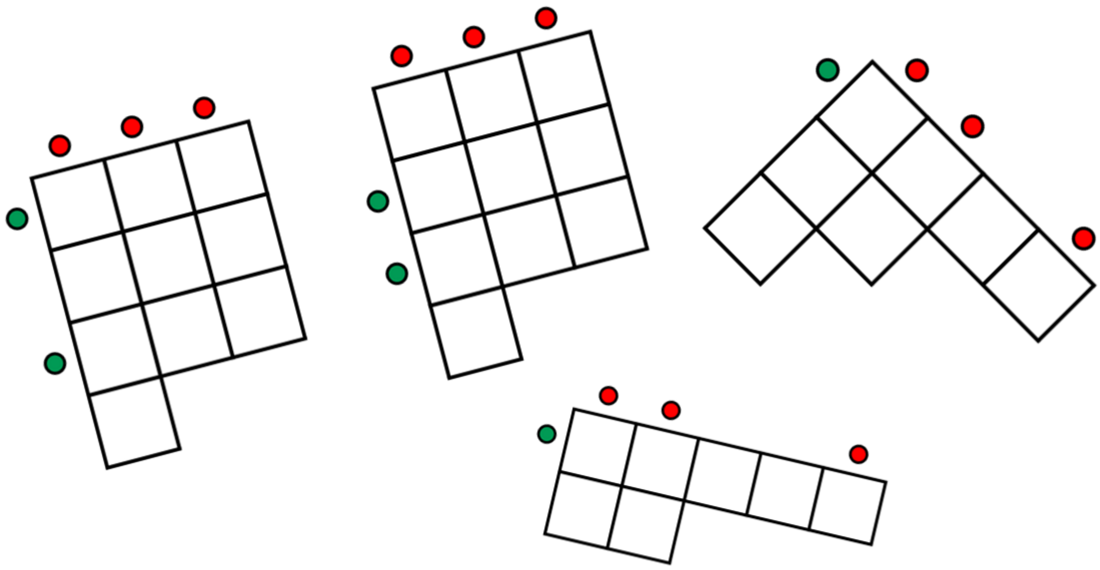
Dualities for q-analogues of multiple zeta values



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Introduction

In this master thesis, a particular type of linear relations among q-analogues of MZVs and some consequences are investigated, so-called duality relations. The thesis is thus integrated into the field of analytic number theory, in parts in that of combinatorics.

From analytic number theory and complex analysis (e.g. [FB], [Za1]) and from some applications in physics (e.g. [RL]), the Riemann Zeta function

$$\zeta(s) := \sum_{m\geq 1} \frac{1}{m^s}$$

is well-known. It's analytic behaviour is a research area in itself (cf. [Za1] for an introduction). For many applications as Planck's law in physics, we only need the values of ζ for $s \in \mathbb{N}_{>1}$, so-called *single zeta values*. Hence, we are interested in examining the algebraic structure of those, especially the structure over $\mathbb{Q}$. In particular, we consider products of single zeta values:

$$\begin{aligned}\zeta(k_1)\zeta(k_2) &= \left(\sum_{m\geq 1} \frac{1}{m^{k_1}}\right)\left(\sum_{m\geq 1} \frac{1}{m^{k_2}}\right) = \left(\sum_{m_1>m_2>0} + \sum_{m_1=m_2>0} + \sum_{m_2>m_1>0}\right) \frac{1}{m_1^{k_1} m_2^{k_2}} \\ &= \zeta(k_1+k_2) + \left(\sum_{m_1>m_2>0} + \sum_{m_2>m_1>0}\right) \frac{1}{m_1^{k_1} m_2^{k_2}}. \qquad (\Delta)\end{aligned}$$

The remaining sums, in general, can not be expressed as $\mathbb{Q}$-linear combination of single zeta values. Hence, these sums are of great interest. This motivates *multiple zeta values (MZVs)*, one of the main objects in this thesis:

$$\zeta(\mathbf{k}) := \zeta(k_1, \ldots, k_r) := \sum_{m_1>\cdots>m_r>0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}, \qquad (*)$$

where $r \geq 0$, and $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{N}^r$ with $k_1 \geq 2$ for convergence reasons. For $r = 0$, we set $\zeta(\emptyset) := 1$. The corresponding indices $\mathbf{k}$ are called *admissible*. MZVs are mathematically interesting objects and build an active area of reasearch as one can see on the webpage by M. Hoffman ([Ho3]), where he collects more than 600 articles about MZVs; for overviews about MZVs have a look on [Ba6], [BF], [Wal] or [Zu1]. But also for theoretical physics when dealing with Feynman diagrams (see [BK] for details), MZVs are important.

With $(*)$, we can express the product in (Δ) as linear combination of multiple zeta values: For every $k_1, k_2 \geq 2$ it is

$$\zeta(k_1)\zeta(k_2) = \zeta(k_1, k_2) + \zeta(k_1 + k_2) + \zeta(k_2, k_1). \tag{0.1}$$

Every monomial in single zeta values can be expressed as $\mathbb{Z}$-linear combination of multiple zeta values, i.e. every algebraic relation in single zeta values transfers into a linear relation among multiple zeta values. This can be shown by induction on the degree of the monomial and by using the definition $(*)$ of MZVs as iterated sums. With the same arguments, it can also be proven that every monomial in MZVs is a $\mathbb{Z}$-linear combination of MZVs again, see [Ba6].

The discussed representation of products of MZVs as linear combination of MZVs is called *stuffle product*. There is a second one, called *shuffle product* and it comes from identifying MZVs as iterated integrals (see Thm. 1.4).

In general, stuffle and shuffle product do not give the same representation in MZVs, which is why we get in general non-trivial $\mathbb{Q}$-linear relations among MZVs. Conjecturally, after *regularization*, those give all $\mathbb{Q}$-linear relations among MZVs. That is why both representations are of interest.

It turns out that there are particular linear relations, called *duality relations*, that cannot be explained with double shuffle relations so far:

Theorem 0.1. *Write an admissible index* $\mathbf{k}$ *in the form*

$$\mathbf{k} = (k_1 + 1, \{1\}^{d_1 - 1}, \ldots, k_r + 1, \{1\}^{d_r - 1})$$

with unique $k_1, \ldots, k_r$, $d_1, \ldots, d_r \geq 1$ *and define its* dual index

$$\mathbf{k}^\vee := (d_r + 1, \{1\}^{k_r - 1}, \ldots, d_1 + 1, \{1\}^{k_1 - 1}).$$

Then we have

$$\zeta(\mathbf{k}) = \zeta(\mathbf{k}^\vee).$$

See Theorem 3.2 for a detailed version of the duality theorem. A first example of such duality relations is

$$\zeta(3) = \zeta(2, 1). \tag{0.2}$$

This relation cannot be obtained from stuffle or shuffle products of MZVs, else $\zeta(1)$ had to be defined. Especially, it cannot be obtained from double shuffle relations.

For studying the algebraic structure of MZVs, it is helpful to consider MZVs on a more algebraically abstract level. This leads to so-called q-analogues of MZVs. These are power series in a formal variable q such that we get an MZV back by taking the limit $q \to 1$ (or by first multiplying with a suitable power of $(1 - q)$). The main advantage is that we can work with $\mathbb{Q}$-algebras instead of vector spaces coming from the span of MZVs. There are several models of q-analogues, such as the one introduced independently by Bradley and Zhao, the one also independently by Schlesinger and Zudilin or the so-called *bi-brackets*, introduced by Bachmann. In this way, for example, transcendence problems about (multiple) zeta values pass over to linear independence problems. Every model of q-analogues has its advantages because every model embodies parts of the structure of MZVs. For example, the Bradley-Zhao model incorporates the duality relation, the Schlesinger-Zudilin model satisfies the stuffle product, and bi-brackets give a strong connection to modular forms.

We give an unified approach to qMZVs by defining

$$\zeta_q(k_1,\ldots,k_r;Q_1,\ldots,Q_r) := \sum_{m_1>\cdots>m_r>0} \frac{Q_1(q^{m_1})}{(1-q^{m_1})^{k_1}} \cdots \frac{Q_r(q^{m_r})}{(1-q^{m_r})^{k_r}} \in \mathbb{Q}[\![q]\!],$$

for $r \geq 0$ ($\zeta_q(\emptyset,\emptyset) := 1$ for $r=0$), $k_1,\ldots,k_r \geq 0$ and polynomials $Q_1 \in X\mathbb{Q}[X]$, $Q_2,\ldots,Q_r \in \mathbb{Q}[X]$ with $\deg Q_j \leq k_j$ for all j. More details of this approach are given in the appendix A and in [BK]. Most of the considered models are obtained by particular choices of the polynomials Q_j.

In this thesis, we examine and compare the most common models of qMZVs for duality relations. Here we will first note that the model of Bradley and Zhao satisfies the same duality as MZVs (Thm. 3.5),

$$\zeta_q^{\mathrm{BZ}}(\mathbf{k}) = \zeta_q^{\mathrm{BZ}}(\mathbf{k}^\vee),$$

that of Schlesinger and Zudilin satisfies a very similar looking, but quite different, relation (Thm. 3.16),

$$\zeta_q^{\mathrm{SZ}}(\mathbf{k}) = \zeta_q^{\mathrm{SZ}}(\mathbf{k}^\dagger),$$

where $\mathbf{k}^\dagger$ is the *SZ-dual index*, which is for $\mathbf{k}$, with $k_1,\ldots,k_r,d_1,\ldots,d_r \geq 1$ uniquely written as

$$\mathbf{k} = (k_1,\{0\}^{d_1-1},\ldots,k_r,\{0\}^{d_r-1}),$$

defined as

$$\mathbf{k}^\dagger := (d_r,\{0\}^{k_r-1},\ldots,d_1,\{0\}^{k_1-1}).$$

Bi-brackets also satisfy a kind of duality, which is called *partition relation* (Thm. 3.21), in other models, we often can 'translate' BZ- or SZ-duality. The goal of the thesis was to elaborate on these different kinds of duality and to draw several conclusions and connections; precisely, there were two main questions:

1.) What is the relation between duality in the BZ- and the one in the SZ-model?

2.) How do BZ- and SZ-duality look in other models of qMZVs?

The following self-reliant results were obtained:

(i) We give some explicit translation of bi-brackets into Schlesinger-Zudilin qMZVs (Thm. 2.19), such as translations between both models on the level of generating series (Thm. 2.18).

(ii) Motivated by Bachmann's proof of the partitions relation (proof of Thm. 3.21), we state and prove SZ-duality similarly on the level of generating series (Thm. 3.18 and Cor. 3.3).

(iii) As a consequence of the mentioned similarity in the proofs of partition relation and SZ-duality, the third result will be the equivalence of SZ-duality and partition relation (Thm. 3.22).

(iv) One of the main results of this thesis is presented in Section 3.5. There we adopt a recent method of proving identities among (q)MZVs, so-called *connected sums*, introduced by Seki and Yamamoto [SY]. We will find using this concept a simultaneous proof of BZ-duality and SZ-duality. Such an accompanying proof was not known so far and is surprising since BZ- and SZ-duality are pretty different, even though they look similar. For this, we prove a more general statement; BZ-duality is the case $x = 1$, SZ-duality is the case $x = 0$. In particular, for both dualities, a new proof is obtained. Furthermore, we get at the same time also a new proof of the q-Ohno relations, of which BZ-duality is a special case.

(v) In Section 3.6, we elucidate a result by Singer, which states that SZ-duality and SZ-stuffle product give some q-shuffle product leading directly to the shuffle product of MZVs (Thm. 3.46). We present an elaboration of Singer's proof idea, which is mainly based on Theorem 3.52.

(vi) Another main result is Section 4, where we develop the concept of so-called *marked partitions*. Every partition of a number is related to its Young diagram. We now understand Young diagrams under a marked partition, where we mark rows or columns with colors. This concept is interesting since we can express qMZVs in terms of marked partitions via writing some qMZV as q-series and interpreting the coefficients of q^N as a weighted sum over partitions of N, where the weights correspond to the number of allowed colorings of the corresponding Young diagrams.

This concept leads to the following results:

a) Visualization of SZ-duality (Prop. 4.14) and a new proof of SZ-duality via marked partitions (Lem. 4.13).

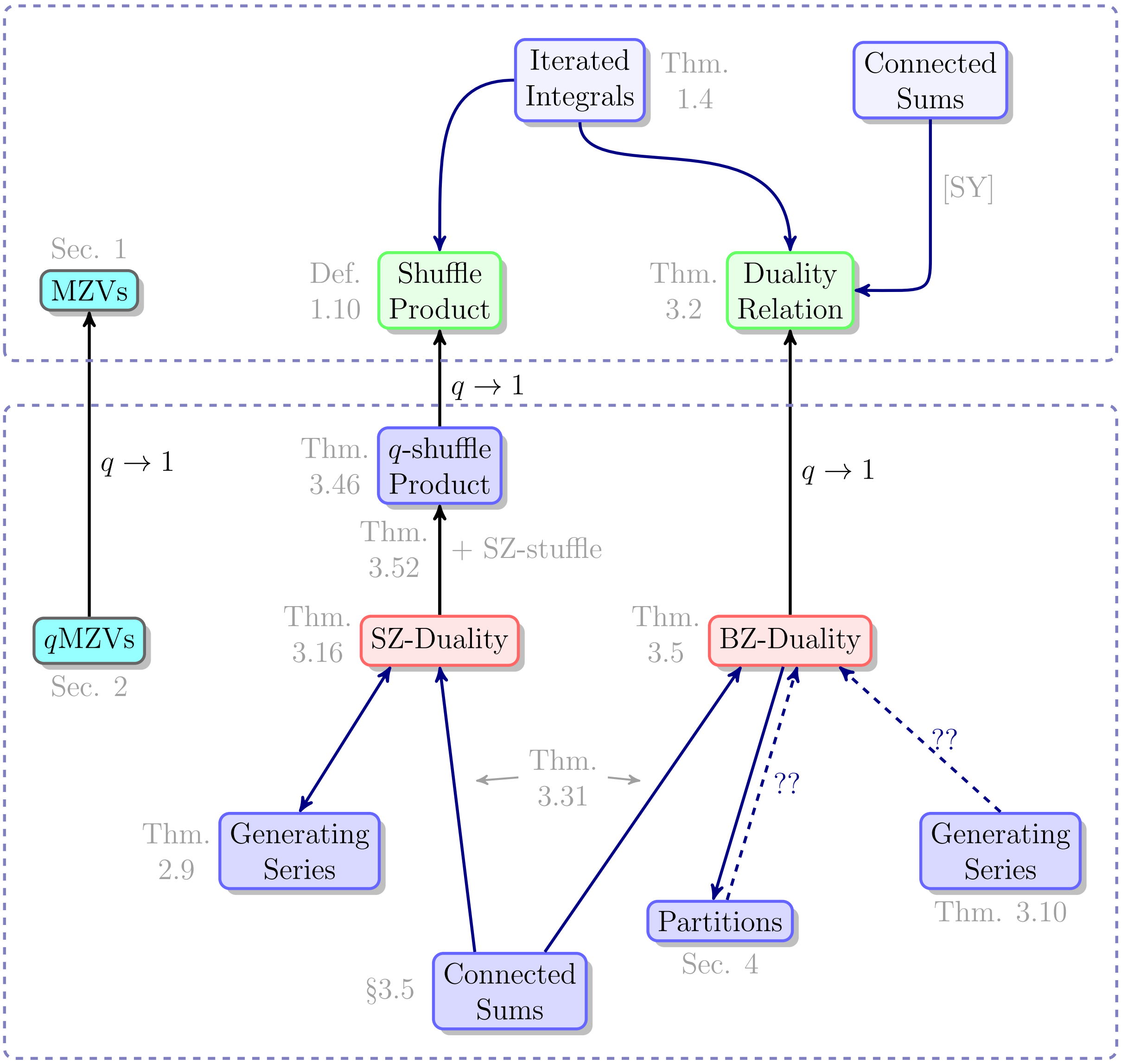
b) Visualization of BZ-duality (Prop. 4.19).

c) Visualization of the number of conjugacy classes of $\mathrm{GL}(n, K)$ for finite fields K (Lem. 4.33) and their connection to qMZVs on the level of generating series (Thm. 4.30).

(vii) To get the desired visualization of SZ-/BZ-duality, we write SZ-qMZVs and BZ-qMZVs as q-series and obtain that the coefficients are sums over partitions. Comparing coefficients in the respective duality relation gives purely combinatorial results, where in particular, the one obtained from BZ-duality is interesting (Lem. 4.13 and Thm. 4.18).

We end this introduction with a tabular overview of how BZ- and SZ-duality in different models look like and with a general summary of the thesis:

	BZ-duality	**SZ-duality**
MZV	Thm. 3.2/Cor. 3.36	Rem. 3.23(i)
BZ	Thm. 3.5	Rem. 3.23(ii)
TBZ	Rem. 3.12	Prop. A.79
SZ	Prop. 3.8/Thm. 3.10	Thm. 3.16
Bi-brackets	Thm. 3.11	Prop. 3.22
OOZ	Thm. 3.24	Thm. 3.24



Structure

The first chapter will provide an initial overview of multiple zeta values, including basic definitions and famous conjectures.

The second chapter gives an introduction to q-analogues of multiple zeta values. In particular, we consider different models of q-analogues of multiple zeta values, namely the Bradley-Zhao (BZ), the Schlesinger-Zudilin (SZ), the bi-brackets, the Ohno-Okuda-Zudilin (OOZ) and the by Takeyama extended Bradley-Zhao model (TBZ).

The third chapter contains the central part of this thesis, namely the duality of MZVs and the duality in selected models of q-analogues. In the first section, we will discuss briefly and at first without proof, the duality for the BZ-model, which is on word algebraic and index level, the same as for MZVs. Afterwards, in the second section, we focus on the duality relation for the SZ-model. There are various proofs for, but we give there a new one, using generating series. In the third section, we will consider one of the main aspects of the thesis. We adopt the new concept of so-called *connected sums*, introduced by Seki and Yamamoto ([SY]), so that we will get a new proof for SZ-duality, BZ-duality and the q-Ohno relation at one go. In the fourth and last section, we prove in detail a consequence of the SZ-duality, namely that, together with the SZ-stuffle product, we can derive the shuffle product for MZVs.

In the first three parts of the fourth chapter, we treat one consequence of the considered dualities of qMZVs in detail. Namely, every qMZV is a q-series, where the coefficients of q^N can be written as a sum over partitions. So a duality relation gives equality of q-series, leading to equality of coefficients. For the SZ-model and bi-brackets, this equality springs elementary from the bijective map ρ, which maps a partition of a natural number N to the one of N with transposed Young diagram. For the duality of the BZ-model, this looks more difficult. However, there - as in the SZ model - the coefficients of q^N are the number of partitions of N where rows and columns are marked in some way we will study. After that, we give in the next section a direct connection between the partition function and SZ-qMZVs resp. bi-brackets. It is already known that the partition function and the number of conjugacy classes of the symmetric group are closely related. That is the motivation and transition to the last section, which states a similar connection between OOZ-qMZVs and the number of conjugacy classes of the general linear group over some finite field K.

The appendix A gives an unified approach to qMZVs. In particular, for every model, we consider fundamental properties and the related shuffle product. Furthermore, we discuss briefly often occurring subalgebras of the algebra $\mathcal{Z}_q$ of qMZVs and several dimension conjectures. The appendix was already written in the framework of the 'Vorbereitungsprojekt'.

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1. Multiple zeta values

In the following, we introduce multiple zeta values (MZVs) and some notations. Furthermore, we give an introduction to quasi-shuffle algebras and point out so-called double shuffle relations. They give after slight modification conjecturally all linear relations among MZVs, which is why they are interesting. We refer to [Ba6], [BF], [Wal] and [Zu1] for several overviews of MZVs.

1.1. Basics

First of all, we have to clarify the term of a multiple zeta value and some related notation:

Definition 1.1. (i) For $r \geq 0$, a tuple $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{N}_0^r$ is called *index*. For $r = 0$ we write $\mathbf{k} = \emptyset$.

(ii) An index $\mathbf{k} = (k_1, \ldots, k_r)$ with either $r = 0$ or $k_1 \geq 2$ and $k_2, \ldots, k_r \geq 1$ is called an *admissible index*.

(iii) Let be $\mathbf{k} = (k_1, \ldots, k_r)$ some index. Then we call

$$\begin{aligned} \mathrm{wt}(\mathbf{k}) &:= k_1 + \cdots + k_r \text{ the } \textit{weight} \text{ of } \mathbf{k} \text{ and} \\ \mathrm{depth}(\mathbf{k}) &:= r \text{ the } \textit{depth} \text{ of } \mathbf{k}. \end{aligned}$$

Definition 1.2 (*Multiple Zeta Value (MZV)*)**.** For an admissible index $\mathbf{k} = (k_1, \ldots, k_r)$ the multiple zeta value of $\mathbf{k}$ is defined as

$$\zeta(\mathbf{k}) := \zeta(k_1, \ldots, k_r) := \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1}} \cdots \frac{1}{m_r^{k_r}},$$

for $r > 0$ and $\zeta(\emptyset) := 1$.

To understand the algebraic structure of MZVs better, we introduce

$$\mathcal{Z} := \langle \zeta(\mathbf{k}) \mid \mathbf{k} \text{ admissible} \rangle_{\mathbb{Q}},$$

the $\mathbb{Q}$-vector space of MZVs and for every $k \geq 0$ we define

$$\mathcal{Z}_k := \langle \zeta(\mathbf{k}) \mid \mathbf{k} \text{ admissible of weight } k \rangle_{\mathbb{Q}}.$$

One obtains that $\mathcal{Z}$ is a $\mathbb{Q}$-algebra with the usual multiplication.

