3 \,\mathrm{g}(3) + (-2\pi i)^5 \,\mathrm{g}(3,2)\,.$$

Question (Gangl-Kaneko-Zagier)

Is there a connection of Goncharovs coproduct and the Fourier expansion of MES?

Answer (B. - Tasaka, (2014) 2017)

Yes.

Proposition (B. - Tasaka 2017)

There exist a $\mathbb{Q}$-algebra homomorphism

$$\mathrm{g}^{ш}\colon \mathfrak{H}^1_{ш} \longrightarrow \mathbb{Q}[[q]]$$
$$z_{k_1}\dots z_{k_r} \longmapsto \mathrm{g}^{ш}(k_1,\dots,k_r)\,,$$

such that $\mathrm{g}^{ш}(k_1,\dots,k_r) = \mathrm{g}(k_1,\dots,k_r)$ for $k_1,\dots,k_r \geq 2$.

These $\mathrm{g}^{ш}$ can be written down explicitly.

Proposition (see my MZV lecture)

For $k_1, k_2 \geq 1$ and $k = k_1 + k_2$ we have

$$\mathrm{g}(k_1)\,\mathrm{g}(k_2) = \sum_{j=1}^{k-1}\left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1}\right)\mathrm{g}(j,k-j)$$
$$+\binom{k-2}{k_1-1}\left(q\frac{d}{dq}\frac{\mathrm{g}(k-2)}{k-2} - \mathrm{g}(k-1)\right) + \delta_{k_1,1}\delta_{k_2,1}\,\mathrm{g}(2)\,.$$

③ Shuffle regularized MES - Definition

Define algebra homomorphism $\hat{\mathrm{g}}^{ш} : \mathfrak{H}^1_{ш} \to \mathbb{Q}[\pi i][[q]]$ by

$$\hat{\mathrm{g}}^{ш}(k_1,\ldots,k_r) = (-2\pi i)^{k_1+\cdots+k_r}\,\mathrm{g}^{ш}(k_1,\ldots,k_r).$$

Definition (Shuffle regularized multiple Eisenstein series (B. - Tasaka 2017))

We define the $\mathbb{Q}$-algebra homomorphism

$$\begin{aligned} G^{ш}\colon \mathfrak{H}^1_{ш} &\longrightarrow \mathcal{Z}[\pi i][[q]] \\ z_{k_1}\ldots z_{k_r} &\longmapsto G^{ш}(k_1,\ldots,k_r)\,, \end{aligned}$$

by $G^{ш} = m \circ (\zeta^{ш} \otimes \hat{\mathrm{g}}^{ш}) \circ \Delta$, where m denotes usual multiplication.

$$\begin{array}{ccc} \mathfrak{H}^1 & \xrightarrow{\quad G^{ш} \quad} & \mathcal{Z}[\pi i][[q]] \\ \Delta \downarrow & & \uparrow m \\ \mathfrak{H}^1 \otimes \mathfrak{H}^1 & \xrightarrow{\zeta^{ш} \otimes \hat{\mathrm{g}}^{ш}} & \mathcal{Z} \otimes \mathbb{Q}[\pi i][[q]] \end{array}$$

③ Shuffle regularized MES - Connection to G & RDSR

Theorem (B.-Tasaka 2017)

For $k_1, \dots, k_r \geq 2$ we have

$$G^{\sqcup\!\sqcup}(k_1, \dots, k_r) = G(k_1, \dots, k_r)\,.$$

Corollary (Restricted double shuffle relations (RDSR))

For $w, v \in \mathfrak{H}^{\geq 2}$ we have

$$G^{\sqcup\!\sqcup}(w \sqcup\!\sqcup v - w * v) = 0\,.$$

Therefore multiple Eisenstein series satisfy some of the relations of MZV and we have

Restricted DSR $\subset$ Finite DSR $\subset$ Extended DSR .

Questions

Are there more relations among $G^{\sqcup\!\sqcup}$ than RDSR?