Now, $\mathcal{Z} = \sum_{k \geq 0} \mathcal{Z}_k$ is obvious, but there is a much more stronger conjecture about the algebraic structure of MZVs:

Conjecture 1.3. *$\mathcal{Z}$ is graded by weight, i.e.*

$$\mathcal{Z} = \bigoplus_{k\geq 0} \mathcal{Z}_k.$$

A consequence of this conjecture would be that every $\mathbb{Q}$-linear relation among MZVs splits up into corresponding relations in fixed weight. A particular implication of this conjecture would be that all $\zeta(k)$ are transcendental.

There are two ways to consider multiple zeta values: On the one hand, as iterated sums as in the definition (Def. 1.2), on the other hand as iterated integrals:

Theorem 1.4 ([BF])**.** *For every $r \geq 1$ and $k_1 \geq 2$, $k_2, \ldots, k_r \geq 1$, we have the so-called* Kontsevich iterated integral representation *of the MZV $\zeta(k_1, \ldots, k_r)$,*

$$\zeta(k_1, \ldots, k_r) = \int\limits_{1>t_1>\cdots>t_k>0} \omega_1(t_1) \ldots \omega_k(t_k),$$

where $k := k_1 + \cdots + k_r$ and

$$\omega_i(t) := \begin{cases} \frac{dt}{1-t}, & \text{if } i \in \{k_1, k_1+k_2, \ldots, k_1+\cdots+k_r\}, \\ \frac{dt}{t}, & \text{else.} \end{cases}$$

□

1.2. Quasi-shuffle algebras

We can view both representations of MZVs, as iterated sums resp. integrals, a bit more abstract on so-called *quasi-shuffle algebras*, introduced by Hoffman ([Ho2]), which are specific word algebras (further references are e.g. [Ba5], [BF], [Wal], [Zu1]). Especially the stuffle and shuffle product of MZVs can be described concretely using quasi-shuffle algebras; both are induced by particular quasi-shuffle products.

We can give expressions of the form '$\zeta(1, k_2, \ldots, k_r)$' some sense as an element in the polynomial ring $\mathcal{Z}[T]$ using the concept of quasi-shuffle algebras. Hence, we can extend the double shuffle relations. Conjecturally, these extended set of relations among MZVs gives all $\mathbb{Q}$-linear relations (see [IKZ]).

Definition 1.5 (*Quasi-shuffle product*, [Ho2])**.** Let be A a set and $\diamond$ an associative and commutative product on $\mathbb{Q}A$. Then the quasi-shuffle product $*_\diamond$ on $\mathbb{Q}\langle A\rangle$ deduced from $\diamond$ is defined via

$$\begin{aligned} &\mathbf{1} *_\diamond w := w *_\diamond \mathbf{1} := w, \\ &au *_\diamond bv := a(u *_\diamond bv) + b(au *_\diamond v) + (a \diamond b)(u *_\diamond v) \end{aligned}$$

for any $a, b \in \mathbb{Q}A$ and words $u, v, w \in \mathbb{Q}\langle A\rangle$. The elements in A are called *letters*.

Theorem 1.6 ([Ho2, Thm. 2.1])**.** *Let be A a set and $\diamond$ an associative and commutative product on $\mathbb{Q}A$. Then the induced quasi-shuffle product $*_\diamond$ makes $(\mathbb{Q}\langle A\rangle, *_\diamond)$ to a commutative graded $\mathbb{Q}$-algebra.* □

From now on, we work with the following free algebras:

Definition 1.7. Define the free non-commutative algebra of two letters,

$$\mathfrak{h} := \mathbb{Q}\langle x_0, x_1\rangle.$$

The unit of $\mathfrak{h}$ is the empty word, $\mathbf{1}$. Monomials in the two *letters* x_0, x_1 are called *words*. They form a $\mathbb{Q}$-basis of $\mathfrak{h}$.
Also, define the subalgebras

$$\mathfrak{h}^1 := \mathbb{Q}\mathbf{1} \oplus \mathfrak{h}x_1,$$

consisting of all words ending in x_1 and

$$\mathfrak{h}^0 := \mathbb{Q}\mathbf{1} \oplus x_0\mathfrak{h}x_1,$$

consisting of all words starting in x_0 and ending in x_1.

Remark 1.8.

(i) We have $\mathfrak{h}^0 \subset \mathfrak{h}^1 \subset \mathfrak{h}$.

(ii) $\mathfrak{h}^1$ is generated by words in $z_k := x_0^{k-1}x_1$, $k \geq 1$, i.e. $\mathfrak{h}^1 = \mathbb{Q}\langle z_1, z_2, \dots\rangle$.

(iii) $\mathfrak{h}^0$ is generated by the words $z_{k_1} \dots z_{k_r}$ with $k_1 \geq 2,\ k_i \geq 1,\ r \geq 0$.

We will give $\mathfrak{h}$ the structure of a $\mathbb{Q}$-algebra in two ways. The products on $\mathfrak{h}$ we consider are special quasi-shuffle products:

Definition 1.9 (*Stuffle product*, [Ba6, Def. 2.11])**.** For $A = \{z_k = x_0^{k-1}x_1 \mid , k \geq 1\}$, i.e. $\mathbb{Q}\langle A\rangle = \mathfrak{h}^1$, and the $z_m \diamond z_n := z_{m+n}$, the resulting quasi-shuffle product $*$ on $\mathfrak{h}^1$ is called *stuffle product.*

Definition 1.10 (*Shuffle product*, [Ba6, Def. 2.8])**.** For $A = \{x_0, x_1\}$, i.e. $\mathbb{Q}\langle A\rangle = \mathfrak{h}$, and $z_m \diamond z_n := 0$, the induced quasi-shuffle product $ш$ on $\mathfrak{h}$ is called the *shuffle product.*

Remark at this point for further applications that $\mathfrak{h}^0$ and $\mathfrak{h}^1$ both are closed under $*$ and $ш$.

The connection to MZVs is now done via the evaluation map that we also denote ζ,

$$\zeta : \mathfrak{h}^0 \longrightarrow \mathbb{R},$$
$$z_{k_1} \dots z_{k_r} = x_0^{k_1-1}x_1 \dots x_0^{k_r-1}x_1 \longmapsto \zeta(k_1, \dots, k_r)$$

with $\zeta(\mathbf{1}) := 1$, and $\mathbb{Q}$-linearity.

The importance of the stuffle and shuffle product yields from the fact that the evaluation map ζ is an algebra homomorphism on $\mathfrak{h}^0$ equipped with $*$ respective $ш$:

Theorem 1.11. *The evaluation maps* $\zeta : (\mathfrak{h}^0, *) \to (\mathbb{R}, \cdot)$ *and* $\zeta : (\mathfrak{h}^0, ш) \to (\mathbb{R}, \cdot)$ *both are algebra homomorphisms, i.e. for all* $w_1, w_2 \in \mathfrak{h}^0$ *we have*

$$\zeta(w_1 * w_2) = \zeta(w_1)\zeta(w_2) = \zeta(w_1 ш w_2).$$

□

Remark 1.12. Comparing both products, we obtain the so-called *double shuffle relations*:

$$\zeta(u * v - u ш w) = 0 \text{ for all } u, v \in \mathfrak{h}^0.$$

We can extend this class of relations: There are isomorphisms ([Ho1, Thm. 3.1])

$$(\mathfrak{h}^1, *) \simeq (\mathfrak{h}^0[T], *), \quad (\mathfrak{h}^1, ш) \simeq (\mathfrak{h}^0[T], ш), \quad \text{where } z_1 \mapsto T.$$

Hence, we can extend the evaluation map ζ to maps from $(\mathfrak{h}^1, *)$ resp. $(\mathfrak{h}^1, ш)$ to the polynomial ring $\mathbb{R}[T]$ such that the extended ζ in both cases maps $z_1 \mapsto T$ and such that the extended ζ is an algebra homomorphism again (cf. [IKZ, Prop. 1]). In this way, we get *extended double shuffle relations* - they have the same shape as the relations in Remark 1.12, but now with $u \in \mathfrak{h}^1$, $v \in \mathfrak{h}^0$.

On the one hand, the procedure of extending ζ is nice since we can give expressions like '$\zeta(1, k_2, \ldots, k_r)$' some sense, on the other, extended double shuffle relations play an important role in the theory of MZVs:

Conjecture 1.13 ([IKZ, Conj. 1])**.** *The extended double shuffle relations,*

$$\zeta(u * v - u ш v) = 0$$

for all $u \in \mathfrak{h}^1$, $v \in \mathfrak{h}^0$, *give all* $\mathbb{Q}$*-linear relations among MZVs.*

A more detailed description of this conjecture and further results the authors give in [IKZ].

2. q-Analogues of Multiple zeta values

We want to focus on some distinguished models of (modified) q-analogues of MZVs presented in the following sections. For a more detailed view of them, including for every model the product structure, we refer to Appendix A. Further references introducing general qMZVs are [BK2], [Ba6] and [Zh3]. Zhao gives in [Zh2] an overview of different models of qMZVs and their history.

In general, a q-analogue of an object is an object in an extra variable q (often a series in q) that returns the original object in the limit $q \to 1$.

For example, a q-analogue of a natural number n is

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}.$$

In this thesis and most common literature about q-analogues (of MZVs), q is a complex variable with $|q| < 1$. By $q \to 1$, we always mean the limit $q \to 1$ taken on the real axis from the left.

Often, it is for MZVs more convenient to consider *modified* q-analogues since the spaces become $\mathbb{Q}$-algebras. These are q-series as defined below that return a multiple zeta value if we multiply the q-series first with a power of $(1-q)$ (usually, $(1-q)^{\mathrm{wt}(\mathbf{k})}$) and then take the limit $q \to 1$. We will not distinguish between the two terms since we focus on *modified* q-analogues of MZVs, which is the reason why we leave out the word *modified.*

We work with the following definition of qMZVs:

Definition 2.1 (*qMZV*, [BK2])**.** (i) Define for $r \geq 0$, $k_1, \ldots, k_r \geq 0$ and polynomials $Q_1 \in X\mathbb{Q}[X]$, $Q_2, \ldots, Q_r \in \mathbb{Q}[X]$ with $\deg Q_j \leq k_j$ for all j

$$\zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r) := \sum_{m_1 > \cdots > m_r > 0} \frac{Q_1(q^{m_1})}{(1-q^{m_1})^{k_1}} \cdots \frac{Q_r(q^{m_r})}{(1-q^{m_r})^{k_r}} \in \mathbb{Q}[\![q]\!],$$

with $\zeta_q(\emptyset, \emptyset) := 1$, where q is a formal variable.

(ii) Furthermore, we define

$$\mathcal{Z}_q := \langle \zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r) \,|\, r \geq 0,\, k_1, \ldots, k_r \geq 0,\, Q_1 \in X\mathbb{Q}[X],\, \deg Q_j \leq k_j \rangle_{\mathbb{Q}}.$$

Remark 2.2. (i) For every $\zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r) \in \mathcal{Z}_q$ with $k_1 \geq 2$, $k_2, \ldots, k_r \geq 1$ we have

$$\lim_{q \to 1} (1-q)^{k_1 + \cdots + k_r} \zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r) = Q_1(1) \cdots Q_r(1) \zeta(k_1, \ldots, k_r).$$

That is why $\zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r)$ indeed can be viewed as a modified q-analogue of $\zeta(k_1, \ldots, k_r)$.

(ii) We need $Q_1 \in X\mathbb{Q}[X]$ for well-definedness.

(iii) Naively, we could think of $k_1 + \cdots + k_r$ as the 'weight' of the qMZV $\zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r)$ in accordance with the definition of weight for MZVs. But this is not well-defined since for example

$$\zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r) = \zeta_q(k_1 + 1, \ldots, k_r; (1 - X)Q_1, Q_2 \ldots, Q_r).$$

Hence, we need another notion of weight. Such one we will consider when talking about bi-brackets (see Def. 2.14). Another possible choice for a 'weight of qMZVs' is the SZ-weight we will mention in the next section (Def. 2.5(ii)).

(iv) Obviously, $\mathcal{Z}_q$ is a $\mathbb{Q}$-algebra, why we call $\mathcal{Z}_q$ also *the algebra of qMZVs.* We can define on $\mathcal{Z}_q$ a quasi-shuffle product (see Prop. A.34) such that $\mathcal{Z}_q$ becomes a particular quasi-shuffle algebra.

For the next definition we use notation from [BK2], where the authors define often needed subspaces of $\mathcal{Z}_q$:

Definition 2.3 ([BK2])**.** Define for $d \geq 0$

$$\mathcal{Z}_{q,d} := \langle \zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r) \in \mathcal{Z}_q |\, r \geq 0,\, k_1, \ldots, k_r \geq 1, \deg(Q_j) \leq k_j - d \rangle_{\mathbb{Q}}$$

and

$$\mathcal{Z}^\circ_{q,d} := \langle \zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r) \in \mathcal{Z}_{q,d} |\, r \geq 0,\, k_1, \ldots, k_r \geq 1, Q_j \in X\mathbb{Q}[X] \rangle_{\mathbb{Q}}$$

with the abbreviation $\mathcal{Z}^\circ_q := \mathcal{Z}^\circ_{q,0}$.

Remark 2.4.

(i) The spaces $\mathcal{Z}_{q,d}$ and $\mathcal{Z}^\circ_{q,d}$ for $d \geq 0$ both are subalgebras of $\mathcal{Z}_q$.

(ii) Under $q\frac{d}{dq}$ the spaces $\mathcal{Z}_q$ and $\mathcal{Z}^\circ_q$ are closed, while $\mathcal{Z}_{q,1}$ and $\mathcal{Z}^\circ_{q,1}$ are only conjecturally closed ([Ba4, Conj. 4.3], [BK1, Prop. 3.14], [Ba7], [Oko, Conj. 1]).

In the following section, we consider particular spanning sets for $\mathcal{Z}_q$, namely the Schlesinger-Zudilin qMZVs, the Takeyama-Bradley-Zhao qMZVs and the bi-brackets and for $\mathcal{Z}^\circ_q$ the brackets and for $\mathcal{Z}_{q,1}$ the Bradley-Zhao qMZVs. We consider for each model its main properties we will need later.

2.1. Schlesinger-Zudilin model

The reason for having a double name for this model is that Schlesinger (in 2001, [Sch]) and Zudilin (in 2003, [Zu1]) introduced it independently. What we present here as Schlesinger-Zudilin model is, in truth, an extended version of the original one due to Ebrahimi-Fard, Manchon and Singer (cf. [EMS]).

An advantage of this model is that it satisfies a canonical analogue of the stuffle product of MZVs and some kind of duality very similar to the duality of MZVs. Comparing both and taking the limit $q \to 1$ will give the shuffle product of MZVs. Ebrahimi-Fard, Manchon and Singer ([EMS, Thm. 5.14]) considered this result first, and we give a proof in detail in Section 3.6.

Also, a difference to MZVs and the BZ-model is that SZ-qMZVs are not only defined for admissible indices. They are for all indices with first entry ≥ 1 and all others ≥ 0 defined, which we call *SZ-admissible*.

Before we define SZ-qMZVs, we briefly give the necessary setup:

Definition 2.5. (i) An index $\mathbf{k} = (k_1, \ldots, k_r)$ with either $r = 0$ or $k_1 \geq 1$ and $k_2, \ldots, k_r \geq 0$ is called an *SZ-admissible index*.

(ii) Let be $\mathbf{k} = (k_1, \ldots, k_r)$ some index. Then we call

$$\mathrm{wt}_{SZ}(\mathbf{k}) := k_1 + \cdots + k_r + \delta_{k_1=0} + \cdots + \delta_{k_r=0} \text{ the } \textit{SZ-weight} \text{ of } \mathbf{k}.$$

(iii) Define the non-commutative free algebra in two variables p and y over $\mathbb{Q}$ by

$$\mathfrak{K} := \mathbb{Q}\langle p, y\rangle.$$

(iv) Define the following subalgebras of $\mathfrak{K}$:

$$\begin{aligned}\mathfrak{K}^1 &:= p\mathfrak{K}y \oplus \mathbb{Q}\mathbf{1},\\ \mathfrak{K}^2 &:= \mathbb{Q}\langle p, py\rangle py \oplus \mathbb{Q}\mathbf{1},\\ \mathfrak{K}^3 &:= p\mathbb{Q}\langle p, py\rangle py \oplus \mathbb{Q}\mathbf{1}.\end{aligned}$$

Definition 2.6. For any SZ-admissible $\mathbf{k} = (k_1, \ldots, k_r)$, i.e. $k_1 \geq 1$, $k_2, \ldots, k_r \geq 0$, its Schlesinger-Zudilin qMZV is defined by

$$\begin{aligned}\zeta_q^{\mathrm{SZ}}(\mathbf{k}) := \zeta_q^{\mathrm{SZ}}(k_1, \ldots, k_r) &:= \zeta_q(k_1, \ldots, k_r; X^{k_1}, \ldots, X^{k_r})\\ &= \sum_{m_1 > \cdots > m_r > 0} \frac{q^{m_1 k_1}}{(1-q^{m_1})^{k_1}} \cdots \frac{q^{m_r k_r}}{(1-q^{m_r})^{k_r}},\end{aligned}$$

where we set $\zeta_q^{\mathrm{SZ}}(\emptyset) := 1$. When identifying an SZ-admissible index $\mathbf{k} = (k_1, \ldots, k_r)$ with the word $p^{k_1}y \ldots p^{k_r}y \in \mathfrak{K}^1$, we can define ζ_q^{SZ} also as the map, uniquely determined through $\mathbf{1} \mapsto 1$, $\mathbb{Q}$-linearity and

$$\begin{aligned}\zeta_q^{\mathrm{SZ}} : \mathfrak{K}^1 &\longrightarrow \mathcal{Z}_q,\\ p^{k_1}y \ldots p^{k_r}y &\longmapsto \zeta_q^{\mathrm{SZ}}(k_1, \ldots, k_r).\end{aligned}$$

Remark 2.7. (i) The advantage of having the free algebra in two generators 'twice' with $\mathfrak{K}$ and $\mathfrak{h}$ will be obtained in Section 3.6 where we will use $\mathfrak{h}$ for MZVs and $\mathfrak{K}$ for Schlesinger and Zudilins model of q-analogues of MZVs. Hence it will be always clear in which 'world' we are by just noting which variables are used.

(ii) Roughly spoken, $\mathfrak{K}^1$ embodies SZ-admissible indices, $\mathfrak{K}^2$ and $\mathfrak{h}^1$ both embody indices $(k_1, \ldots, k_r)$ with $k_1, \ldots, k_r \geq 1$ and $\mathfrak{K}^3$ and $\mathfrak{h}^0$ embody admissible indices.

(iii) It is to note that if one of the indices is 0 in an SZ-qMZV, i.e. $k_j = 0$ for some j, then the summand is independent of m_j. Therefore, it is often helpful to distinguish between zero and non-zero indices.

(iv) An index $\mathbf{k}$ is SZ-admissible iff $\mathbf{k} + \mathbf{1}$ (every argument of $\mathbf{k}$ is increased by 1) is admissible in the sense of Definition 1.1.

Proposition 2.8 (Prop. A.39)**.** *SZ-qMZVs span* $\mathcal{Z}_q$*, i.e.*

$$\mathcal{Z}_q = \langle \zeta_q^{SZ}(k_1, \ldots, k_r) \,|\, r \geq 0,\ k_1 \geq 1,\ k_i \geq 0 \rangle_{\mathbb{Q}}.$$

□

We now consider the generating series of Schlesinger-Zudilin qMZVs. One reason why we do this is that a generating series is a reasonably compact written term that includes all information of all SZ-qMZVs. We will use this for writing duality in compact form and for connections to different models. As in Remark 2.7(i) mentioned, it is often helpful to distinguish between zero and non-zero indices, which is the justification why we take two types of variables.

Theorem 2.9 (Thm. A.45)**.** *Define for* $r \geq 1$ *the generating series*

$$\mathfrak{s}\binom{X_1, \ldots, X_r}{Y_1, \ldots, Y_r} := \sum_{\substack{k_1, \ldots, k_r \geq 1 \\ d_1, \ldots, d_r \geq 0}} \zeta_q^{SZ}(k_1, \{0\}^{d_1}, \ldots, k_r, \{0\}^{d_r}) \cdot X_1^{k_1 - 1} \ldots X_r^{k_r - 1} Y_1^{d_1} \ldots Y_r^{d_r}.$$

Then, for every $r \geq 1$ *it is:*

$$\mathfrak{s}\binom{X_1, \ldots, X_r}{Y_1, \ldots, Y_r} = \sum_{\substack{m_1 > \cdots > m_r > 0 \\ n_1, \ldots, n_r \geq 1}} \prod_{j=1}^{r} (1 + X_j)^{n_j - 1} (1 + Y_j)^{m_j - m_{j+1} - 1} q^{m_j n_j},$$

where we set $m_{r+1} := 0$. □

2.2. Bradley-Zhao model

In depth one, this model of qMZVs was first considered by Kaneko, Kurokawa and Wakayama in 2002 ([KKW]). The general model then was independently introduced by Zhao in 2003 ([Zh1]) and Bradley in 2004 ([Bra]), clarifying its name. We will later see that BZ-qMZVs satisfy the same duality as MZVs.

Definition 2.10 (*Bradley-Zhao-qMZVs*)**.** For every admissible index $\mathbf{k} = (k_1, \ldots, k_r)$, i.e. $k_1 \geq 2$, $k_2, \ldots, k_r \geq 1$, we define the Bradley-Zhao qMZV

$$\begin{aligned}\zeta_q^{\mathrm{BZ}}(\mathbf{k}) := \zeta_q^{\mathrm{BZ}}(k_1, \ldots, k_r) &:= \zeta_q(k_1, \ldots, k_r; X^{k_1-1}, \ldots, X^{k_r-1}) \\ &= \sum_{m_1 > \cdots > m_r > 0} \frac{q^{m_1(k_1-1)}}{(1-q^{m_1})^{k_1}} \cdots \frac{q^{m_r(k_r-1)}}{(1-q^{m_r})^{k_r}}\end{aligned}$$

and set $\zeta_q^{\mathrm{BZ}}(\emptyset) := 1$. We define ζ_q^{BZ} also as map, defined via $\mathbf{1} \mapsto 1$, $\mathbb{Q}$-linearity and

$$\begin{aligned}\zeta_q^{\mathrm{BZ}} : \mathfrak{h}^0 &\longrightarrow \mathcal{Z}_q, \\ z_{k_1} \ldots z_{k_r} &\longmapsto \zeta_q^{\mathrm{BZ}}(k_1, \ldots, k_r).\end{aligned}$$

Proposition 2.11 (Prop. A.52)**.** *BZ-qMZVs span $\mathcal{Z}_{q,1}$, i.e.*

$$\mathcal{Z}_{q,1} = \langle \zeta_q^{BZ}(k_1, \ldots, k_r) \,|\, r \geq 0,\ k_1 \geq 2,\ k_i \geq 1 \rangle_{\mathbb{Q}}.$$

□

Remark 2.12. BZ-qMZVs satisfy a quasi-shuffle product, in analogy to the stuffle product of MZVs. For details, we refer to the appendix A.

It is hard to find a generating series that gives BZ-duality easily. Using the generating series of SZ-qMZVs and the translation of BZ-qMZVs into the SZ-model (cf. in the appendix), we can rewrite BZ-duality in terms of $\mathfrak{s}$, the generating series of SZ-qMZVs:

Theorem 2.13 (Thm. A.55)**.** *Define for $r \geq 1$ the generating series of BZ-qMZVs*

$$\mathfrak{b}\binom{X_1, \ldots, X_r}{Y_1, \ldots, Y_r} := \sum_{\substack{k_1, \ldots, k_r \geq 1 \\ d_1, \ldots, d_r \geq 1}} \zeta_q^{BZ}(k_1+1, \{1\}^{d_1-1}, \ldots, k_r+1, \{1\}^{d_r-1}) X^{k_1} Y_1^{d_1} \cdots X_r^{k_r} Y_r^{d_r}.$$

Then, we have for every $r \geq 1$

$$\begin{aligned}&\mathfrak{b}\binom{X_1, \ldots, X_r}{Y_1, \ldots, Y_r} \\ &= \sum_{\substack{l_1, \ldots, l_r \geq 1 \\ \delta_1, \ldots, \delta_r \in \{0,1\}}} (-1)^{r-(\delta_1+\cdots+\delta_r)} \mathfrak{s}\binom{\delta_1 X_1, 0, \ldots, 0, \ldots, \delta_r X_r, 0, \ldots, 0}{Y_1, \underbrace{Y_1, \ldots, Y_1}_{l_1-1}, \ldots, Y_r, \underbrace{Y_r, \ldots, Y_r}_{l_r-1}} \prod_{j=1}^{r} (1+\delta_j X_j) Y_j^{l_j}.\end{aligned}$$

□

2.3. Bi-brackets

Another interesting model of q-analogues are so-called brackets (introduced in Bachmann's master thesis [Ba1], further investigated in [BK1]) and their generalization, bi-brackets, introduced by Bachmann in his PhD thesis ([Ba2]). One of the goals of this section is to find a relationship between bi-brackets and the Schlesinger-Zudilin qMZVs. We will translate both models into the other each. First, we do this on the level of power series, then explicitly.

The motivation for introducing bi-brackets comes from their connection to quasi-modular forms. These is obtained via examining the Fourier expansion of Eisenstein series and their generalization, Multiple Eisenstein series and their derivatives (studied in [Ba4]), which motivated their original definition:

Definition 2.14. (i) Define for $r \geq 0$, $k_1, \ldots, k_r \geq 1$, $d_1, \ldots, d_r \geq 0$ the bi-bracket

$$\mathrm{g}\binom{k_1, \ldots, k_r}{d_1, \ldots, d_r} := \sum_{\substack{m_1 > \cdots > m_r > 0 \\ n_1, \ldots, n_r > 0}} \frac{m_1^{d_1}}{d_1!} \cdots \frac{m_r^{d_r}}{d_r!} \frac{n_1^{k_1-1} \ldots n_r^{k_r-1}}{(k_1-1)! \ldots (k_r-1)!} q^{m_1 n_1 + \cdots + m_r n_r},$$

where $\mathrm{g}\binom{\emptyset}{\emptyset} := 1$ as usually.
We call r the *depth* and $k_1 + \cdots + k_r + d_1 + \cdots + d_r$ the *weight*.

(ii) Define for $r \geq 0$, $k_1, \ldots, k_r \geq 1$ the bracket

$$\mathrm{g}(k_1, \ldots, k_r) := \sum_{\substack{m_1 > \cdots > m_r > 0 \\ n_1, \ldots, n_r > 0}} \frac{n_1^{k_1-1}}{(k_1-1)!} \cdots \frac{n_r^{k_r-1}}{(k_r-1)!} q^{m_1 n_1 + \cdots + m_r n_r}$$

with $\mathrm{g}(\emptyset) := 1$.

Remark 2.15. (i) The name *(bi-) bracket* comes from the original notation where (bi-) brackets were denoted by $[\ldots]$ instead of $\mathrm{g}(\ldots)$.

(ii) Every bracket $\mathrm{g}(k_1, \ldots, k_r)$ is a bi-bracket since $\mathrm{g}(k_1, \ldots, k_r) = \mathrm{g}\binom{k_1, \ldots, k_r}{0, \ldots, 0}$.

(iii) Let be P_k the k-th *Eulerian polynomial*, defined via

$$\frac{P_k(X)}{(1-X)^k} := \sum_{n>0} \frac{n^{k-1}}{(k-1)!} X^n.$$

Then (cf. [BK2, (2.7)]), for every bracket we have

$$\begin{aligned} \mathrm{g}(k_1, \ldots, k_r) &= \zeta_q(k_1, \ldots, k_r; P_{k_1}, \ldots, P_{k_r}) \\ &= \sum_{m_1 > \cdots > m_r > 0} \frac{P_{k_1}(q^{m_1})}{(1-q^{m_1})^{k_1}} \cdots \frac{P_{k_r}(q^{m_r})}{(1-q^{m_r})^{k_r}} \end{aligned}$$

and for every bi-bracket

$$\mathrm{g}\binom{k_1,\ldots,k_r}{d_1,\ldots,d_r} = \sum_{m_1>\cdots>m_r>0} \frac{m_1^{d_1}}{d_1!}\cdots\frac{m_r^{d_r}}{d_r!}\frac{P_{k_1}(q^{m_1})}{(1-q^{m_1})^{k_1}}\cdots\frac{P_{k_r}(q^{m_r})}{(1-q^{m_r})^{k_r}}.$$

Theorem 2.16 ([BK2, Thm. 2.3])**.** *Bi-brackets span $\mathcal{Z}_q$, i.e.*

$$\mathcal{Z}_q = \left\langle \mathrm{g}\binom{k_1,\ldots,k_r}{d_1,\ldots,d_r} \,\middle|\, r\geq 0,\ k_i\geq 1,\ d_i\geq 0 \right\rangle_{\mathbb{Q}}.$$

□

Bi-brackets generalize Eisenstein series since for even k, $\mathrm{g}\binom{k}{0}$ is the usual Eisenstein series of weight k, G_k, minus the constant term, i.e.

$$G_k = -\frac{B_k}{2k!} + \mathrm{g}\binom{k}{0}.$$

Furthermore, for every $d>0$, it is

$$\left(q\frac{d}{dq}\right)^d G_k = \frac{(k+d-1)!d!}{(k-1)!}\,\mathrm{g}\binom{k+d}{d}.$$

With the observation done before, one can obtain that the space of quasi-modular forms, which is $\mathbb{Q}[G_2,G_4,G_6]$, is a proper subspace of $\mathcal{Z}_{q,1}$. In this way, we get a connection to modular forms, which play an essential role in the theory of MZVs as considered e.g. in [GKZ].

Since SZ-qMZVs, as well as bi-brackets, span $\mathcal{Z}_q$ (Prop. 2.8, Thm. 2.16), we can write bi-brackets as SZ-qMZVs and vice versa.

A very elegant, but in return not explicit, way to translate bi-brackets into Schlesinger-Zudilin qMZVs is to consider their generating series:

Theorem 2.17 ([Ba4, Theorem 2.3], Thm. A.65)**.** *For every $r\geq 1$, let be*

$$\mathfrak{g}\binom{X_1,\ldots,X_r}{Y_1,\ldots,Y_r} := \sum_{\substack{k_1,\ldots,k_r>0\\ d_1,\ldots,d_r\geq 0}} \mathrm{g}\binom{k_1,\ldots,k_r}{d_1,\ldots,d_r} X_1^{k_1-1}\cdots X_r^{k_r-1}Y_1^{d_1}\cdots Y_r^{d_r}.$$

Then we have

$$\mathfrak{g}\binom{X_1,\ldots,X_r}{Y_1,\ldots,Y_r} = \sum_{\substack{m_1>\cdots>m_r>0\\ n_1,\ldots,n_r\geq 1}} \prod_{j=1}^r e^{m_jY_j}e^{n_jX_j}q^{m_jn_j}.$$

Theorem 2.18 (Translation bi-brackets-SZ-model, Thm. A.68). *(i) For every $r \geq 1$ we have*

$$\prod_{j=1}^{r} e^{X_j} e^{Y_1+\cdots+Y_j} \cdot \mathfrak{s}\binom{e^{X_1}-1,\ldots,e^{X_r}-1}{e^{Y_1}-1,\ldots,e^{Y_1+\cdots+Y_r}-1} = \mathfrak{g}\binom{X_1,\ldots,X_r}{Y_1,\ldots,Y_r}.$$

(ii) For every $r \geq 1$ we have

$$\mathfrak{s}\binom{X_1,\ldots,X_r}{Y_1,\ldots,Y_r} = \left(\prod_{j=1}^{r}(1+X_j)(1+Y_1+\cdots+Y_j)\right)^{-1} \mathfrak{g}\binom{\ln(X_1+1),\ldots,\ln(X_r+1)}{\ln(Y_1+1),\ldots,\ln(Y_1+\cdots+Y_r+1)}.$$

□

We obtain an explicit, but less elegant, translation of bi-brackets into SZ-qMZVs when using identities among special rational functions and elementary calculations:

Theorem 2.19 (Thm. A.70). *For every $r \in \mathbb{N}$, $k_1,\ldots,k_r \in \mathbb{N}$, $d_1,\ldots,d_r \in \mathbb{N}_0$, we have:*

$$\begin{aligned}
\mathrm{g}\binom{k_1,\ldots,k_r}{d_1,\ldots,d_r} &= \sum_{\substack{1\leq n_j\leq p_j\leq k_j\\ 0\leq f_j\leq d_j\\ 1\leq j\leq r}} c_{\mathbf{k}}^{\mathbf{d}}(\mathbf{n}) \cdot \left[\prod_{j=1}^{r}\binom{d_j}{f_j}\binom{k_j-n_j}{p_j-n_j}\right] \\
&\times \prod_{j=1}^{r} \sum_{g_j=0}^{F_{\mathbf{h},\mathbf{g}}^{\mathbf{f}}(j)} \binom{F_{\mathbf{h},\mathbf{g}}^{\mathbf{f}}(j)}{g_j} \sum_{l_j=0}^{g_j} \left(\delta_{g_j=0} + \sum_{\substack{s_1+\cdots+s_{l_j}=g_j\\ s_i\geq 1}} \binom{g_j}{s_1,\ldots,s_{l_j}}\right) \sum_{h_j=0}^{H_{\mathbf{h},\mathbf{g}}^{\mathbf{f}}(j)} \binom{H_{\mathbf{h},\mathbf{g}}^{\mathbf{f}}(j)}{h_j} \\
&\times \zeta_q^{SZ}(p_1,\{0\}^{l_1},\ldots,p_r,\{0\}^{l_r}) \\
&= \sum_{\substack{1\leq n_j\leq p_j\leq k_j\\ 0\leq f_j\leq d_j\\ 1\leq j\leq r}} c_{\mathbf{k}}^{\mathbf{d}}(\mathbf{n}) \left(\prod_{j=1}^{r}\binom{d_j}{f_j}\binom{k_j-n_j}{p_j-n_j}\right) \\
&\quad \times \sum_{g_1=0}^{f_1}\binom{f_1}{g_1}\sum_{l_1=0}^{g_1}\left[\delta_{g_1=0} + \sum_{\substack{s_1+\cdots+s_{l_1}=g_1\\ s_i\geq 1}}\binom{g_1}{s_1,\ldots,s_{l_1}}\right]\sum_{h_1=0}^{f_1-g_1}\binom{f_1-g_1}{h_1} \\
&\quad \times \ldots \\
&\quad \times \cdots \times \sum_{h_{r-1}=0}^{\substack{f_1+\cdots+f_{r-1}\\ -(g_1+\cdots+g_{r-1})\\ -(h_1+\cdots+h_{r-2})}} \binom{f_1+\cdots+f_{r-1}-(g_1+\ldots g_{r-1}+h_1+\ldots h_{r-2})}{h_{r-1}}
\end{aligned}$$

$$\times \sum_{l_r=0}^{\substack{f_1+\dots+f_r\\ -(g_1+\dots+g_{r-1})\\ -(h_1+\dots+h_{r-1})}} \left[\delta_{\substack{f_1+\dots+f_r\\ -(g_1+\dots+g_{r-1})\\ -(h_1+\dots+h_{r-1})=0}} + \sum_{\substack{s_1+\dots+s_{l_r}\\ =f_1+\dots+f_r\\ -(g_1+\dots+g_{r-1})\\ -(h_1+\dots+h_{r-1})\\ s_i\geq 1}} \binom{f_1+\dots+f_r-(g_1+\dots+g_{r-1}\\ +h_1+\dots+h_{r-1})}{s_1,\dots,s_{l_r}} \right]$$

$$\times \zeta_q^{SZ}(p_1, \{0\}^{l_1}, \dots, p_r, \{0\}^{l_r})$$

with $c_{\mathbf{k}}^{\mathbf{d}}(\mathbf{n}) := \prod_{l=1}^{r} \frac{1}{d_l!(k_l-1)!} \left(\sum_{i=0}^{n_l-1} (-1)^i \binom{k_l}{i} (n_l-i)^{k_l-1} \right) \in \mathbb{Q}$ *and*

$$F_{\mathbf{h},\mathbf{g}}^{\mathbf{f}}(j) := f_j + \sum_{i=1}^{j-1}(f_i - g_i - h_i), \quad H_{\mathbf{h},\mathbf{g}}^{\mathbf{f}}(j) := F_{\mathbf{h},\mathbf{g}}^{\mathbf{f}}(j) - g_j = f_j - g_j + \sum_{i=1}^{j-1}(f_i - g_i - h_i)$$

for all $1 \leq j \leq r$.

2.4. Takeyama-Bradley-Zhao model

The span of the BZ-qMZVs is a proper subspace of $\mathcal{Z}_q$ (see Prop. A.52 for a proof). But for some situations, like comparing linear relations in different models, it is comfortable to extend the BZ-model of qMZVs, especially when the elements of the model should span $\mathcal{Z}_q$.

This extension for the BZ-model is follows Takeyama's work [Ta1], who introduces an extra index, called $\overline{1}$:

Definition 2.20. (*Takeyama-Bradley-Zhao-qMZVs*)
Denote $\overline{\mathbb{N}} := \{\overline{1}\} \cup \mathbb{N} = \{\overline{1}, 1, 2, 3, \dots\}$ and define for every $r \geq 1$, $k_1, \dots, k_r \in \overline{\mathbb{N}}$, $k_1 \neq 1$,

$$\zeta_q^{\mathrm{TBZ}}(k_1, \dots, k_r) := \sum_{m_1 > \dots > m_r > 0} f(k_1, m_1) \dots f(k_r, m_r),$$

$$f(\overline{1}, m) := \frac{q^m}{1-q^m}, \; f(k,m) := \frac{q^{(k-1)m}}{(1-q^m)^k}, \text{ for } k \geq 1.$$

Proposition 2.21. *TBZ-qMZVs span* $\mathcal{Z}_q$*, i.e.*

$$\mathcal{Z}_q = \langle \zeta_q^{TBZ}(k_1, \dots, k_r) \,|\, r \geq 0,\, k_i \in \overline{\mathbb{N}},\, k_1 \geq 1 \rangle_{\mathbb{Q}}.$$

□

Remark 2.22. Also, this extended version of the BZ-model satisfies a quasi-shuffle product. It is compatible with those of the BZ-model. For the definition and further details, we refer to Appendix A.

We give now translations between the SZ- and TBZ-model (in particular, this proves that the TBZ-model spans $\mathcal{Z}_q$):

Proposition 2.23 (Translation TBZ-model into SZ-model, Prop. A.80)**.**
For every $d_1, \ldots, d_r \in \mathbb{N}_0$, $k_1, \ldots, k_{r-1} \in \mathbb{N}$ (with $k_1 \geq 2$ if $d_1 = 0$) we have

$$\begin{aligned}&\zeta_q^{TBZ}(\{\overline{1}\}^{d_1}, k_1, \ldots, k_{r-1}, \{\overline{1}\}^{d_r})\\ =&\sum_{\substack{\delta_j \in \{0,1\}\\ 1 \leq j \leq r-1}} \zeta_q^{SZ}(\{1\}^{d_1}, k_1 - \delta_1, \ldots, \{1\}^{d_{r-1}}, k_{r-1} - \delta_{r-1}, \{1\}^{d_r}).\end{aligned}$$

□

Example 2.24 (Ex. A.81)**.** Consider $\zeta_q^{\mathrm{TBZ}}(\overline{1}, 2, 1)$. We have

$$\begin{aligned}\zeta_q^{\mathrm{TBZ}}(\overline{1}, 2, 1) &= \sum_{m_1 > m_2 > m_3 > 0} \frac{q^{m_1}}{1 - q^{m_1}} \frac{q^{m_2}}{(1 - q^{m_2})^2} \frac{1}{1 - q^{m_3}}\\ &= \sum_{m_1 > m_2 > m_3 > 0} \frac{q^{m_1}}{1 - q^{m_1}} \left(\frac{q^{m_2}}{1 - q^{m_2}} + \frac{q^{2m_2}}{(1 - q^{m_2})^2}\right)\left(1 + \frac{q^{m_3}}{1 - q^{m_3}}\right)\\ &= \zeta_q^{\mathrm{SZ}}(1, 2, 1) + \zeta_q^{\mathrm{SZ}}(1, 1, 1) + \zeta_q^{\mathrm{SZ}}(1, 2, 0) + \zeta_q^{\mathrm{SZ}}(1, 1, 0).\end{aligned}$$

Proposition 2.25 (Translation SZ-model into TBZ-model, Prop. A.82)**.**
For every $k_1, \ldots, k_r \in \mathbb{N}$, $d_1, \ldots, d_r \in \mathbb{N}_0$ we have

$$\begin{aligned}&\zeta_q^{SZ}(k_1, \{0\}^{d_1}, \ldots, k_r, \{0\}^{d_r})\\ =&\sum_{\substack{1 \leq j_i \leq k_i\\ \varepsilon_i \in \{\overline{1}, 1\}^{d_i}\\ 1 \leq i \leq r}} (-1)^{\sum_{i=1}^{r} k_i - j_i + |\varepsilon_i|} \zeta_q^{TBZ}(j_1 \delta_{j_1 \neq 1} + \overline{1}\delta_{j_1 = 1}, \varepsilon_1, \ldots, j_r \delta_{j_r \neq 1} + \overline{1}\delta_{j_r = 1}, \varepsilon_r),\end{aligned}$$

where $|\varepsilon|$ counts the $\overline{1}$'s in ε. □

Example 2.26. We have

$$\begin{aligned}&\zeta_q^{\mathrm{SZ}}(3, 0, 1) = \sum_{m_1 > m_2 > m_3 > 0} \frac{q^{3m_1}}{(1 - q^{m_1})^3} \frac{q^{m_3}}{1 - q^{m_3}}\\ =&\sum_{m_1 > m_2 > m_3 > 0} \left(\frac{q^{m_1}}{1 - q^{m_1}} - \frac{q^{m_1}}{(1 - q^{m_1})^2} + \frac{q^{2m_1}}{(1 - q^{m_1})^3}\right)\left(\frac{1}{1 - q^{m_2}} - \frac{q^{m_1}}{1 - q^{m_1}}\right) \frac{q^{m_3}}{1 - q^{m_1}}\\ =&\zeta_q^{\mathrm{TBZ}}(\overline{1}, 1, \overline{1}) - \zeta_q^{\mathrm{TBZ}}(\overline{1}, \overline{1}, \overline{1}) - \zeta_q^{\mathrm{TBZ}}(2, 1, \overline{1}) + \zeta_q^{\mathrm{TBZ}}(2, \overline{1}, \overline{1}) + \zeta_q^{\mathrm{TBZ}}(3, 1, \overline{1}) - \zeta_q^{\mathrm{TBZ}}(3, \overline{1}, \overline{1}).\end{aligned}$$

2.5. Ohno-Okuda-Zudilin model

Another model for q-analogues of MZVs is the one introduced in 2012 ([OOZ]) and named after Ohno, Okuda and Zudilin. We describe in Section 4.6 an application of this model. There, in Theorem 4.30, we will see that for finite fields K, the generating series of the number of conjugacy classes of $\mathrm{GL}(n, K)$ is a sum of OOZ-qMZVs. In [EMS] the authors discuss the algebraic structure of OOZ-qMZVs.