③ Shuffle regularized MES - Example of RDSR

Proposition (Homework in my MZV course)

For all $n \geq 1$ we have

$$\sum_{j=-n}^{n} (-1)^j z_2^{n-j} ш z_2^{n+j} - \sum_{j=-n}^{n} (-1)^j z_2^{n-j} * z_2^{n+j} = 4^n (z_3 z_1)^n - z_4^n \,.$$

In particular, this gives

$$4^n G^{ш}(\{3,1\}^n) = G^{ш}(\{4\}^n)\,.$$

This implies that $G^{ш}(\{3,1\}^n)$ is a modular form of weight $4n$, which follows from

$$\sum_{n \geq 0} G(\{4\}^n) X^n = \exp\left(\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} G(4m) X^m \right)\,.$$

This fact can be seen as an analogue of the classical 3-1 formula of MZV

$$\zeta(\{3,1\}^n) = \frac{2\pi^{4n}}{(4n+2)!}\,.$$

③ Shuffle regularized MES - More relations of $G^{ш}$

In weight 5 there are relations among $G^{ш}$ which do not come from RDSR. For example, one can check that the following FDSR holds

$$G^{ш}(z_2 ш z_2 z_1 - z_2 * z_2 z_1) = G^{ш}(2,2,1) + 6G^{ш}(3,1,1) - G^{ш}(2,3) - G^{ш}(4,1) = 0\,.$$

Proposition (B., 2020+)

For $k_1, k_2 \geq 2$ we have $G^{ш}(z_{k_1} ш z_{k_2} z_1 - z_{k_1} * z_{k_2} z_1) = 0\,.$

Proof sketch: There exist a stuffle regularized version G^*, which satisfies $G^*(k,1) = G^{ш}(k,1)$ for $k \geq 2$.

Questions

Do $G^{ш}$ satifsy all FDSH? ... No! There seems to be an unknown set of relations

$$\text{Restricted DSR} \subsetneq \text{relations satisfied by } G^{ш} \subsetneq \text{Extended DSR}\,.$$

Maybe finite double shuffle relations among words, which do not contain the substring $z_1 z_1$?

Definition

We define the **multiple Eisenstein coefficients** $\mathcal{E}_n(k_1, \dots, k_r)$ by

$$G^{⧢}(k_1, \dots, k_r) = \sum_{n \geq 0} \mathcal{E}_n(k_1, \dots, k_r) q^n \, .$$

These can also be seen as maps $\mathcal{E}_n : \mathfrak{H}^1 \to \mathcal{Z}[\pi i] = \mathcal{Z} + \pi i \mathcal{Z}$.

Basic facts

- $\mathcal{E}_0(k_1, \dots, k_r) = \zeta^{⧢}(k_1, \dots, k_r)$.
- For all $n \geq 0$ and $w, v \in \mathfrak{H}^{\geq 2}$ we have

$$\mathcal{E}_n(w ⧢ v - w * v) = 0 \, .$$

Relations among $\mathcal{E}_n \rightsquigarrow$ Relations among elements in $\mathcal{Z}[\pi i]$.

We write $P = -2\pi i$.

Examples In depth one we have for $k \geq 2, n \geq 1$

$$\mathcal{E}_0(k) = \zeta(k)\,, \qquad \mathcal{E}_n(k) = \sigma_{k-1}(n)\frac{P^k}{(k-1)!}\,.$$

In depth two we get $\mathcal{E}_0(k_1,k_2) = \zeta^{ш}(k_1,k_2)$ and for $k_1, k_2 \geq 2$

$$\mathcal{E}_1(k_1,k_2) = \frac{P^{k_1}}{(k_1-1)!}\zeta(k_2) + \sum_{\substack{m_1+m_2=k_1+k_2\\ m_1,m_2\geq 2}} C^{m_2}_{k_1,k_2}\frac{P^{m_1}}{(m_1-1)!}\zeta(m_2)\,,$$

where

$$C^{m_2}_{k_1,k_2} = (-1)^{k_1}\binom{m_2-1}{k_2-1} + (-1)^{m_2-k_1}\binom{m_2-1}{k_1-1}\,.$$

General: For $n \geq 1$ the $\mathcal{E}_n(k_1,\dots,k_r)$ are products of P and MZV in depth $< r$.