Definition 2.27. (i) We will work with an extended version of OOZ-qMZVs: Define for $r \geq 1$ and all integers $k_1, \ldots, k_r \geq 0$ with $k_1 \geq 1$

$$\begin{aligned}\zeta_q^{\mathrm{OOZ}}(k_1, \ldots, k_r) &:= \zeta_q(k_1, \ldots, k_r; X, 1 \ldots, 1) \\ &= \sum_{m_1 > \cdots > m_k > 0} \frac{q^{m_1}}{(1-q^{m_1})^{k_1} \cdots (1-q^{m_r})^{k_r}}\end{aligned}$$

and set as usual $\zeta_q^{\mathrm{OOZ}}(\emptyset) := 1$.

Identif an SZ-admissible index $(k_1, \ldots, k_r)$ with $p^{k_1} y \cdots p^{k_r} y \in \mathfrak{K}^1$, we can define ζ_q^{OOZ} also as the map, uniquely determined through $\mathbf{1} \mapsto 1$, $\mathbb{Q}$-linearity and

$$\begin{aligned}\zeta_q^{\mathrm{OOZ}} : \mathfrak{K}^1 &\longrightarrow \mathcal{Z}_q, \\ p^{k_1} y \ldots p^{k_r} y &\longmapsto \zeta_q^{\mathrm{OOZ}}(k_1, \ldots, k_r).\end{aligned}$$

(ii) For a connection to the BZ-model, it is sometimes useful to restrict to admissible indices: Hence, we could define ζ_q^{OOZ} also on $\mathfrak{h}^0$ as the map

$$\begin{aligned}\zeta_q^{\mathrm{OOZ}} : \mathfrak{h}^0 &\longrightarrow \mathcal{Z}_q, \\ z_{k_1} \ldots z_{k_r} &\longmapsto \zeta_q^{\mathrm{OOZ}}(k_1, \ldots, k_r),\end{aligned}$$

extended to $\mathfrak{h}^0$ by $\mathbb{Q}$-linearity and mapping $\mathbf{1} \mapsto 1$.

It is natural to expect and necessary when working with the OOZ-model that the $\mathbb{Q}$-span of the (extended) OOZ-model is $\mathcal{Z}_q$:

Proposition 2.28 (Prop. A.86). *We have for the span of the OOZ-model*

$$\begin{aligned}\mathcal{Z}_{q,1} &= \langle \zeta_q^{OOZ}(k_1, \ldots, k_r) \,|\, r \geq 0,\, k_1 \geq 2,\, k_i \geq 1 \rangle_{\mathbb{Q}}, \\ \mathcal{Z}_q &= \langle \zeta_q^{OOZ}(k_1, \ldots, k_r) \,|\, r \geq 0,\, k_1 \geq 1,\, k_i \geq 0 \rangle_{\mathbb{Q}}.\end{aligned}$$

□

Remark 2.29. OOZ-qMZVs satisfy a quasi-shuffle product. The definition and main statement can be found in the appendix A. For further details we refer to [CEM] and [EMS].

When describing some kind of duality in the OOZ-model (see §3.4), we need the translation of the OOZ-model into the SZ- and BZ-model. The existence of such translations secures the following proposition. For more details, like the explicit maps, we refer to Proposition A.92 in the appendix.

Proposition 2.30 ([EMS, Prop. 5.7, Rem. 5.8])**.** *There are linear isomorphisms* $\mathrm{U} : \mathfrak{h}^0 \to \mathfrak{h}^0$ *and* $\mathrm{V} : \mathfrak{K}^1 \to \mathfrak{K}^1$ *such that they translate the OOZ-model into the BZ- and SZ-model, i.e.*

(i) for all $w \in \mathfrak{h}^0$ *we have*

$$\zeta_q^{OOZ}(w) = (\zeta_q^{BZ} \circ \mathrm{U})(w) \quad \text{and} \quad \zeta_q^{BZ}(w) = (\zeta_q^{OOZ} \circ \mathrm{U}^{-1})(w)$$

(ii) and for all $w \in \mathfrak{K}^1$ *we have*

$$\zeta_q^{OOZ}(w) = (\zeta_q^{SZ} \circ \mathrm{V})(w) \quad \text{and} \quad \zeta_q^{SZ}(w) = (\zeta_q^{OOZ} \circ \mathrm{V}^{-1})(w).$$

□

Notice that on the level of indices in (i) only admissible indices $\mathbf{k} = (k_1, \ldots, k_r)$ occur, i.e. $k_1 \geq 2, k_2, \ldots, k_r \geq 1$, while in (ii) all SZ-admissible indices, $(k_1 \geq 1, k_2, \ldots, k_r \geq 0)$, occur.

2.6. Comparison of the models

Every of the presented models of qMZVs has its advantages. We want to point out the main advantages and differences to other models.

The SZ-model has the advantage that it satisfies a product similar to the stuffle product (Thm. 3.42). Another - also not yet comprehensible - benefit is that SZ-qMZVs fulfil a kind of duality relation, similar to the one of MZVs (see Thm. 3.16). It is not the similarity to the duality of MZVs itself that is so fascinating, but the fact that this duality together with the SZ-stuffle product implies the shuffle product of MZVs (Thm. 3.46).

In contrast to the SZ-model, BZ-qMZVs satisfy exactly (i.e. on word algebraic or index level) the duality relation of MZVs (Thm. 3.5). Furthermore, it is the only of the considered models with this property. However, they span not whole $\mathcal{Z}_q$ (Prop. 2.11), which can be seen as a disadvantage of the BZ-model. One way to fix this is to consider an extension of the BZ-model as Takeyama did. That is also the reason why the TBZ-model is important.

What only one of the models - bi-brackets - fulfils is being closely related to (quasi) modular forms. Even the original definition of and motivation for bi-brackets contains this connection since they are defined as a generalization of Eisenstein series. That deep connections of MZVs and modular forms exist, also Gangl, Kaneko and Zagier ([GKZ]) have shown, such as Broadhurst and Kreimer ([BK]). Furthermore, they satisfy some kind of duality - called partitions relation (Thm. 3.21) - which is equivalent to SZ-duality (Prop. 3.22).

Remarkable for the SZ-model and bi-brackets is also that there are pretty nice generating series (Thm. 2.9, Thm. 2.17). It is often more clear, e.g. how translations into other models work or how to prove identities such as SZ-duality (Thm. 3.18) or the partitions relation (Thm. 3.21). For the other models, no (good) generating series to work with is known so far.

A bit surprising and hence interesting is the OOZ-model since the corresponding shuffle product can be extended to indices of arbitrary integers (see [OOZ], [EMS] for this extension). That can not be done for the other models we consider.

3. Dualities for MZVs and qMZVs

We will mention in this chapter the duality of MZVs and duality or kinds of duality in various models of qMZVs. We refer to Zagier's work [Za2] for the duality of MZVs. For duality relations in several models of qMZVs, see the given references in the introduction of the corresponding section.

3.1. MZV-duality

We have seen with the (extended) double shuffle relations that MZVs satisfy many $\mathbb{Q}$-linear relations. One special type of linear relation is the so-called *duality relation*. First non-trivial examples are

$$\zeta(3) = \zeta(2,1) \quad \text{or} \quad \zeta(5) = \zeta(2,1,1,1).$$

They are very surprising when we recall the definition of MZVs as iterated sums, since e.g. the first example above gives the equality

$$\zeta(3) = \sum_{m>0} \frac{1}{m^3} = \sum_{m_1>m_2>0} \frac{1}{m_1^2 m_2} = \zeta(2,1).$$

These relations become more clear when we use Kontsevich's integral representation (Thm. 1.4). Then we are able to prove this example by some suitable substitution of variables:

$$\zeta(3) \overset{Thm.1.4}{=} \int\limits_{1>t_1>t_2>t_3>0} \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{1-t_3} \overset{t_i\mapsto 1-t_i}{=} \int\limits_{1>t_3>t_2>t_1>0} \frac{dt_1}{1-t_1}\frac{dt_2}{1-t_2}\frac{dt_3}{t_3} \overset{Thm.1.4}{=} \zeta(2,1).$$

We want to describe duality algebraically. Therefore, we introduce an anti-automorphism on $\mathfrak{h} = \mathbb{Q}\langle x_0, x_1\rangle$, which corresponds to the change of variables

Definition 3.1. (i) Define the anti-automorphism τ on $\mathfrak{h}$ via

$$\begin{aligned} \tau : \mathfrak{h} &\longrightarrow \mathfrak{h}, \\ x_0 &\longmapsto x_1, \\ x_1 &\longmapsto x_0. \end{aligned}$$

The restriction to $\mathfrak{h}^0$ will be called τ too.

(ii) For an admissible index $\mathbf{k}$, uniquely written as

$$\mathbf{k} = (k_1 + 1, \{1\}^{d_1 - 1}, \ldots, k_r + 1, \{1\}^{d_r - 1})$$

with $k_1, \ldots, k_r, d_1, \ldots, d_r \geq 1$, we define its dual index $\mathbf{k}^\vee$ as

$$\mathbf{k}^\vee := (d_r + 1, \{1\}^{k_r - 1}, \ldots, d_1 + 1, \{1\}^{k_1 - 1}).$$

When we define for any admissible index $\mathbf{k} = (k_1, \ldots, k_r)$

$$z_{\mathbf{k}} := z_{k_1} \ldots z_{k_r} \in \mathfrak{h}^0 = x_0 \mathfrak{h} x_1 + \mathbb{Q},$$

then the restriction of τ to $\mathfrak{h}^0$ and building the dual index correspond to each other via

$$\tau(z_{\mathbf{k}}) = z_{\mathbf{k}^\vee}.$$

The general duality of MZVs can be now described as follows:

Theorem 3.2 (Duality of MZVs, [Za2, §9])**.** *For all $w \in \mathfrak{h}^0$ we have*

$$(\zeta \circ \tau)(w) = \zeta(w),$$

which is equivalent to the statement that for every admissible index $\mathbf{k}$ *we have*

$$\zeta(\mathbf{k}^\vee) = \zeta(\mathbf{k}).$$

Proof sketch of Theorem 3.2. The proof is an application of the Kontsevich integral representation of MZVs (Thm. 1.4): Namely, we use in this representation the substitution of variables $(t_1, \ldots, t_k) \mapsto (1 - t_1, \ldots, 1 - t_k)$, which leaves the integral invariant since changing an integration variable t with $1 - t$ corresponds here to get the negative integral with interchanged limits, i.e. the same integral. Furthermore, changing t with $1 - t$ corresponds here to interchange x_0 with x_1 and vice versa; also, the order of variables is interchanged, why we get the MZV of dual index indeed. □

Later, we will give a new proof of the duality theorem using connected sums for qMZVs (Cor. 3.36).

Example 3.3. (i) With the duality, we can prove especially equation (0.2) since

$$\zeta(\{3\}^n) = \zeta((x_0^2 x_1)^n) = \zeta(\tau((x_0^2 x_1)^n)) = \zeta((x_0 x_1^2)^n) = \zeta(\{2, 1\}^n).$$

For $n = 1$ in particular, we get the famous equation

$$\zeta(2, 1) = \zeta(3).$$

(ii) Duality for single zeta value gives ($k \geq 2$)

$$\zeta(k) = \zeta(x_0^{k-1}x_1) = \zeta(\tau(x_0^{k-1}x_1)) = \zeta(x_0x_1^{k-1}) = \zeta(2, \{1\}^{k-2}).$$

The duality relation is a distinguished $\mathbb{Q}$-linear relation among MZVs. Hence, by Conjecture 1.13, the duality should be implied by the extended double shuffle relations. However, this could not be proven, and only partial results exist by Kawasaki and Tanaka ([KT]) and Li ([Li]). In these works, it is shown that special cases of the duality relation can be proven by the so-called derivation relations, for which we know that they follow from the extended double shuffle relations ([IKZ]). Recently these results have been generalized by Kimura ([Kim]), who was able to describe the exact intersection of the derivation relations and the duality relations.

Conjecture 3.4. *The extended double shuffle relations imply all duality relations among MZVs.*

If Ihara-Kaneko-Zagier's conjecture that extended double shuffle implies all linear relations among MZVs could be proven, this conjecture would be a direct consequence since duality relations are distinguished $\mathbb{Q}$-linear relations among MZVs. But also as alone-standing statement, this conjecture could not be proven so far.

We can ask whether also q-analogues of MZVs satisfy a relation similar to the duality relation, Theorem 3.2. When considering the BZ-model, the answer is yes and the SZ-model satisfies another duality relation. But for other models the answer is more complicated, in some models like the TBZ-model or OOZ-model, the only 'duality relation' comes from translating into the BZ- or SZ-model, applying there duality and then translating back. That is what we discuss in the following, the table below gives a brief overview:

	BZ-duality	**SZ-duality**
MZV	Thm. 3.2/Cor. 3.36	Rem. 3.23(i)
BZ	Thm. 3.5	Rem. 3.23(ii)
TBZ	Rem. 3.12	Prop. A.79
SZ	Prop. 3.8/Thm. 3.10	Thm. 3.16
Bi-brackets	Thm. 3.11	Prop. 3.22
OOZ	Thm. 3.24	Thm. 3.24

3.2. BZ-duality

When considering the Bradley-Zhao model, it turns out that BZ-qMZVs satisfy the same duality as MZVs.

Theorem 3.5 (BZ-Duality; Bradley [Bra, Thm. 5], Seki-Yamamoto [SY, Thm. 1.2])**.** *For all $w \in \mathfrak{h}^0$ we have*

$$(\zeta_q^{BZ} \circ \tau)(w) = \zeta_q^{BZ}(w).$$

In other words, we have for every admissible index $\boldsymbol{k}$

$$\zeta_q^{BZ}(\boldsymbol{k}^\vee) = \zeta_q^{BZ}(\boldsymbol{k}).$$

Proof. A proof can be found in [Bra] or [SY], but we will give in Section 3.5 also a new one. □

We can state BZ-duality also on generating series:

Lemma 3.6. *For all $r \geq 1$ we have*

$$\mathfrak{b}\binom{X_1, \dots, X_r}{Y_1, \dots, Y_r} = \mathfrak{b}\binom{Y_r, \dots, Y_1}{X_r, \dots, X_1},$$

which is equivalent to BZ-duality. □

Example 3.7. The analogous relations as the one obtained from duality of MZVs hold, i.e. for example we have

$$\zeta_q^{\mathrm{BZ}}(\{3\}^n) = \zeta_q^{\mathrm{BZ}}(\{2,1\}^n) \quad \text{and} \quad \zeta_q^{\mathrm{BZ}}(k) = \zeta_q^{\mathrm{BZ}}(2, \{1\}^{k-2})$$

for all $n \geq 0$ and all $k \geq 2$.

With the explicit translation of BZ-qMZVs into the SZ-model (Prop. 2.23), we can express BZ-duality in the SZ-model:

Proposition 3.8. *For all $r \geq 1$ and $k_1, \dots, k_r, d_1, \dots, d_r \geq 1$ we have*

$$\sum_{\substack{\delta_1, \dots, \delta_r \in \{0,1\} \\ \varepsilon_i \in \{0,1\}^{d_i - 1}}} \zeta_q^{SZ}(k_1 + \delta_1, \varepsilon_1, \dots, k_r + \delta_r, \varepsilon_r) = \sum_{\substack{\delta_1, \dots, \delta_r \in \{0,1\} \\ \varepsilon_i \in \{0,1\}^{k_i - 1}}} \zeta_q^{SZ}(d_r + \delta_r, \varepsilon_r, \dots, d_1 + \delta_1, \varepsilon_1).$$

□

Example 3.9. Remark first that $(2,2,1)^\vee = (3,2)$ and consider

$$\zeta_q^{\mathrm{BZ}}(2,2,1)$$

$$
\begin{aligned}
&= \sum_{m_1>m_2>m_3>0} \frac{q^{m_1}}{(1-q^{m_1})^2}\frac{q^{m_2}}{(1-q^{m_2})^2}\frac{1}{1-q^{m_3}} \\
&= \sum_{m_1>m_2>m_3>0} \left(\frac{q^{m_1}}{1-q^{m_1}}+\frac{q^{2m_1}}{(1-q^{m_1})^2}\right)\left(\frac{q^{m_2}}{1-q^{m_2}}+\frac{q^{2m_2}}{(1-q^{m_2})^2}\right)\left(1+\frac{q^{m_3}}{1-q^{m_3}}\right) \\
&= \zeta_q^{\mathrm{SZ}}(1,1,0)+\zeta_q^{\mathrm{SZ}}(1,1,1)+\zeta_q^{\mathrm{SZ}}(1,2,0)+\zeta_q^{\mathrm{SZ}}(1,2,1) \\
&\quad +\zeta_q^{\mathrm{SZ}}(2,1,0)+\zeta_q^{\mathrm{SZ}}(2,1,1)+\zeta_q^{\mathrm{SZ}}(2,2,0)+\zeta_q^{\mathrm{SZ}}(2,2,1) \\
&= \sum_{\substack{\delta_1,\delta_2\in\{0,1\}\\ \varepsilon_1\in\{0,1\}^0,\varepsilon_1\in\{0,1\}^1}} \zeta_q^{\mathrm{SZ}}(1+\delta_1,1+\delta_2,\varepsilon_1).
\end{aligned}
$$

Analogously, we find

$$
\zeta_q^{\mathrm{BZ}}(3,2) = \sum_{\substack{\delta_1,\delta_2\in\{0,1\}\\ \varepsilon_1\in\{0,1\}^0,\varepsilon_1\in\{0,1\}^0}} \zeta_q^{\mathrm{SZ}}(2+\delta_1,1+\delta_2).
$$

Comparing both equations (by BZ-duality, it is $\zeta_q^{\mathrm{BZ}}(2,2,1)=\zeta_q^{\mathrm{BZ}}(3,2)$), gives the proposition for $r=2$, $k_1=k_2=1$, $d_1=1$, $d_2=2$.

We can formulate BZ-duality also on generating series using the SZ-model:

Theorem 3.10. *For all $r\geq 1$ we have*

$$
\begin{aligned}
&\sum_{\substack{l_1,\dots,l_r\geq 1\\ \delta_1,\dots,\delta_r\in\{0,1\}}} (-1)^{r-(\delta_1+\cdots+\delta_r)}\mathfrak{s}\begin{pmatrix}\delta_1X_1,0,\dots,0,\dots,\delta_rX_r,0,\dots,0\\ Y_1,\underbrace{Y_1,\dots,Y_1}_{l_1-1},\dots,Y_r,\underbrace{Y_r,\dots,Y_r}_{l_r-1}\end{pmatrix}\prod_{j=1}^r(1+\delta_jX_j)Y_j^{l_j} \\
&= \sum_{\substack{l_1,\dots,l_r\geq 1\\ \delta_1,\dots,\delta_r\in\{0,1\}}} (-1)^{r-(\delta_1+\cdots+\delta_r)}\mathfrak{s}\begin{pmatrix}\delta_rY_r,0,\dots,0,\dots,\delta_1Y_1,0,\dots,0\\ X_r,\underbrace{X_r,\dots,X_r}_{l_1-1},\dots,X_1,\underbrace{X_1,\dots,X_1}_{l_r-1}\end{pmatrix}\prod_{j=1}^r(1+\delta_jY_j)X_j^{l_j},
\end{aligned}
$$

which is equivalent to BZ-duality.

Proof. This is a direct consequence of BZ-duality on the level of power series (Lem. 3.6),

$$
\mathfrak{b}\begin{pmatrix}X_1,\dots,X_r\\ Y_1,\dots,Y_r\end{pmatrix}=\mathfrak{b}\begin{pmatrix}Y_r,\dots,Y_1\\ X_r,\dots,X_1\end{pmatrix}
$$

for all $r\geq 1$ and writing $\mathfrak{b}$ in terms of $\mathfrak{s}$ (Thm. 2.13) on both sides of the equation. □

We can state BZ-duality also on the level of bi-brackets using the translation of SZ into bi-brackets, Theorem 2.18(ii):

Theorem 3.11. *We have for all* $r \geq 1$

$$\sum_{\substack{l_1,\dots,l_r \geq 1 \\ \delta_1,\dots,\delta_r \in \{0,1\}}} \mathfrak{g}\begin{pmatrix} a_{1,1},\dots,a_{1,l_1},\dots,a_{r,1},\dots,a_{r,l_r} \\ b_{1,1},\dots,b_{1,l_1},\dots,b_{r,1},\dots,b_{r,l_r} \end{pmatrix} \prod_{i=1}^{r} (-1)^{1-\delta_i} Y_i^{l_i} e^{-\sum\limits_{j=1}^{l_i} b_{i,j}}$$

$$= \sum_{\substack{l_1,\dots,l_r \geq 1 \\ \delta_1,\dots,\delta_r \in \{0,1\}}} \mathfrak{g}\begin{pmatrix} \alpha_{1,1},\dots,\alpha_{1,l_1},\dots,\alpha_{r,1},\dots,\alpha_{r,l_r} \\ \beta_{1,1},\dots,\beta_{1,l_1},\dots,\beta_{r,1},\dots,\beta_{r,l_r} \end{pmatrix} \prod_{i=1}^{r} (-1)^{1-\delta_i} X_{r-i+1}^{l_i} \cdot e^{-\sum\limits_{j=1}^{l_i} \beta_{i,j}}$$

with the abbreviations

$$a_{i,j} := \begin{cases} \ln(1+\delta_i X_i), & \text{if } j=1, \\ 0, & \text{else}, \end{cases}$$

$$b_{i,j} := \ln\left(1 + jY_i + \sum_{k=1}^{i-1} l_k Y_k\right),$$

$$\alpha_{i,j} := \begin{cases} \ln(1+\delta_i Y_{r-i+1}), & \text{if } j=1, \\ 0, & \text{else}, \end{cases}$$

$$\beta_{i,j} := \ln\left(1 + jX_{r-i+1} + \sum_{k=1}^{i-1} l_k X_{r-k+1}\right),$$

defined for all $1 \leq i \leq r,\ 1 \leq j \leq l_i$.

Proof. This follows from Theorem 3.10 by writing $\mathfrak{s}$ in terms of $\mathfrak{g}$ (Thm. 2.18(ii)). □

Finally, we take a look at the TBZ-model:

Remark 3.12. The TBZ-model is an extension of the BZ-model, but so far, there is no known way of how to extend BZ-duality to the TBZ-model.

3.3. SZ-Duality

As mentioned in the introduction, SZ-qMZVs satisfy a relation similar to the duality of MZVs resp. BZ-qMZVs called *SZ-duality* in the following. The main difference is that 0 gets the special role, as the 1 did in the BZ-duality.

Definition 3.13. (i) Define the anti-automorphism $\tilde{\tau}$ on $\mathfrak{K} = \mathbb{Q}\langle p, y\rangle$ via

$$\begin{aligned} \tilde{\tau} : \mathfrak{K} &\longrightarrow \mathfrak{K}, \\ p &\longmapsto y, \\ y &\longmapsto p. \end{aligned}$$

The restriction to subalgebras will always be called $\tilde{\tau}$ too.

(ii) Any SZ-admissible index $\mathbf{k}$ can be uniquely written as

$$\mathbf{k} = (k_1, \{0\}^{d_1-1}, \dots, k_r, \{0\}^{d_r-1})$$

with $k_1, \dots, k_r, d_1, \dots, d_r \geq 1$. We define its SZ-dual index $\mathbf{k}^\dagger$ by

$$\mathbf{k}^\dagger := (d_r, \{0\}^{k_r-1}, \dots, d_1, \{0\}^{k_1-1}).$$

When we define for any SZ-admissible index $\mathbf{k} = (k_1, \dots, k_r)$

$$z_{\mathbf{k}} := p^{k_1} y \cdots p^{k_r} y \in \mathfrak{K}^1 = p\mathfrak{K}y + \mathbb{Q},$$

then the restriction of $\tilde{\tau}$ to $\mathfrak{K}^1$ and building the SZ-dual index correspond to each other via

$$\tilde{\tau}(z_{\mathbf{k}}) = z_{\mathbf{k}^\dagger}.$$

Proposition 3.14. *The dual and SZ-dual index are very closely related: For every SZ-admissible index* $\mathbf{k}$ *we have*

$$(\mathbf{k} + \mathbf{1})^\vee = \mathbf{k}^\dagger + \mathbf{1},$$

where $\mathbf{k} + \mathbf{1}$ *means that every entry of* $\mathbf{k}$ *is increased by 1.* $\mathbf{k} - \mathbf{1}$ *is similarly defined.*

This is equivalent to the statement that for every admissible index $\mathbf{k}$ *we have*

$$\mathbf{k}^\vee - \mathbf{1} = (\mathbf{k} - \mathbf{1})^\dagger.$$

□

Example 3.15. It is $(3)^\vee = (2, 1)$ and $(1, 0)^\dagger = (2)$, which coincides with

$$((3)^\vee - \mathbf{1}) = ((2, 1) - \mathbf{1}) = (1, 0) = (2)^\dagger = ((3) - \mathbf{1})^\dagger.$$

Proof (of Prop. 3.14). Let be $\mathbf{k} = (k_1, \{0\}^{d_1-1}, \dots, k_r, \{0\}^{d_r-1})$ an arbitrary SZ-admissible index, i.e. $k_1, \dots, k_r, d_1, \dots, d_r \geq 1$ arbitrary. Then we have

$$\begin{aligned} (\mathbf{k} + \mathbf{1})^\vee &= (k_1 + 1, \{1\}^{d_1-1}, \dots, k_r + 1, \{1\}^{d_r-1})^\vee \\ &= (d_r + 1, \{1\}^{k_r-1}, \dots, d_1 + 1, \{1\}^{k_1-1}) = (d_r, \{0\}^{k_r-1}, \dots, d_1, \{0\}^{k_1-1}) + \mathbf{1} = \mathbf{k}^\dagger + \mathbf{1}, \end{aligned}$$

where we only used the definition of the dual respective SZ-dual index.

The second statement follows by remarking that $\mathbf{k}$ is admissible iff $\mathbf{k} - \mathbf{1}$ is SZ-admissible, resp. $\mathbf{k}$ is SZ-admissible iff $\mathbf{k} + \mathbf{1}$ is admissible. □

Theorem 3.16 (SZ-Duality; Zhao [Zh2, Thm. 8.3])**.** *For all* $w \in \mathfrak{K}^1$ *we have*

$$\zeta_q^{SZ}(w) = (\zeta_q^{SZ} \circ \tilde{\tau})(w).$$

In other words: For every SZ-admissible index $\mathbf{k}$ *we have*

$$\zeta_q^{SZ}(\mathbf{k}) = \zeta_q^{SZ}(\mathbf{k}^\dagger).$$

Example 3.17. Since $(1,1,0)^\dagger = (2,1)$ and $(3,2)^\dagger = (1,0,1,0,0)$, we have

$$\zeta_q^{\mathrm{SZ}}(1,1,0) = \zeta_q^{\mathrm{SZ}}(2,1), \qquad \zeta_q^{\mathrm{SZ}}(3,2) = \zeta_q^{\mathrm{SZ}}(1,0,1,0,0).$$

The original proof of SZ-duality, given in [Zh2], works with identities among Rota-Baxter operators (RBO) and the fact that SZ-qMZVs are particular values of some iterated RBOs (see the end of §A.2.1 and [Zh2], [Sin] for more details).

We will give in the following three new proofs of SZ-duality. One of the new ones works with generating series (Thm. 3.18 and Cor. 3.3), the second with so-called connected sums (Cor. 3.33) and the third uses combinatorial arguments in interplay with partitions (Lem. 4.13, Rem. 4.17).

Theorem 3.18 (SZ-duality on generating series)**.** *For every $r \geq 0$ we have*

$$\mathfrak{s}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} = \mathfrak{s}\binom{Y_r,\dots,Y_1}{X_r,\dots,X_1}. \tag{3.1}$$

Proof. Key of the proof is the observation that an ordered summation over all natural numbers $m_1 > \cdots > m_r > 0$ is the same as an unordered summation over all natural numbers $n_1,\dots,n_r > 0$, where n_j corresponds to $m_j - m_{j+1}$ (we set here and in the following $m_{r+1} := 0 =: n_{r+1}$). The proof hence, follows Bachmann's idea of proving Theorem 3.21 ([Ba4, Thm. 2.3]):

$$\mathfrak{s}\binom{Y_r,\dots,Y_1}{X_r,\dots,X_1} \overset{\mathit{Thm.}\,2.9}{=} \sum_{\substack{m_1>\cdots>m_r>0\\ n_1,\dots,n_r>0}} \prod_{j=1}^{r}(1+Y_{r-j+1})^{n_j-1}(1+X_{r-j+1})^{m_j-m_{j+1}-1}q^{m_jn_j}$$

$$= \sum_{\substack{m_1,\dots,m_r>0\\ n_1>\cdots>n_r>0}} \prod_{j=1}^{r}(1+Y_{r-j+1})^{n_{r-j+1}-n_{r-j+2}-1}(1+X_{r-j+1})^{m_{r-j+1}-1}q^{(m_1+\cdots+m_{r-j+1})(n_{r-j+1}-n_{r-j+2})}$$

$$= \sum_{\substack{m_1,\dots,m_r>0\\ n_1>\cdots>n_r>0}} \prod_{j=1}^{r}(1+Y_j)^{n_j-n_{j+1}-1}(1+X_j)^{m_j-1}q^{(m_1+\cdots+m_j)(n_j-n_{j+1})}$$

$$= \sum_{\substack{m_1,\dots,m_r>0\\ n_1>\cdots>n_r>0}} \prod_{j=1}^{r}(1+Y_j)^{n_j-n_{j+1}-1}(1+X_j)^{m_j-1}q^{n_jm_j} \overset{\mathit{Thm.}\,2.9}{=} \mathfrak{s}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r},$$

where the second last equality comes from the telescopic sum

$$\sum_{j=1}^{r}\left(\sum_{l=1}^{j} m_l\right)(n_j - n_{j+1}) = \sum_{j=1}^{r} m_jn_j.$$

□

Corollary 3.19. *From Theorem 3.18, SZ-duality (Thm. 3.16) follows.*

Proof. The coefficient of $X_1^{k_1-1}Y_1^{d_1}\dots X_r^{k_r-1}Y_r^{d_r}$ in $\mathfrak{s}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r}$ is

$$\zeta_q^{\mathrm{SZ}}(k_1,\{0\}^{d_1-1},\dots,k_r,\{0\}^{d_r-1}),$$

the coefficient of $X_1^{k_1-1}Y_1^{d_1}\dots X_r^{k_r-1}Y_r^{d_r}$ in $\mathfrak{s}\binom{Y_r,\dots,Y_1}{X_r,\dots,X_1}$ is

$$\zeta_q^{\mathrm{SZ}}(d_r,\{0\}^{k_r-1},\dots,d_1,\{0\}^{k_1-1}) = \zeta_q^{\mathrm{SZ}}((k_1,\{0\}^{d_1-1},\dots,k_r,\{0\}^{d_r-1})^{\dagger}).$$

Hence, SZ-duality follows by comparing coefficients in Theorem 3.18. □

Remark 3.20. At this point, we can ask whether BZ-duality (Thm. 3.10) could also be proven on the level of generating series, in particular, when writing them in terms of SZ-generating series as in Theorem 3.18 and using SZ-duality. However, no such proof is known so far.

In contrast to the BZ-model, SZ-duality responses exactly to the 'duality' of bi-brackets which is better known as *partition relation* and discovered from the following identity of power series:

Theorem 3.21 (Partition relation, [Ba4, Theorem 2.3])**.** *For all $r \geq 1$ we have*

$$\mathfrak{g}\begin{pmatrix}X_1,\dots,X_r\\ Y_1,\dots,Y_r\end{pmatrix} = \mathfrak{g}\begin{pmatrix}Y_1+\dots+Y_r,\dots,Y_1+Y_2,Y_1\\ X_r,X_{r-1}-X_r,\dots,X_1-X_2\end{pmatrix}. \tag{3.2}$$

Proof. The proof follows via using Theorem 2.17:

$$\mathfrak{g}\begin{pmatrix}X_1,\dots,X_r\\ Y_1,\dots,Y_r\end{pmatrix} = \sum_{\substack{m_1>\dots>m_r>0\\ n_1,\dots,n_r>0}} \prod_{j=1}^{r} e^{m_jY_j}e^{n_jX_j}q^{m_jn_j}$$

$$= \sum_{\substack{m_1,\dots,m_r>0\\ n_1>\dots>n_r>0}} \prod_{j=1}^{r} e^{(m_j-m_{j+1})Y_j}e^{(n_1+\dots+n_j)X_j}q^{m_jn_j} = \mathfrak{g}\begin{pmatrix}Y_1+\dots+Y_r,\dots,Y_1+Y_2,Y_1\\ X_r,X_{r-1}-X_r,\dots,X_1-X_2\end{pmatrix}.$$

□

Remarkable is now the aforementioned connection between SZ-duality and partition relation:

Theorem 3.22. *SZ-duality and the partition relation are equivalent.*

Proof. We compute, using the translation of $\mathfrak{g}$ into $\mathfrak{s}$ (Thm. 2.18(i)):

$$\mathfrak{g}\begin{pmatrix}Y_1+\dots+Y_r,\dots,Y_1+Y_2,Y_1\\ X_r,X_{r-1}-X_r,\dots,X_1-X_2\end{pmatrix}$$

$$\overset{Thm.\,2.18(i)}{=} \prod_{j=1}^{r} e^{Y_1+\dots+Y_{r-j+1}}e^{X_r+(X_{r-1}-X_r)+\dots+(X_{r-j+1}-X_{r-j+2})}$$

$$
\begin{aligned}
&\times \mathfrak{s}\binom{e^{Y_1+\cdots+Y_r}-1,\dots,e^{Y_1+Y_2}-1,e^{Y_1}-1}{e^{X_r}-1,e^{X_r+(X_{r-1}-X_r)}-1,\dots,e^{X_r+(X_{r-1}-X_r)+\dots(X_1-X_2)}-1}\\
&=\prod_{j=1}^{r} e^{Y_1+\cdots+Y_j}e^{X_j}\mathfrak{s}\binom{e^{Y_1+\cdots+Y_r}-1,\dots,e^{Y_1+Y_2}-1,e^{Y_1}-1}{e^{X_r}-1,e^{X_{r-1}}-1,\dots,e^{X_1}-1}\\
&\overset{SZ-duality}{=}\prod_{j=1}^{r} e^{Y_1+\cdots+Y_j}e^{X_j}\mathfrak{s}\binom{e^{X_1}-1,\dots,e^{X_{r-1}}-1,e^{X_r}-1}{e^{Y_1}-1,e^{Y_1+Y_2}-1,\dots,e^{Y_1+\cdots+Y_r}-1}\\
&\overset{Thm.\,2.18(i)}{=}\mathfrak{g}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r}.
\end{aligned}
$$

This proves that the partition relation follows from SZ-duality.

The other direction is now obtained by reading the above chain of equalities in another direction:

$$
\begin{aligned}
&\mathfrak{s}\binom{e^{X_1}-1,\dots,e^{X_{r-1}}-1,e^{X_r}-1}{e^{Y_1}-1,e^{Y_1+Y_2}-1,\dots,e^{Y_1+\cdots+Y_r}-1}\\
&\overset{Thm.\,2.18(i)}{=}\left(\prod_{j=1}^{r} e^{Y_1+\cdots+Y_j}e^{X_j}\right)^{-1}\mathfrak{g}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r}\\
&\overset{\substack{partition\\ relation}}{=}\left(\prod_{j=1}^{r} e^{Y_1+\cdots+Y_j}e^{X_j}\right)^{-1}\mathfrak{g}\binom{Y_1+\cdots+Y_r,\dots,Y_1+Y_2,Y_1}{X_r,X_{r-1}-X_r,\dots,X_1-X_2}\\
&\overset{Thm.\,2.18(i)}{=}\mathfrak{s}\binom{e^{Y_1+\cdots+Y_r}-1,\dots,e^{Y_1+Y_2}-1,e^{Y_1}-1}{e^{X_r}-1,e^{X_{r-1}}-1,\dots,e^{X_1}-1}.
\end{aligned}
$$

We substitute

$$
X_j\mapsto \ln(X_j+1),\quad Y_1+\cdots+Y_j\mapsto \ln(Y_j+1)-\ln(Y_{j-1}+1)
$$

for all $1\le j\le r$, where we set $Y_0:=0$, and get

$$
\mathfrak{s}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r}=\mathfrak{s}\binom{Y_r,\dots,Y_1}{X_r,\dots,X_1},
$$

i.e. SZ-duality by Theorem 3.18 . □

Remark 3.23. (i) In current research, it is investigated how a regularized limit of an arbitrary SZ-qMZV one has to take such that we get back MZVs. In this way, SZ-duality will give relations among MZVs. This regularized limit is not expressed explicitly so far, which is the reason why the resulting relations among MZVs cannot be given explicitly at this point.

(ii) Since the anti-automorphism inducing SZ-duality leaves the span of the BZ-model - a proper subspace of $\mathcal{Z}_q$ - on algebraic level not invariant, it is not possible to express SZ-duality via the BZ-model.

3.4. OOZ-model and BZ-/SZ-duality

Using SZ- and BZ-duality, we can also find in the OOZ-model two kinds of duality. We obtain them by translating OOZ-qMZVs into the SZ- respective BZ-model (depending on the alphabet, we consider OOZ-qMZVs on), applying the known SZ-/BZ-duality, then translating back into the OOZ-model. In [EMS] the authors call those relations among OOZ-qMZVs 'duality relation (in the OOZ-model)' too. However, in the classical sense, these relations are not duality relations that would say that every OOZ-qMZV equals another one, with a different index in general.

Those 'duality relations' in the OOZ-model hence can be indeed viewed as duality in the OOZ-model. Still, they should be rather compared with the partition relation in the model of bi-brackets: We have seen in Proposition 3.22 that the partition relation is the same as when translating bi-brackets into SZ-qMZVs, applying SZ-duality and then translating back into bi-brackets. Let be therefore $U : \mathfrak{h}^0 \to \mathfrak{h}^0$ and $V : \mathfrak{K}^1 \to \mathfrak{K}^1$ the maps from Proposition 2.30, explicit given in Proposition A.92.

Theorem 3.24 ([EMS, Thm. 5.9]). *(i) For all $w \in \mathfrak{h}^0$ we have*

$$\zeta_q^{OOZ}(w) = (\zeta_q^{OOZ} \circ \mathrm{U}^{-1} \circ \tau \circ \mathrm{U})(w)$$

(ii) For all $w \in \mathfrak{K}^1$ we have

$$\zeta_q^{OOZ}(w) = (\zeta_q^{OOZ} \circ \mathrm{V}^{-1} \circ \tilde{\tau} \circ \mathrm{V})(w).$$

□

Example 3.25. Consider $z_2 z_1 \in \mathfrak{h}^0$. Then, (i) of our theorem gives

$$\begin{aligned}\zeta_q^{\mathrm{OOZ}}(2,1) &= \zeta_q^{\mathrm{OOZ}}(z_2 z_1) = \left(\zeta_q^{\mathrm{OOZ}} \circ U^{-1} \circ \tau\right)(U(z_2 z_1)) = (\zeta_q^{\mathrm{OOZ}} \circ U^{-1})(\tau(z_2 z_1)) \\ &= \zeta_q^{\mathrm{OOZ}}(U^{-1}(z_3)) = \zeta_q^{\mathrm{OOZ}}(-z_2 + z_3) = \zeta_q^{\mathrm{OOZ}}(3) - \zeta_q^{\mathrm{OOZ}}(2).\end{aligned}$$

Furthermore, (ii) gives

$$\begin{aligned}\zeta_q^{\mathrm{OOZ}}(2,1) &= (\zeta_q^{\mathrm{OOZ}}(p^2ypy) = \zeta_q^{\mathrm{OOZ}} \circ V^{-1} \circ \tilde{\tau})(V(p^2ypy)) \\ &= (\zeta_q^{\mathrm{OOZ}} \circ V^{-1})(\tilde{\tau}(pyy + pypy + p^2yy + p^2ypy)) \\ &= \zeta_q^{\mathrm{OOZ}}(V^{-1}(p^2y + pypy + p^2yy + pypyy)) \\ &= \zeta_q^{\mathrm{OOZ}}(p^2y - py + pypy - 2pyy + p^2yy + pypyy - pyyy) \\ &= \zeta_q^{\mathrm{OOZ}}(2) - \zeta_q^{\mathrm{OOZ}}(1) + \zeta_q^{\mathrm{OOZ}}(1,1) - \zeta_q^{\mathrm{OOZ}}(1,0) - 2\zeta_q^{\mathrm{OOZ}}(1,0) \\ &\quad + \zeta_q^{\mathrm{OOZ}}(2,0) + \zeta_q^{\mathrm{OOZ}}(1,1,0) - \zeta_q^{\mathrm{OOZ}}(1,0,0).\end{aligned}$$

In [EMS] the authors give another relation among qMZVs that they also include in the family of duality relations, but which is translation of different models:

Theorem 3.26 ([EMS, Thm. 5.5])**.** *For any $w \in \mathfrak{H}^1$ we have*

$$\zeta_q^{OOZ}(w) = (\zeta_q^{SZ,\star} \circ \tilde{\tau})(w),$$

where $\zeta_q^{SZ,\star}$ is the map of multiple zeta star values, defined via $\mathbf{1} \mapsto 1$, $\mathbb{Q}$-linearity and

$$\begin{aligned} \zeta_q^{SZ,\star} : \mathfrak{H}^1 &\longrightarrow \mathcal{Z}_q, \\ p^{k_1}y \cdots p^{k_r}y &\longmapsto \sum_{m_1 \geq \cdots \geq m_r > 0} \frac{q^{m_1 k_1}}{(1-q^{m_1})^{k_1}} \cdots \frac{q^{m_r k_r}}{(1-q^{m_r})^{k_r}}. \end{aligned}$$

□

Example 3.27. Consider $w = p^2ypy$, i.e. $\tilde{\tau}(w) = pypyy$. It is

$$\begin{aligned} \zeta_q^{\mathrm{OOZ}}(w) &= \zeta_q^{\mathrm{OOZ}}(2,1) = \sum_{m_1 > m_2 > 0} \frac{q^{m_1}}{(1-q^{m_1})^2(1-q^{m_2})} \\ &= \sum_{m_1 > m_2 > 0} \left(\frac{q^{m_1}}{1-q^{m_1}} + \frac{q^{2m_1}}{(1-q^{m_1})^2} \right) \left(1 + \frac{q^{m_2}}{1-q^{m_2}} \right) \\ &= \zeta_q^{\mathrm{SZ}}(1,0) + \zeta_q^{\mathrm{SZ}}(1,1) + \zeta_q^{\mathrm{SZ}}(2,0) + \zeta_q^{\mathrm{SZ}}(2,1) \\ &\overset{SZ-duality}{=} \zeta_q^{\mathrm{SZ}}(2) + \zeta_q^{\mathrm{SZ}}(1,1) + \zeta_q^{\mathrm{SZ}}(2,0) + \zeta_q^{\mathrm{SZ}}(1,1,0) \\ &= \left(\sum_{m_1 = m_2 = m_3 > 0} + \sum_{m_1 > m_2 = m_3 > 0} + \sum_{m_1 = m_2 > m_3 > 0} + \sum_{m_1 > m_2 > m_3 > 0} \right) \frac{q^{m_1}}{1-q^{m_1}} \frac{q^{m_2}}{1-q^{m_2}} \\ &= \sum_{m_1 \geq m_2 \geq m_3 > 0} \frac{q^{m_1}}{1-q^{m_1}} \frac{q^{m_2}}{1-q^{m_2}} = \zeta_q^{\mathrm{SZ},\star}(1,1,0) = \zeta_q^{\mathrm{SZ},\star}(\tilde{\tau}(w)), \end{aligned}$$

verifying the theorem in this case.