$P = -2\pi i.$

Examples We saw that $4G^{\sqcup\!\sqcup}(3,1) = G^{\sqcup\!\sqcup}(4)$. Since we have

$$4\mathcal{E}_1(3,1) = -4\zeta(2)P^2,$$

$$\mathcal{E}_1(4) = \frac{P^4}{6},$$

we obtain $\zeta(2) = -\frac{P^2}{24} = \frac{\pi^2}{6}$.

Remark

The stuffle product formula for $G(2)G(k)$ can be used to get Euler's formula

$$\zeta(2m) = -\frac{B_{2m}}{2(2m)!}P^{2m},$$

by using an explicit "stuffle-product" analogue for $\mathrm{g}(k_1)\,\mathrm{g}(k_2)$.

(The proofs of these do not use Euler's formula)

$P = -2\pi i.$

Examples We saw that $G^{ш}(2,2,1) + 6G^{ш}(3,1,1) - G^{ш}(2,3) - G^{ш}(4,1) = 0.$ This gives

$$\mathcal{E}_1(2,2,1) + 6\mathcal{E}_1(3,1,1) - \mathcal{E}_1(2,3) - \mathcal{E}_1(4,1) = 3\zeta(3)P^2 - 3\zeta(2,1)P^2 = 0$$

from which we deduce $\zeta(2,1) = \zeta(3)$.

This example show that relations among the $\mathcal{E}$ can give (some) extended double shuffle relations.

Questions

Can we obtain all extended double shuffle relations of MZV?

Open problems & Future work

Possible future joint works & student projects:

- Understand which relations are satisfied by $G^{\sqcup\!\sqcup}$.
- Give explicit formulas for $\mathcal{E}_n$.
- Understand the relationship between relations among $\mathcal{E}_n$ and MZV & π^2.
- Study the operator $q\frac{d}{dq}$. Conjecturally the spaced spanned by $G^{\sqcup\!\sqcup}$ is closed under this operator.
- Consider a rational version $C^{\sqcup\!\sqcup}$ of $G^{\sqcup\!\sqcup}$. Assume we have a map $Z : \mathfrak{H}^1 \to \mathbb{Q}$, which maps $z_{k_1} \dots z_{k_r}$ to the coefficient of an rational associator. Then define

$$\begin{array}{ccc} \mathfrak{H}^1 & \xrightarrow{\quad C^{\sqcup\!\sqcup} \quad} & \mathbb{Q}[[q]] \\ \Delta \big\downarrow & & \big\uparrow m \\ \mathfrak{H}^1 \otimes \mathfrak{H}^1 & \xrightarrow{\; Z \otimes \mathrm{g}^{\sqcup\!\sqcup} \;} & \mathbb{Q} \otimes \mathbb{Q}[[q]] \end{array}$$

  This construction should be closely related to ongoing projects with A. Burmester, U. Kühn and N. Matthes.

Proposition

For $k_1, k_2 \geq 1$ and we have

$$\mathrm{g}(k_1)\,\mathrm{g}(k_2) = \mathrm{g}(k_1, k_2) + \mathrm{g}(k_2, k_1) + \mathrm{g}(k_1 + k_2) + \sum_{j=1}^{k_1+k_2-1} \left(\lambda^j_{k_1,k_2} + \lambda^j_{k_2,k_1}\right) \mathrm{g}(j),$$

where the rational numbers $\lambda^j_{k_1,k_2}$ are given by

$$\lambda^j_{k_1,k_2} = (-1)^{k_2-1} \binom{k_1 + k_2 - 1 - j}{k_1 - j} \frac{B_{k_1+k_2-j}}{(k_1 + k_2 - j)!}\,.$$