3.5. Connected sums & proof of dualities

As a new tool for proving identities among (q-)multiple zeta values, Seki and Yamamoto introduced the concept of *connected sums* (this notion is independent of connected sums in topology). With this concept, they have proven the duality of MZVs, Hoffman's identity or the q-analogue of Ohno's relation, cf. [Sek] or [SY].

In this section, using connected sums, we give a new proof of the duality of Schlesinger-Zudilin qMZVs (Thm. 3.16), the duality of Bradley-Zhao qMZVs (Thm. 3.5) and the usual duality of MZVs (Thm. 3.2). It turns out that the connected sum defined below has the power to prove all three statements at once. As a by-product, we also get a proof for the q-Ohno relation (Thm. A.58).

Definition 3.28 (Connected sum)**.**
Let be $r, s \geq 0$, $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{N}_0^r$, $\mathbf{l} = (l_1, \ldots, l_s) \in \mathbb{N}_0^s$ and x real with $|x| < 1$. Define the connected sum as

$$Z_q(\mathbf{k}; \mathbf{l}; x) := \sum_{\substack{m_1 > \cdots > m_r > m_{r+1} = 0 \\ n_1 > \cdots > n_s > n_{s+1} = 0}} \prod_{i=1}^{r} \frac{q^{m_i k_i}}{(1 - q^{m_i} x)(1 - q^{m_i})^{k_i}} \prod_{j=1}^{s} \frac{q^{n_j l_j}}{(1 - q^{n_j} x)(1 - q^{n_j})^{l_j}} \cdot \frac{q^{m_1 n_1} f_q(m_1; x) f_q(n_1; x)}{f_q(m_1 + n_1; x)},$$

where $f_q(m; x) := \prod_{h=1}^{m} (1 - q^h x)$.

Remark 3.29. (i) Z_q is symmetric in $\mathbf{k}$ and $\mathbf{l}$ by definition.

(ii) We should also remark that the connected sum is well-defined in the sense that it is a series over positive real numbers and hence either a positive real number (if convergent) or $+\infty$ (if not convergent).

(iii) If $k_1 \geq 1$ then $Z_q(\mathbf{k}; \emptyset; 0) = \zeta_q^{\mathrm{SZ}}(\mathbf{k})$.

(iv) If $k_1 \geq 1$ then $\lim_{x \to 1} Z_q(\mathbf{k}; \emptyset; x) = \zeta_q^{\mathrm{BZ}}(\mathbf{k} + \mathbf{1})$, where $\mathbf{k} + \mathbf{1}$ means that every argument of $\mathbf{k}$ is increased by 1 ($\mathbf{k} - \mathbf{1}$ is similarly defined).

Proposition 3.30 (Boundary conditions)**.**
If $k_1 \geq 1$, then $Z_q(\mathbf{k}; \emptyset; x) < \infty$ and is hence a well-defined real number for every x ($|x| < 1$) and $0 < q < 1$.

Proof. It is

$$\begin{aligned} Z_q(\mathbf{k}; \emptyset; x) &= \sum_{m_1 > \cdots > m_{r+1} := 0} \prod_{i=1}^{r} \frac{q^{m_i k_i}}{(1 - q^{m_i} x)(1 - q^{m_i})^{k_i}} \\ &\leq \frac{1}{(1 - qx)^r} \sum_{m_1 > \cdots > m_{r+1} := 0} \prod_{i=1}^{r} \frac{q^{m_i k_i}}{(1 - q^{m_i})^{k_i}} \\ &= \frac{1}{(1 - qx)^r} \zeta_q^{\mathrm{SZ}}(k_1, \ldots, k_r), \end{aligned}$$

which is well-defined since $k_1 \geq 1$. □

Theorem 3.31 (Transport relations)**.**
Let $r, s \geq 0$ and $k_1, \ldots, k_r, l_1, \ldots, l_s \geq 0$.
If $s > 0$,

$$Z_q((0, k_1, \ldots, k_r); (l_1, \ldots, l_s); x) = Z_q((k_1, \ldots, k_r); (l_1 + 1, l_2, \ldots, l_s); x) \tag{3.3}$$

and if $r > 0$,

$$Z_q((k_1 + 1, k_2, \ldots, k_r); (l_1, \ldots, l_s); x) = Z_q((k_1, \ldots, k_r); (0, l_1, l_2, \ldots, l_s); x). \tag{3.4}$$

Proof. The second equality follows from the first one by symmetry and the first one is obtained from

$$\begin{aligned}
&\sum_{a>m} \frac{1}{1-q^a x} \cdot \frac{q^{an} f_q(a;x) f_q(n;x)}{f_q(a+n;x)} \\
&= \frac{q^n}{1-q^n} \sum_{a>m} \left(\frac{q^{(a-1)n} f_q(a-1;x) f_q(n;x)}{f_q(a+n-1;x)} - \frac{q^{an} f_q(a;x) f_q(n;x)}{f_q(a+n;x)} \right) \\
&= \frac{q^n}{1-q^n} \frac{q^{mn} f_q(m;x) f_q(n;x)}{f_q(m+n;x)}
\end{aligned}$$

and setting $m = m_1$, $n = n_1$, $a = m_0$. □

Corollary 3.32. *For every SZ-admissible index* $\mathbf{k}$ *and real* x *with* $|x| < 1$ *we have*

$$Z_q(\mathbf{k}; \emptyset; x) = Z_q(\emptyset; \mathbf{k}^\dagger; x).$$

Proof. For every indices $\mathbf{k}$ and $\mathbf{l}$ and $k \geq 1, d \geq 0$ we obtain (with $(k, \{0\}^d, \mathbf{k})$ we mean the concatination of the indices $(k, \{0\}^d)$ and $\mathbf{k}$)

$$Z_q((k, \{0\}^d, \mathbf{k}); \mathbf{l}; x) \overset{(3.4)}{=} Z_q((\{0\}^{d+1}, \mathbf{k}); (\{0\}^k, \mathbf{l}); x) \overset{(3.3)}{=} Z_q(\mathbf{k}, (d+1, \{0\}^{k-1}; \mathbf{l}); x). \tag{3.5}$$

Now, set $\mathbf{l} = \emptyset$ and write an SZ-admissible index $\mathbf{k}$ in the form

$$\mathbf{k} = (k_1, \{0\}^{d_1}, \dots, k_r, \{0\}^{d_r}).$$

Then we obtain the corollary by induction on r and using (3.5) in the induction step. □

Corollary 3.33. *SZ-duality (Thm. 3.16) follows from Theorem 3.31.*

Proof. Take some SZ-admissible index $\mathbf{k}$. With the symmetry of Z_q and setting $x = 0$, we are done by using Corollary 3.32:

$$\zeta_q^{\mathrm{SZ}}(\mathbf{k}) \overset{\textit{Rem.} 3.29\,(iii)}{=} Z_q(\mathbf{k}; \emptyset; 0) \overset{\textit{Cor.} 3.32}{=} Z_q(\emptyset; \mathbf{k}^\dagger; 0) \overset{\textit{Rem.} 3.29\,(i)}{=} Z_q(\mathbf{k}^\dagger; \emptyset; 0) \overset{\textit{Rem.} 3.29\,(iii)}{=} \zeta_q^{\mathrm{SZ}}(\mathbf{k}^\dagger).$$

□

Corollary 3.34. *BZ-duality (Thm. 3.5) follows from Theorem 3.31.*

Proof. For an admissible index $\mathbf{k}$ we have:

$$\begin{aligned}
\zeta_q^{\mathrm{BZ}}(\mathbf{k}) &\overset{\textit{Rem.} 3.29\,(iv)}{=} \lim_{x \to 1} Z_q(\mathbf{k} - \mathbf{1}; \emptyset; x) \overset{\textit{Cor.} 3.32}{=} \lim_{x \to 1} Z_q(\emptyset; (\mathbf{k} - \mathbf{1})^\dagger; x) \\
&\overset{\textit{Rem.} 3.29\,(i)}{=} \lim_{x \to 1} Z_q((\mathbf{k} - \mathbf{1})^\dagger; \emptyset; x) \overset{\textit{Rem.} 3.29\,(iv)}{=} \zeta_q^{\mathrm{BZ}}((\mathbf{k} - \mathbf{1})^\dagger + \mathbf{1}) \\
&\overset{\textit{Prop.} 3.14}{=} \zeta_q^{\mathrm{BZ}}(\mathbf{k}^\vee).
\end{aligned}$$

□

Example 3.35. We give two concrete examples of applying the transport relations step by step to make clear what happens:

$$\begin{aligned} &Z_q((1,0);\emptyset;x) = Z_q((0,0);(0);x) \\ &= Z_q((0);(1);x) = Z_q(\emptyset;(2);x) \\ &\Rightarrow \zeta_q^{SZ}(1,0) = \zeta_q^{SZ}(2),\ \zeta_q^{BZ}(2,1) = \zeta_q^{BZ}(3). \end{aligned}$$

Remark that $(1,0)^\dagger = (2)$ and $(2,1)^\vee = (3)$, why the two results indeed correspond to SZ- resp. BZ-duality.

$$\begin{aligned} &Z_q((2,1);\emptyset;x) = Z_q((1,1);(0);x) \\ &= Z_q((0,1);(0,0);x) = Z_q((1);(1,0);x) \\ &= Z_q((0);(0,1,0);x) = Z_q(\emptyset;(1,1,0);x) \\ &\Rightarrow \zeta_q^{SZ}(2,1) = \zeta_q^{SZ}(1,1,0),\ \zeta_q^{BZ}(3,2) = \zeta_q^{BZ}(2,2,1). \end{aligned}$$

Remark also here that we get SZ-duality and BZ-duality in the mentioned example since $(2,1)^\dagger = (1,1,0)$ and $(3,2)^\vee = (2,2,1)$.

Corollary 3.36. *MZVs satisfy the usual duality, Theorem 3.2, i.e.* $\zeta(\mathbf{k}) = \zeta(\mathbf{k}^\vee)$ *for every admissible index* $\mathbf{k}$.

Proof. The proof is done in the usual way when obtaining the duality for MZVs from the duality of BZ-qMZVs:

Let $\mathbf{k}$ be any admissible index. Then we have

$$\zeta(\mathbf{k}) = \lim_{q\to 1}(1-q)^{\mathrm{wt}(\mathbf{k})}\zeta_q^{\mathrm{BZ}}(\mathbf{k}) \overset{\substack{Cor.\,3.34,\\ \mathrm{wt}(\mathbf{k})=\mathrm{wt}(\mathbf{k}^\vee)}}{=} \lim_{q\to 1}(1-q)^{\mathrm{wt}(\mathbf{k}^\vee)}\zeta_q^{\mathrm{BZ}}(\mathbf{k}^\vee) = \zeta(\mathbf{k}^\vee).$$

□

Corollary 3.37. *The q-Ohno relation follows from Theorem 3.31, i.e. for any admissible index* $\mathbf{k} = (k_1,\ldots,k_r)$ *and any* $c \in \mathbb{N}_0$ *we have*

$$\sum_{|\mathbf{c}|=c} \zeta_q^{BZ}(\mathbf{k}+\mathbf{c}) = \sum_{|\mathbf{c}|=c} \zeta_q^{BZ}(\mathbf{k}^\vee+\mathbf{c}),$$

where we sum over all $\mathbf{c} = (c_1,\ldots,c_r) \in \mathbb{N}_0^r$ *with* $|\mathbf{c}| := c_1 + \cdots + c_r = c$.

Proof. The proof consists of considering connected sums of the form $Z_q(\mathbf{k};\emptyset;x)$ and the related one of the form $Z_q(\emptyset;\mathbf{l};x)$ using transport relations. In both, we will develop all occurring terms as a Taylor series at $x=1$, mainly we use

$$\frac{1}{1-q^m x} = \frac{1}{1-q^m}\frac{1}{1-\frac{q^m}{1-q^m}(x-1)} = \frac{1}{1-q^m}\sum_{c\geq 0}\left(\frac{q^m}{1-q^m}\right)^c (x-1)^c$$

$$= \sum_{c \geq 0} \frac{q^{mc}}{(1-q^m)^{c+1}} (x-1)^c.$$

Now, let be $\mathbf{k} = (k_1, \dots, k_r)$ an admissible index. Then we have

$$\begin{aligned} Z_q(\mathbf{k}-\mathbf{1}; \emptyset; x) &= \sum_{m_1 > \cdots > m_r > 0} \prod_{j=1}^{r} \frac{1}{1-q^{m_j}x} \frac{q^{m_j(k_j-1)}}{(1-q^{m_j})^{k_j-1}} \\ &= \sum_{m_1 > \cdots > m_r > 0} \prod_{j=1}^{r} \left(\sum_{c_j \geq 0} \frac{q^{m_j c_j}}{(1-q^{m_j})^{c_j+1}} (x-1)^{c_j} \frac{q^{m_j(k_j-1)}}{(1-q^{m_j})^{k_j-1}} \right) \\ &= \sum_{c_1, \dots, c_r \geq 0} \sum_{m_1 > \cdots > m_r > 0} \left(\prod_{j=1}^{r} \frac{q^{m_j(k_j+c_j-1)}}{(1-q^{m_j})^{k_j+c_j}} \right) (x-1)^{c_1+\cdots+c_r} \\ &= \sum_{c_1, \dots, c_r \geq 0} \zeta_q^{\mathrm{BZ}}(\mathbf{k}+\mathbf{c})(x-1)^{|\mathbf{c}|}. \end{aligned}$$

Since $\mathbf{k}$ was an arbitrary admissible index and $\mathbf{k}^\vee$ is admissible too, we get

$$Z_q(\emptyset; \mathbf{k}^\vee - \mathbf{1}; x) = \sum_{c_1, \dots, c_r \geq 0} \zeta_q^{\mathrm{BZ}}(\mathbf{k}^\vee + \mathbf{c})(x-1)^{|\mathbf{c}|}.$$

Now, since $Z_q(\mathbf{k}-\mathbf{1}; \emptyset; x) = Z_q(\emptyset; \mathbf{k}^\vee - \mathbf{1}; x)$ for every admissible index by using the transport relations, the result follows by comparing the coefficient of $(x-1)^c$ on both sides. □

In the same way, we can consider $Z_q(\mathbf{k}-\mathbf{1}; \emptyset; x)$ when developing the $\frac{1}{1-q^m x}$ around some $a \in \mathbb{R}$, i.e.

$$\frac{1}{1-q^m x} = \frac{1}{1-aq^m - q^m(x-a)} = \frac{1}{1-aq^m} \frac{1}{1 - \frac{q^m}{1-aq^m}(x-a)} = \sum_{c \geq 0} \frac{q^{mc}}{(1-aq^m)^{c+1}} (x-a)^c.$$

Then it is

$$\begin{aligned} Z_q(\mathbf{k}; \emptyset; x) &= \sum_{m_1 > \cdots > m_r > 0} \prod_{j=1}^{r} \frac{1}{1-q^{m_j}x} \frac{q^{m_j k_j}}{(1-q^{m_j})^{k_j}} \\ &= \sum_{m_1 > \cdots > m_r > 0} \prod_{j=1}^{r} \sum_{c_j \geq 0} \frac{q^{m_j c_j}}{(1-aq^{m_j})^{c_j+1}} \frac{q^{m_j k_j}}{(1-q^{m_j})^{k_j}} (x-a)^{c_j}. \end{aligned}$$

Remark 3.38. The series

$$\sum_{m_1 > \cdots > m_r > 0} \prod_{j=1}^{r} \frac{q^{m_j c_j}}{(1-aq^{m_j})^{c_j+1}} \frac{q^{m_j k_j}}{(1-q^{m_j})^{k_j}}$$

for $c_1, \dots, c_r \geq 0$, $k_1 \geq 2$, $k_2, \dots, k_r \geq 1$ and $a \in [0,1]$ can be seen as q-analogue of MZVs: For $a = 1$ we have seen already by proving the q-Ohno relation, how this works (Cor. 3.37). For arbitrary a, it is more difficult to see whether and why this series is an qMZV. In particular, for $a = 0$, we see that the series is not an element of $\mathcal{Z}_q$.

Remark 3.39. In [SY], the authors use very similar-looking connected sums to prove the q-Ohno relation. The main difference is that the authors consider non-modified qMZVs, while we always work with modified objects.

3.6. Consequences of duality (shuffle product)

SZ-qMZVs satisfy a similar stuffle product as MZVs and the SZ-duality. Both combined will give the shuffle product of qMZVs and then after $q \to 1$ the shuffle product of MZVs, giving an application of SZ-qMZVs. That was already mentioned in [EMS] and [Sin], and in this section, we provide details of their proof. There exist similar statements in other models, like in Bachmann's model of bi-brackets or Zudilin's model of q-brackets ([Zu2]).

Definition 3.40 (SZ-stuffle product)**.**

(i) Define $u_k := p^k y \in \mathfrak{K}$ for all $k \geq 0$.

(ii) Consider on $\mathfrak{K}$ the *SZ-stuffle product*, i.e. define recursively $*_{\mathrm{SZ}} : \mathfrak{K} \otimes \mathfrak{K} \to \mathfrak{K}$ via distributivity and

a) $\mathbf{1} *_{\mathrm{SZ}} w = w *_{\mathrm{SZ}} \mathbf{1} := w$,

b) $u_s v *_{\mathrm{SZ}} u_t w := u_s(v *_{\mathrm{SZ}} u_t v) + u_t(u_s v *_{\mathrm{SZ}} w) + u_{s+t}(v *_{\mathrm{SZ}} w)$

for all words $v, w \in \mathfrak{K}$ and $s, t \geq 0$.

Remark at this point that $\mathfrak{K}^1$ is closed under $*_{\mathrm{SZ}}$, and that $\mathfrak{K}^1$ is generated by the words starting in an u_k with $k \geq 1$.

Definition 3.41. Define the evaluation map

$$\begin{aligned} Z_q^{\mathrm{SZ},*} : \mathfrak{K}^1 &\longrightarrow \mathcal{Z}_q, \\ u_{k_1} \dots u_{k_r} &\longmapsto \zeta_q^{\mathrm{SZ}}(k_1, \dots, k_r) \end{aligned}$$

and extend $Z_q^{\mathrm{SZ},*}$ linearly to $\mathfrak{K}^1$.

Theorem 3.42 ([Sin, Thm. 6.3])**.** *The map $Z_q^{SZ,*}$ is an algebra homomorphism, i.e. for all words $v, w \in \mathfrak{K}^1$, we have*

$$Z_q^{SZ,*}(v) Z_q^{SZ,*}(w) = Z_q^{SZ,*}(v *_{SZ} w).$$

□

But we can consider SZ-qMZVs also in a second way on $\mathfrak{K}$:

Definition 3.43 (SZ-$q$shuffle product)**.** Define recursively $ш_{\mathrm{SZ}} : \mathfrak{K} \times \mathfrak{K} \to \mathfrak{K}$ via

(i) $\mathbf{1} ш_{\mathrm{SZ}} w = w ш_{\mathrm{SZ}} \mathbf{1} := w$,

(ii) $yu ш_{\mathrm{SZ}} v = u ш_{\mathrm{SZ}} yv := y(u ш_{\mathrm{SZ}} v)$,

(iii) $pu ш_{\mathrm{SZ}} pv := p(u ш_{\mathrm{SZ}} pv) + p(pu ш_{\mathrm{SZ}} v) + p(u ш_{\mathrm{SZ}} v)$

for all $u, v, w \in \mathfrak{K}$ and $\mathbb{Q}$-bilinearity.

Remark that $\mathfrak{K}^1$ is closed under $ш_{\mathrm{SZ}}$ and define:

Definition 3.44. We define the evaluation map

$$\begin{aligned} Z_q^{\mathrm{SZ},ш} : (\mathfrak{K}^1, ш_{SZ}) &\longrightarrow \mathcal{Z}_q, \\ p^{k_1}y \dots p^{k_r}y &\longmapsto \zeta_q^{\mathrm{SZ}}(k_1, \dots, k_r) \end{aligned}$$

and extend $Z_q^{\mathrm{SZ},ш}$ linearly to $\mathfrak{K}^1$.

As Singer proved (in [Sin]) using that SZ-qMZVs are just evaluations of iterated Rota-Baxter operators, also $Z_q^{\mathrm{SZ},ш}$ is an algebra homomorphism:

Theorem 3.45 ([Sin, Thm. 6.2])**.** *The map $Z_q^{SZ,ш}$ is an algebra homomorphism, i.e. for all words $v, w \in \mathfrak{K}^1$, we have*

$$Z_q^{SZ,ш}(v) Z_q^{SZ,ш}(w) = Z_q^{SZ,ш}(v ш_{SZ} w).$$

□

We give a detailed proof of the following theorem which is the mainpoint of the proof that $*_{\mathrm{SZ}}$ and SZ-duality imply the shuffle product of MZVs:

Theorem 3.46 ([Sin, §8])**.** *The following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{K}^1 \otimes_{\mathbb{Q}} \mathfrak{K}^1 & \xrightarrow{ш_{SZ}} & \mathfrak{K}^1 \\ {\scriptstyle \tilde{\tau} \otimes_{\mathbb{Q}} \tilde{\tau}} \big\downarrow & & \big\uparrow {\scriptstyle \tilde{\tau}} \\ \mathfrak{K}^1 \otimes_{\mathbb{Q}} \mathfrak{K}^1 & \xrightarrow{*_{SZ}} & \mathfrak{K}^1 \end{array}$$

Recall that MZVs satisfy the shuffle product, in the sense that if we identify an admissible index $(k_1, \dots, k_r)$ with the word $x_0^{k_1-1}x_1 \dots x_0^{k_r-1}x_1 \in \mathfrak{h}^1$, then we have for all words $u, v \in \mathfrak{h}^1$

$$\zeta(u)\zeta(v) = \zeta(u ш v).$$

The SZ-qMZV $\zeta_q^{\mathrm{SZ}}(k_1, \dots, k_r)$ is on word algebraic level identified with $p^{k_1}y \dots p^{k_r}y$. On the other hand, the MZV $\zeta(k_1, \dots, k_r)$ (for admissible indices) is on word algebraic level identified with $x_0^{k_1-1}x_1 \dots x_0^{k_r-1}x_1$. This fact motivates the definition of the according translation map:

Definition 3.47. Define the bijection

$$\begin{aligned}\Psi\,:\,\mathfrak{K}^3 &\longrightarrow \mathfrak{h}^1,\\ p^{k_1}y\dots p^{k_r}y &\longmapsto x_0^{k_1-1}x_1\dots x_0^{k_r-1}x_1\end{aligned}$$

and extend it to $\mathfrak{K}^3$ linearly together with $\Psi(\mathbf{1}) := \mathbf{1}$.

Definition 3.48. For $u \in \mathfrak{K}$ a word let be $\mathrm{w}(u)$ the number of p's occurring in u and $\mathrm{l}(u)$ the number of y's occurring in u.

Proposition 3.49. *For all words $u \in \mathfrak{K}^3$ we have*

(i) $x_0\Psi(u) = \Psi(pu)$,

(ii) $x_1\Psi(u) = \Psi(pyu)$.

Proof. The proof is a direct calculation: Let be $u = p^{k_1}y\dots p^{k_r}y \in \mathfrak{K}^3$ a word. Then we have

$$\begin{aligned}\Psi(pu) &= \Psi(p^{k_1+1}yp^{k_2}y\dots p^{k_r}y) = x_0^{k_1}x_1x_0^{k_2-1}x_1\dots x_0^{k_r-1}x_1 = x_0\Psi(u)\\ \Psi(pyu) &= \Psi(pyp^{k_1}y\dots p^{k_r}y) = x_1x_0^{k_1-1}x_1\dots x_0^{k_r-1}x_1 = x_1\Psi(u).\end{aligned}$$

□

Definition 3.50. Define for $k \geq 0$ the map $\mod k : \mathfrak{K} \to \mathfrak{K}$ via

$$(\mod k(u)) := u \mod k := \begin{cases} u, & \text{if } \mathrm{w}(u) \geq k\\ 0, & \text{else}\end{cases}$$

for all words $u \in \mathfrak{K}$ and extend this map linearly to $\mathfrak{K}$.

Proposition 3.51. *Let $u, v \in \mathfrak{K}^1$ words. Then the number of p's occurring in a word, that occurs in $u \sqcup\!\sqcup_{SZ} v$, is at most $(\mathrm{w}(u) + \mathrm{w}(v))$, i.e. for all words $u, v \in \mathfrak{K}$ we have*

$$u \sqcup\!\sqcup_{SZ} v \mod (\mathrm{w}(u) + \mathrm{w}(v) + 1) = 0.$$

Proof. Formally, the proof is done by induction on $(\mathrm{w}(u) + \mathrm{w}(v))$. □

After the above preliminary work, we are able to prove the most important theorem of this section, which will give us afterwards the stated result that $*_{\mathrm{SZ}}$ and SZ-duality imply the shuffle product of MZVs:

Theorem 3.52. *For all words $w_1, w_2 \in \mathfrak{h}^1$ we have*

$$w_1 \sqcup\!\sqcup w_2 = \Psi(\Psi^{-1}(w_1) \sqcup\!\sqcup_{SZ} \Psi^{-1}(w_2) \mod (\mathrm{w}(\Psi^{-1}(w_1)) + \mathrm{w}(\Psi^{-1}(w_2)))).$$

Proof. Write $\Psi^{-1}(w_1) := u$, $\Psi^{-1}(w_2) := v$.
We distinguish between three cases:

(i) $\mathrm{l}(u) = \mathrm{l}(v) = 1$,

(ii) $\mathrm{l}(u) = 1$, $\mathrm{l}(v) > 1$,

(iii) $\mathrm{l}(u), \mathrm{l}(v) > 1$.

Remark that the case $\mathrm{l}(u) > 1$, $\mathrm{l}(v) = 1$ follows from the second one by symmetry of $ш_{SZ}$ and $ш$.

Because of $\Psi(\mathbf{1}) = \mathbf{1}$ and $\mathrm{w}(\mathbf{1}) = 0$, we see that the proof of the first two cases is done exactly as the proof of the third case when we notice that in (i) and (ii) we have to set $u' := \mathbf{1}$ or $v' = \mathbf{1}$ (the proof of (iii) is independent of $\mathrm{l}(u)$ and $\mathrm{l}(v)$).

Hence, for the remaining proof we are allowed to assume w.l.o.g. $\mathrm{l}(u), \mathrm{l}(v) > 1$.

Write now $u = p^{k_1}y \dots p^{k_r}y$, $v = p^{l_1}y \dots p^{l_s}y$.

We prove again by induction on $n := \mathrm{w}(u) + \mathrm{w}(v)$:

The base case, $n = 1 + 1 = 2$, is done direct: Then $u = v = py$ and we compute:

$$\begin{aligned}&\Psi(py \, ш_{SZ} \, py \mod 2) = \Psi(p(y \, ш_{SZ} \, py) + p(py \, ш \, y) + p(y \, ш_{SZ} \, y \mod 2))\\ &= \Psi(2pypy + py^2 \mod 2) = 2\Psi(pypy) = 2y^2 = y \, ш \, y = \Psi(py) \, ш \, \Psi(py).\end{aligned}$$

So, lets take words $u, v \in \mathfrak{K}^2$ with $n := \mathrm{w}(u) + \mathrm{w}(v) > 2$ and assume that the theorem is proven for all words $u', v' \in \mathfrak{K}^2$ with $\mathrm{w}(u') + \mathrm{w}(v') < n$.

Note that the k_i and l_j are all ≥ 0, why $p^{k_2}y \dots p^{k_r}y$, $p^{l_2}y \dots p^{l_s}y \in \mathfrak{K}^2$ again.

We distinguish between the three cases

1.) $k_1 = k_2 = 1$,

2.) $k_1 = 1$, $l_1 > 1$,

3.) $k_1, l_1 > 1$.

Remark that the case $k_1 > 1$, $l_1 = 1$ follows from (ii) by symmetry of $ш_{SZ}$ and $ш$.

1.) In the first write $u = pyu'$, $v = pyv'$ with $u', v' \in \mathfrak{K}^2$ again. Then it is

$$\begin{aligned}&\Psi(\Psi^{-1}(w_1) \, ш_{SZ} \, \Psi^{-1}(w_2) \mod n) = \Psi(pyu' \, ш \, pyv' \mod n)\\ &= \Psi(p(yu' \, ш_{SZ} \, pyv') + p(pyu' \, ш_{SZ} \, yv') + p(yu' \, ш_{SZ} \, yv') \mod n)\\ &= \Psi(py(u' \, ш_{SZ} \, v) + py(u \, ш_{SZ} \, v') \mod n)\\ &= x_1\Psi(u' \, ш_{SZ} \, v \mod (n-1)) + x_1\Psi(u \, ш_{SZ} \, v' \mod (n-1))\\ &\overset{Induction}{=} x_1(\Psi(u') \, ш \, \Psi(v)) + x_1(\Psi(u) \, ш \, \Psi(v')) = \Psi(u) \, ш \, \Psi(v)\\ &= w_1 \, ш \, w_2.\end{aligned}$$

2.) For the second case, write $u = pyu', \ v = pv'$ and note that $u', v' \in \mathfrak{K}^2$. Then we compute

$$\begin{aligned}
&\Psi(\Psi^{-1}(w_1) \sqcup\!\sqcup_{SZ} \Psi^{-1}(w_2) \mod n) = \Psi(pyu' \sqcup\!\sqcup_{SZ} pv' \mod n) \\
=&\Psi(p(yu' \sqcup\!\sqcup_{SZ} v) + p(u \sqcup\!\sqcup_{SZ} v') + p(yu' \sqcup\!\sqcup_{SZ} v') \mod n) \\
=&\Psi(py(u' \sqcup\!\sqcup_{SZ} v) + p(u \sqcup\!\sqcup_{SZ} v') + py(u' \sqcup\!\sqcup_{SZ} v') \mod n) \\
=&x_1\Psi(u' \sqcup\!\sqcup_{SZ} v \mod (n-1)) + x_0\Psi(u \sqcup\!\sqcup_{SZ} v' \mod (n-1)) \\
\overset{Induction}{=}& x_1(\Psi(u') \sqcup\!\sqcup \Psi(v)) + x_0(\Psi(u) \sqcup\!\sqcup \Psi(v')) = \Psi(u) \sqcup\!\sqcup \Psi(v) \\
=&w_1 \sqcup\!\sqcup w_2.
\end{aligned}$$

since $\Psi(u)\sqcup\!\sqcup\Psi(v) = (y\Psi(u'))\sqcup\!\sqcup(p\Psi(v')) = y(\Psi(u')\sqcup\!\sqcup(p\Psi(v'))+p((y\Psi(u'))\sqcup\!\sqcup\Psi(v'))$.

3.) Write $u = pu', \ v = pv'$ and note that - since $k_1, l_1 > 1$ - $u', v' \in \mathfrak{K}^2$. We compute again:

$$\begin{aligned}
&\Psi(\Psi^{-1}(w_1) \sqcup\!\sqcup_{SZ} \Psi^{-1}(w_2) \mod n) = \Psi(pu' \sqcup\!\sqcup_{SZ} pv' \mod n) \\
=&\Psi(p(u' \sqcup\!\sqcup_{SZ} v) + p(u \sqcup\!\sqcup_{SZ} v') + p(u' \sqcup\!\sqcup_{SZ} v') \mod n) \\
=&\Psi(p(u' \sqcup\!\sqcup_{SZ} v) + p(u \sqcup\!\sqcup_{SZ} v') + \mod n) \\
=&x_0\Psi(u' \sqcup\!\sqcup_{SZ} v \mod (n-1)) + x_0\Psi(u \sqcup\!\sqcup_{SZ} v' \mod (n-1)) \\
\overset{Induction}{=}& x_0(\Psi(u') \sqcup\!\sqcup \Psi(v)) + x_0(\Psi(u) \sqcup\!\sqcup \Psi(v')) = \Psi(u) \sqcup\!\sqcup \Psi(v) \\
=&w_1 \sqcup\!\sqcup w_2.
\end{aligned}$$

□

Lemma 3.53. *Let be $w \in \mathfrak{K}^3$ and $k \geq 1$. If $w \mod k = 0$, then*

$$\lim_{q\to 1}(1-q)^k \zeta_q^{SZ}(w) = 0.$$

Proof. Let be $k \geq 1$ and choose a word $w \in \mathfrak{K}^3$ arbitrary and let be $\mathbf{k}$ the corresponding admissible index ($\mathfrak{K}^3$ embodies the admissible indices). Then we get

$$\lim_{q\to 1}(1-q)^{\mathrm{wt}(\mathbf{k})} \zeta_q^{\mathrm{SZ}}(w) = \zeta(w).$$

Furthermore, since $w \in \mathfrak{K}^3$, the index $\mathbf{k}$ contains no zeros, i.e. we have $\mathrm{wt}(\mathbf{k}) = \mathrm{wt}_{SZ}(\mathbf{k})$. But since $w \mod k = 0$, this means $\mathrm{wt}_{SZ}(k) < k$ and so

$$\lim_{q\to 1}(1-q)^k \zeta_q^{\mathrm{SZ}}(w) = 0$$

as we wanted to show. □

Corollary 3.54. *The shuffle product of MZVs follows by considering the product of SZ-qMZVs (of admissible indices), applying SZ-duality, then the SZ-stuffle product and SZ-duality again.*

Proof. Apply in Theorem 3.52 (for admissible indices) the evaluation map $Z_q^{\mathrm{SZ},\sqcup\!\sqcup}$, multiply with $(1-q)^{\mathrm{w}(\Psi^{-1}(w_1))+\mathrm{w}(\Psi^{-1}(v))}$ and take in compliance with Lemma 3.53 then the limit $q \to 1$. □

4. Partitions of numbers for qMZVs

We give in this section a combinatorial view of the considered dualities of q-analogues of MZVs. For that, we will use partitions intensively. A good reference on partitions, in general, is, for example, [FH]. For Stanley coordinates, we refer to Stanley's original work [Sta].

A partition of a natural number N is usually defined as an in descending order sorted tuple of natural numbers $\lambda = (\lambda_1, \dots, \lambda_h)$ (i.e. $\lambda_1 \geq \dots \geq \lambda_h$) with

$$|\lambda| := \lambda_1 + \dots + \lambda_h = N.$$

We often write $\lambda \dashv N$ to say that λ is partition of N.

We can characterise $\lambda \dashv N$ also in a different way via summarizing the λ_i that are equal: So we can identify λ with two tuples of natural numbers, $\mathbf{m} = (m_1, \dots, m_r)$ and $\mathbf{n} = (n_1, \dots, n_r)$, where $\mathbf{m}$ contains the in λ appearing values in strict descending order (i.e. $m_1 > \dots > m_r > 0$) and $\mathbf{n}$ their multiplicities, i.e. n_i describes the number of λ_j being equal to m_i and it is $N = \sum\limits_{j=1}^{r} m_j n_j$.

Definition 4.1 (*Stanley Coordinates*)**.** A partition of length r of some $N \in \mathbb{N}$ in Stanley coordinates is a pair of r natural numbers $(\mathbf{m}, \mathbf{n}) = ((m_1, \dots, m_r), (n_1, \dots, n_r))$ such that

(i) $m_1 > \dots > m_r > 0$,

(ii) $m_1 n_1 + \dots + m_r n_r = N$.

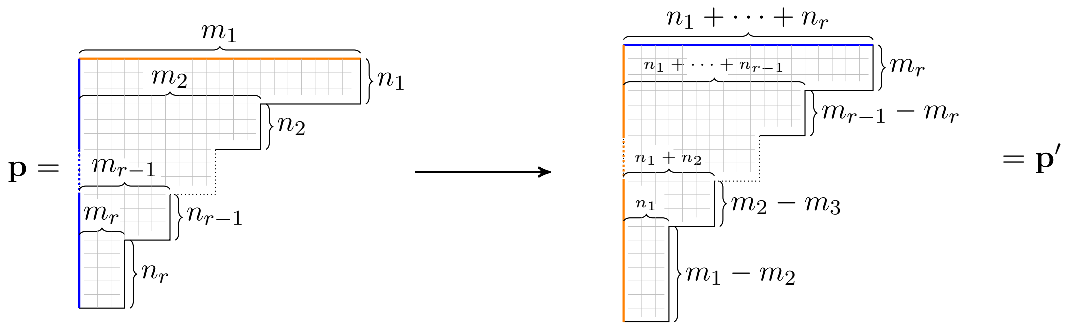
In the following, we will use mainly Stanley coordinates. We will call r the length of the partition $\mathbf{p} = (\mathbf{m}, \mathbf{n})$.

By $\mathbf{p}'$ the *transposed partition* of $\mathbf{p}$ is denoted, i.e. the partition with at the main diagonal reflected Young-diagram. More formally:

Definition 4.2 (*Transposed partition*)**.** Let be $\mathbf{p} = (\mathbf{m}, \mathbf{n}) \dashv N$ a partition of a natural number N. Then the transposed partition of $\mathbf{p}$ is the partition

$$\mathbf{p}' := ((n_1 + \dots + n_r, \dots, n_1 + n_2, n_1), (m_r, m_r - m_{r-1} \dots, m_1 - m_2)).$$

The following illustrates the formal definition of the transposed partition:



Remark 4.3. Note that the transposed partition is indeed a partition of the same number and same length again because

(i) $n_1 + \cdots + n_j > n_1 + \cdots + n_{j-1}$ since $n_j > 0$ for all $1 < j \leq r$,

(ii) $m_j - m_{j-1} > 0$ since $m_j > m_{j-1}$ for all $1 < j \leq r$,

(iii) (set $m_{r+1} := 0$)

$$\sum_{j=1}^{r} \left(\sum_{k=1}^{r-j+1} n_k \right) (m_{r-j+1} - m_{r-j+2}) = \sum_{j=1}^{r} m_j \left(\sum_{k=1}^{r-(r-j+1)+1} n_k - \sum_{k=1}^{r-(r-j+2)+1} n_k \right)$$
$$= \sum_{j=1}^{r} m_j n_j = N.$$

We will often consider sums over all partitions of a fixed number and with fixed length and therefore give the following definitions:

Definition 4.4. Define for every $N \in \mathbb{N}$ and $r \geq 1$

(i) the set of partitions of N of length r,

$$\mathcal{P}_r(N) := \{((m_1, \dots, m_r), (n_1, \dots, n_r)) \in \mathbb{N}^r \times \mathbb{N}^r : m_1 > \cdots > m_r, \sum_{j=1}^{r} m_j n_j = N\},$$

(ii) the set of partitions of N of length $\leq r$,

$$\mathcal{P}_{\leq r}(N) := \bigcup_{s=1}^{r} \mathcal{P}_s(N),$$

(iii) the set of partitions of natural numbers of length r,

$$\mathcal{P}_r := \bigcup_{N>0} \mathcal{P}_r(N),$$

(iv) the set of partitions of natural numbers of length $\leq r$,

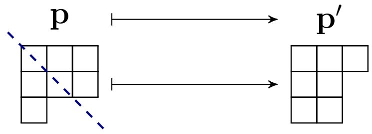
$$\mathcal{P}_{\leq r} := \bigcup_{N>0} \mathcal{P}_{\leq r}(N),$$

(v) the set of partitions of natural numbers,

$$\mathcal{P} := \bigcup_{r \geq 1} \mathcal{P}_r.$$

Remark at this point that the map

$$\begin{aligned} \rho : \mathcal{P} &\longrightarrow \mathcal{P}, \\ \mathbf{p} &\longmapsto \mathbf{p}' \end{aligned}$$



that maps a partition to its transposed (= reflected at the main diagonal (dark blue)) is an involution, and the restriction to one of the sets of Def. 4.4(i)-(iv) is also an involution. By abuse of notation, the restricted maps will also be denoted by ρ.

Remark 4.5. That ρ is indeed an involution can be proven by direct calculation or by making clear that ρ reflects the corresponding Young diagram. Hence, applying rho twice, i.e. double reflecting, will give the original one again.

Theorem 4.6. *For every qMZV* $\zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r) \in \mathcal{Z}_q$ *with* $r \geq 1$*, there are rational numbers* $a_{\mathbf{p}} \in \mathbb{Q}$ *for all partitions* $\mathbf{p} \in \mathcal{P}_{\leq r}$ *such that*

$$\zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r) = \sum_{((m_1, \ldots, m_{r'}), (n_1, \ldots, n_{r'})) \in \mathcal{P}_{\leq r}} a_{\substack{m_1, \ldots, m_{r'} \\ n_1, \ldots, n_{r'}}} q^{m_1 n_1 + \cdots + m_{r'} n_{r'}}.$$

Moreover, these are polynomials in $m_1, \ldots, m_{r'}, n_1, \ldots, n_{r'}$.

Proof. This follows from the expansion

$$\frac{q^{ml}}{(1-q^m)^k} = q^{ml} \sum_{n \geq 0} \binom{n+k-1}{k-1} q^{mn} = \sum_{n \geq l} \binom{n-l+k-1}{k-1} q^{mn}$$

for all $m \geq 1$, $0 \leq l \leq k$ and from $\deg Q_j \leq k_j$, that all Q_j have rational coefficients and from the fact that binomial coefficients in particular are rational numbers too. That the coefficients are polynomial in $m_1, \ldots, m_{r'}, n_1, \ldots, n_{r'}$ follows also dirct from this expansion. □

Theorem 4.6 implies that for every element $S \in \mathcal{Z}_q$ there is a map

$$\mathbf{a} : \mathcal{P} \longrightarrow \mathbb{Q}$$

and rational a_0 such that

$$S = a_0 + \sum_{N \geq 1} \left(\sum_{\mathbf{p} \in \mathcal{P}(N)} \mathbf{a}(\mathbf{p}) \right) q^N$$

and all but finite many of the projections $\mathbf{a}_r := \mathbf{a}|_{\mathcal{P}_r}$, $r \geq 1$, are constant zero.

The mappings $\mathbf{a}$ do not have to be unique, but we can find a nice one for each element $S \in \mathcal{Z}_q$ because they are polynomial:

Theorem 4.7. *A q-series S is in $\mathcal{Z}_q$ iff there exists $f = (f_r)_{r \geq 0}$ with $f_r \in \mathbb{Q}[X_1, \ldots, X_r, Y_1, \ldots, Y_r]$ for $r \geq 1$ and $f_0 \in \mathbb{Q}$ such that*

(i) $f_r \equiv 0$ *for all but finite many* r,

(ii)

$$S = f_0 + \sum_{N \geq 1} \left(\sum_{r \geq 1} \sum_{((\boldsymbol{m}, \boldsymbol{n})) \in \mathcal{P}_r(N)} f_r(m_1, \ldots, m_r, n_1, \ldots, n_r) \right) q^N.$$

Proof. Remark first that for every bi-bracket such an f exists by the original definition of bi-brackets,

$$\mathrm{g}\binom{k_1, \ldots, k_r}{d_1, \ldots, d_r} := \sum_{\substack{m_1 > \cdots > m_r > 0 \\ n_1, \ldots, n_r > 0}} \frac{m_1^{d_1}}{d_1!} \cdots \frac{m_r^{d_r}}{d_r!} \frac{n_1^{k_1 - 1} \ldots n_r^{k_r - 1}}{(k_1 - 1)! \ldots (k_r - 1)!} q^{m_1 n_1 + \cdots + m_r n_r}.$$

There, $f_s \equiv 0$ for all s except $s = r$, the depth of the bi-bracket. And f_r is a monomial (up to a rational factor) in $\mathbb{Q}[X_1, \ldots, Y_r]$. Indeed, since $d_i \geq 0$, $k_j \geq 1$ can take all values, f_r coming from a bi-bracket can be every monomial in $\mathbb{Q}[X_1, \ldots, Y_r]$. Furthermore, this holds for every $r \geq 1$.

Now, if $S \in \mathcal{Z}_q$, S is a rational linear combination of bi-brackets since they span $\mathcal{Z}_q$. In this case, we see that S indeed is of the desired shape since a possible f is a finite rational linear combination of monomials by the remark above. Hence, in particular, it is a polynomial again.

Conversely, if S is of the shape in the theorem, the monomials occurring in f correspond to bi-brackets as remarked, i.e. S is a rational linear combination of bi-brackets and hence an element of $\mathcal{Z}_q$. □

Remark 4.8. Functions like $\mathbf{a}$ provide a direct connection from qMZVs to so-called q-brackets. For a function $\mathbf{a} : \mathcal{P} \to \mathbb{Q}$, the q-bracket of $\mathbf{a}$ is defined as

$$\langle \mathbf{a} \rangle_q := \frac{\sum_{\lambda \in \mathcal{P}} \mathbf{a}(\lambda) q^{|\lambda|}}{\sum_{\lambda \in \mathcal{P}} q^{|\lambda|}}.$$

They were first introduced by Bloch and Okounkov in [BO] and are of interest in current research since, under certain conditions on $\mathbf{a}$, the q-bracket is quasi-modular. Recall at this point that every quasi-modular form is, in particular, an element of $\mathcal{Z}_q$.

The exact connection between qMZVs and q-brackets will be described in [BvI]. For further research details on q-brackets, we refer to the works by Zagier ([Za3]) and van Ittersum ([vIt]).

Lemma 4.9. *For all $r, N \geq 1$ and maps $\boldsymbol{a}_r : \mathcal{P}_r(N) \to \mathbb{Q}$ we have the equation*

$$\sum_{\mathbf{p} \in \mathcal{P}_r(N)} \boldsymbol{a}_r(\mathbf{p}) = \sum_{\mathbf{p} \in \mathcal{P}_r(N)} \boldsymbol{a}_r(\rho(\mathbf{p})). \tag{4.1}$$

Proof. The map ρ is an involution on $\mathcal{P}_r(N)$. □

The importance of this lemma will be derived later, e.g. in Lemma 4.13 and is one of the main points of this section.

Example 4.10. Consider

$$\begin{aligned}
&\zeta_q(1,0,2;X,1,1+X) = \sum_{m_1>m_2>m_3>0} \frac{q^{m_1}}{1-q^{m_1}} \frac{1+q^{m_3}}{(1-q^{m_3})^2} \\
&= \sum_{m_1>m_3>0} (m_1-m_3-1) \frac{q^{m_1}}{1-q^{m_1}} \frac{1+q^{m_3}}{(1-q^{m_3})^2} \\
&= \sum_{\substack{m_1>m_2>0 \\ n_1>0,\, n_2 \geq 0}} (m_1-m_2-1) \binom{n_2+1}{1} q^{m_1 n_1} \left(q^{m_2 n_2} + q^{m_2(n_2+1)} \right) \\
&= \sum_{\substack{m_1>m_2>0 \\ n_1>0,\, n_2 \geq 0}} (m_1-m_2-1)(n_2+1) q^{m_1 n_1 + m_2 n_2} \\
&\quad + \sum_{\substack{m_1>m_2>0 \\ n_1,n_2>0}} (m_1-m_2-1)(n_2+1) q^{m_1 n_1 + m_2 n_2} \\
&= 2 \sum_{\substack{m_1>m_2>0 \\ n_1,n_2>0}} (m_1-m_2-1)(n_2+1) q^{m_1 n_1 + m_2 n_2} + \sum_{\substack{m_1>m_2>0 \\ n_1>0}} (m_1-m_2-1) q^{m_1 n_1} \\
&= 2 \sum_{\substack{m_1>m_2>0 \\ n_1,n_2>0}} (m_1-m_2-1)(n_2+1) q^{m_1 n_1 + m_2 n_2} + \sum_{\substack{m_1>0 \\ n_1>0}} \left(\sum_{m_2=1}^{m_1} (m_1-m_2-1) \right) q^{m_1 n_1}
\end{aligned}$$

$$= 2 \sum_{\substack{m_1>m_2>0\\ n_1,n_2>0}} (m_1-m_2-1)(n_2+1)q^{m_1n_1+m_2n_2} + \sum_{\substack{m_1>0\\ n_1>0}} \frac{(m_1-2)(m_1-1)}{2} q^{m_1n_1}.$$

In terminology of maps $a : \mathcal{P} \to \mathbb{Q}$ we find now for $\zeta_q(1,0,2;X,1,1+X)$ a suitable map as the following:

$$\begin{aligned} \mathbf{a} : \mathcal{P} &\longrightarrow \mathbb{Q}, \\ (\mathbf{m},\mathbf{n}) &\longmapsto \delta_{r=1}\frac{(m_1-2)(m_1-1)}{2} + \delta_{r=2}\cdot 2(m_1-m_2-1)(n_2+1). \end{aligned}$$

Especially, we verify Theorem 4.7 in this example since we can choose $f = (f_r)_{r\geq 1}$ with

$$f_1(m_1,n_1) := \frac{(m_1-2)(m_1-1)}{2}, \quad f_2(m_1,m_2,n_1,n_2) := 2(m_1-m_2-1)(n_2+1)$$

and $f_r :\equiv 0$ for $r > 2$, which are all polynomial.

Furthermore, Lemma 4.9 gives us in this example the identity

$$\begin{aligned} &2 \sum_{\substack{m_1>m_2>0\\ n_1,n_2>0}} (m_1-m_2-1)(n_2+1)q^{m_1n_1+m_2n_2} + \sum_{\substack{m_1>0\\ n_1>0}} \frac{(m_1-2)(m_1-1)}{2} q^{m_1n_1} \\ = &2 \sum_{\substack{m_1>m_2>0\\ n_1,n_2>0}} (n_2-1)(m_1-m_2+1)q^{m_1n_1+m_2n_2} + \sum_{\substack{m_1>0\\ n_1>0}} \frac{(n_1-2)(n_1-1)}{2} q^{m_1n_1}. \end{aligned}$$

Now, the duality of the considered q-analogues of MZVs gives similar identities of q-series as considered in the above example. Especially, we get relations among the coefficients of q^N $(N \geq 1)$ on both sides of the duality relation. For SZ-duality, BZ-duality and the partition relation (bi-brackets), we will see that these weights are non-negative integers. Therefore, we can interpret the coefficient of q^N as the number of partitions of N, every counted with some multiplicity. These multiplicities can be visualized as markings of rows and columns in the Young diagram of the corresponding partition, as we will see.

A natural problem is constructing a bijection between the marked partitions of N on both sides of the duality relation. For SZ-duality/Takeyama resummation and the partition relation for bi-brackets, we get this bijection as transposing the Young diagram (including the markings) of the partitions in accordance with (4.1). But on the other hand, for BZ-duality, the construction of this bijection turns out to be more complicated and is still an open question.

4.1. Bi-brackets

For every bi-bracket $g\binom{k_1,\dots,k_r}{d_1,\dots,d_r}$ the coefficient of q^N can be easily derived by the original definition:

$$\begin{aligned} g\begin{pmatrix}k_1,\dots,k_r\\ d_1,\dots,d_r\end{pmatrix} &= \sum_{\substack{m_1>\dots>m_r>0\\ n_1,\dots,n_r>0}} \frac{m_1^{d_1}}{d_1!}\cdots\frac{m_r^{d_r}}{d_r!}\frac{n_1^{k_1-1}\dots n_r^{k_r-1}}{(k_1-1)!\dots(k_r-1)!}q^{m_1n_1+\dots+m_rn_r} \\ &= \frac{1}{\prod\limits_{j=1} d_j!(k_j-1)!}\sum_{N>0}\left(\sum_{(\mathbf{m},\mathbf{n})\in\mathcal{P}_r(N)} m_1^{d_1}\cdots m_r^{d_r}n_1^{k_1-1}\cdots n_r^{k_r-1}\right)q^N. \end{aligned}$$

Recall that the SZ-duality of bi-brackets, the partition relation (Lemma 3.21), was proven on the level of generating series using the involution ρ. On the level of bi-brackets, we can express the partition relation now using Lemma 4.9 explicit:

Lemma 4.11. *For all $r\geq 1$, $d_1,\dots,d_r\geq 0$, $k_1,\dots,k_r\geq 1$, we have*

$$\begin{aligned} &g\begin{pmatrix}k_1,\dots,k_r\\ d_1,\dots,d_r\end{pmatrix} \\ =&\sum_{\substack{0\leq k_i'\leq k_i-1\\ d'_{i,j}\geq 0,\,1\leq i\leq r,\,1\leq j\leq r-i+1\\ d'_{i,1}+\dots+d'_{i,r-i+1}=d_i}} \prod_{j=1}^{r}\frac{(d'_{1,j}+\dots+d'_{r-j+1,j})!(k'_{r-j+1}+k_{r-j+2}-1-k'_{r-j+2})!}{d_j!(k_j-1)!} \\ &\times\begin{pmatrix}d_j\\ d'_{j,1},\dots,d'_{j,r-j+1}\end{pmatrix}\begin{pmatrix}k_j-1\\ k'_j\end{pmatrix} g\begin{pmatrix}d'_{1,1}+\dots+d'_{1,r},\dots,d'_{r-1,1}+d'_{r-1,2},d'_{r,1}\\ k'_r-k'_{r+1}-1+k_{r+1},\dots,k'_1-k'_2-1+k_2\end{pmatrix}. \end{aligned}$$

with $k_{r+1}:=k'_{r+1}:=0$.

Proof. Apply Lemma 4.9 to the map

$$\begin{aligned} a_r:\mathcal{P}_r(N)&\longrightarrow\mathbb{Q}, \\ (\mathbf{m},\mathbf{n})&\longmapsto m_1^{d_1}\cdots m_r^{d_r}n_1^{k_1-1}\cdots n_r^{k_r-1}. \end{aligned}$$

We get with $m_{r+1}:=0$

$$\begin{aligned} &\sum_{(\mathbf{m},\mathbf{n})\in\mathcal{P}_r(N)} m_1^{d_1}\cdots m_r^{d_r}n_1^{k_1-1}\cdots n_r^{k_r-1} \\ =&\sum_{(\mathbf{m},\mathbf{n})\in\mathcal{P}_r(N)} (n_1+\dots+n_r)^{d_1}\cdots n_1^{d_r}(m_r-m_{r+1})^{k_1-1}\cdots(m_1-m_2)^{k_r-1} \end{aligned}$$

and by multiplying out,

$$\sum_{(\mathbf{m},\mathbf{n})\in\mathcal{P}_r(N)} m_1^{d_1}\cdots m_r^{d_r}n_1^{k_1-1}\cdots n_r^{k_r-1}$$

$$= \sum_{(\mathbf{m},\mathbf{n})\in\mathcal{P}_r(N)} \sum_{\substack{0\le k_i'\le k_i-1 \\ d'_{i,j},1\le i\le r,1\le j\le r-i+1 \\ d'_{i,1}+\dots+d'_{i,r-i+1}=d_i}} \prod_{j=1}^{r} \binom{d_j}{d'_{j,1},\dots,d'_{j,r-j+1}}\binom{k_j-1}{k'_j}$$
$$\times\, n_j^{d'_{1,j}+\dots+d'_{r-j+1,j}} m_j^{k'_{r-j+1}+k_{r-j+2}-1-k'_{r-j+2}},$$

we obtain Lemma 4.11 when we use the definition of bi-brackets. □

Interesting in the context of bi-brackets and Theorem 4.7 is the following reformulation of Bachmann's conjecture which says that brackets and bi-brackets span the same space:

Conjecture 4.12 (Refinement of [Ba4, Conj. 4.3])**.** *Let $P \in \mathbb{Q}[X_1,\dots,X_r,Y_1,\dots,Y_r]$, $r\ge 1$, be a polynomial. Then there exists $Q=(Q_j)_{j\ge1}$ with $Q_j\in\mathbb{Q}[X_1,\dots,X_j]$ and $Q_j\equiv 0$ for all but finite many j such that for every $N\ge1$ we have*

$$\sum_{(\mathbf{m},\mathbf{n})\in\mathcal{P}_r(N)} P(m_1,\dots,m_r,n_1,\dots,n_r) = \sum_{j=1}^{M}\sum_{(\mathbf{m},\mathbf{n})\in\mathcal{P}_j(N)} Q_j(n_1,\dots,n_r),$$

where $M := r+\sum_{i=1}^{r}\deg_{Y_i}(P)$.

4.2. SZ-model

For SZ-duality: Consider some SZ-admissible index

$$\mathbf{k}=(k_1+1,\{0\}^{d_1},\dots,k_r+1,\{0\}^{d_r})$$

(i.e. $k_1,\dots,k_r,d_1,\dots,d_r\ge0$) and obtain

$$\zeta_q^{\mathrm{SZ}}(\mathbf{k}) = \sum_{\substack{m_1>n_1>\dots>n_{d_1}>\dots \\ >m_r>n_{d_1+\dots+d_{r-1}+1}>\dots>n_{d_1+\dots+d_r}>0}} \frac{q^{m_1(k_1+1)}}{(1-q^{m_1})^{k_1+1}}\cdots\frac{q^{m_r(k_r+1)}}{(1-q^{m_r})^{k_r+1}}$$
$$= \sum_{m_1>\dots>m_r>0}\binom{m_1-m_2-1}{d_1}\cdots\binom{m_r-m_{r+1}-1}{d_r}\frac{q^{m_1(k_1+1)}}{(1-q^{m_1})^{k_1+1}}\cdots\frac{q^{m_r(k_r+1)}}{(1-q^{m_r})^{k_r+1}}$$
$$= \sum_{\substack{m_1>\dots>m_r>0\\ n_1,\dots,n_r>0}}\binom{m_1-m_2-1}{d_1}\cdots\binom{m_r-m_{r+1}-1}{d_r}\binom{n_1-1}{k_1}\cdots\binom{n_r-1}{k_r}q^{m_1n_1+\dots+m_rn_r}.$$

Hence, analogously we can rewrite $\zeta_q^{\mathrm{SZ}}(\mathbf{k}^\dagger)$ as

$$\sum_{\substack{m_1>\dots>m_r>0\\ n_1,\dots,n_r>0}}\binom{m_1-m_2-1}{k_r}\cdots\binom{m_r-m_{r+1}-1}{k_1}\binom{n_1-1}{d_r}\cdots\binom{n_r-1}{d_1}q^{m_1n_1+\dots+m_rn_r}.$$

Therefore, SZ-duality is equivalent to the following lemma:

Lemma 4.13. *For all $r \geq 1$, $k_1, \dots, k_r, d_1, \dots, d_r \geq 0$ and all $N > 0$ we have*

$$\sum_{\substack{m_1 > \cdots > m_r > 0 \\ n_1, \dots, n_r > 0 \\ m_1 n_1 + \cdots + m_r n_r = N}} \binom{m_1 - m_2 - 1}{d_1} \cdots \binom{m_r - m_{r+1} - 1}{d_r} \binom{n_1 - 1}{k_1} \cdots \binom{n_r - 1}{k_r}$$
$$= \sum_{\substack{m_1 > \cdots > m_r > 0 \\ n_1, \dots, n_r > 0 \\ m_1 n_1 + \cdots + m_r n_r = N}} \binom{m_1 - m_2 - 1}{k_r} \cdots \binom{m_r - m_{r+1} - 1}{k_1} \binom{n_1 - 1}{d_r} \cdots \binom{n_r - 1}{d_1}.$$

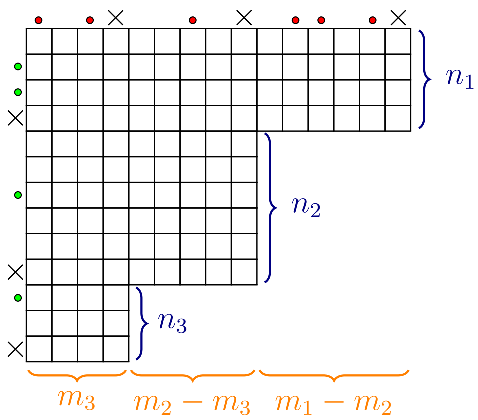
We can interpret both sides of the lemma as the number of partitions of N with rows and columns in the Young diagram marked in some way:

Proposition 4.14. *The first sum of Lemma 4.13 is the number of partitions of N with exactly r parts, where d_i rows of the $i-th$ part without the last row in the corresponding Young diagram are marked, such as k_j columns lying between the j-th and $(j+1)$-th rightmost corner of the Young diagram for all $1 \leq i, j \leq r$.*

Example 4.15. For $N = 126$ one of the in Proposition 4.14 described marked partitions with exactly $r = 3$ parts and with

$$k_1 = 2,\ k_2 = k_3 = 1,$$
$$d_1 = 2,\ d_2 = 1,\ d_3 = 3$$

is the following:



The crosses $\times$ stand for the corresponding row/column not being allowed to be colored. That there is in every part a fixed row/column (we always fix the lowest row/rightest column) comes from the -1's in the binomial coefficients that we consider in Lemma 4.13.

Proof (of Lemma 4.14). Considering the index of the first sum in Lemma 4.13, we get that it is just the number of partitions of N with exactly r parts, every partition counted with the respective multiplicity, given as the product of binomial coefficients we have seen.

Now, given such a partition of N, the j-the part consists of n_j rows, why marking d_j of the rows of the j-th part without the last one gives a multiplicity of $\binom{n_j-1}{k_j}$. Since those markings of rows in some part are independent of the markings in the other parts this rows coloring gives a multiplicity of

$$\binom{n_1-1}{k_1}\cdots\binom{n_r-1}{k_r}$$

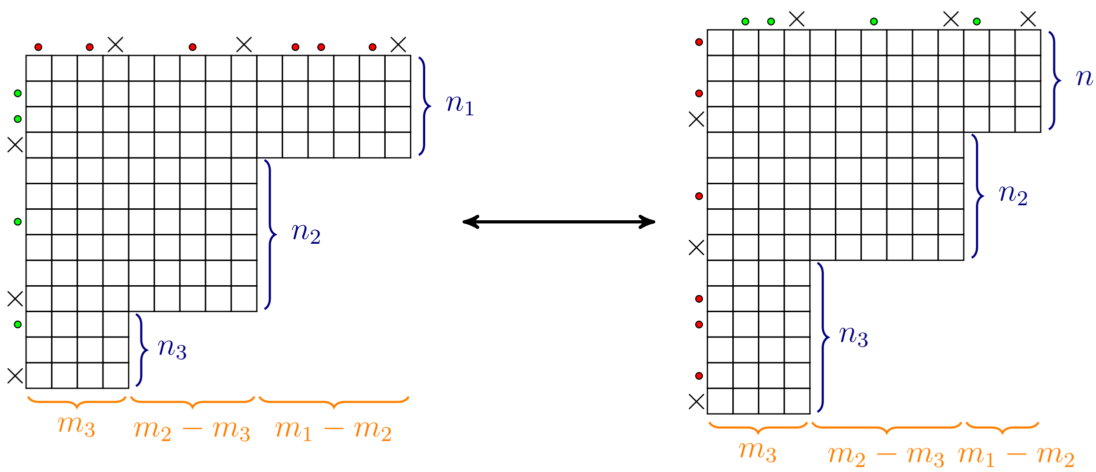
of the given partition.

For the column markings we have the same argument: Since the j-th part of the Young diagram has length m_j and the $(j+1)$-th has length m_{j+1}, between the j-th and $(j+1)$-th corner, counted from the right, there are $m_j - m_{j+1} - 1$ columns. Hence, with marking d_j of them, we get an additional factor $\binom{m_j-m_{j+1}-1}{d_j}$ for the multiplicity of the given partition (for every $1 \leq j \leq r$) and so exactly the first sum in Lemma 4.13. □

Since Lemma 4.13 is equivalent to SZ-duality, the following proof of the Lemma gives the third new proof in this thesis:

Proof (of Lemma 4.13/Theorem 3.16). Fix some $r \geq 1$, $N > 0$ and $k_1, \ldots, k_r, d_1, \ldots, d_r \geq 0$. How both sides of the desired equation can be interpreted as marked partitions, we have seen already in Proposition 4.14. Now, a bijection between both sets of partitions is given by transposing the Young diagram together with the markings (so column markings will be convicted in row markings and vice versa). In particular, the cardinality of the sets is the same and hence, Lemma 4.13 is proven. □

Example 4.16. We get the identification via transposing the Young diagram together with markings as said in the proof before. For example, we identify (in accordance to Example 4.15):



Remark 4.17.

(i) SZ-duality/Lemma 4.13 is a direct consequence of the fact that (4.1) is an involution.

(ii) As remarked, Lemma 4.13 can be proven by some index shift, why on both sides of the lemma the same products of binomial coefficients occur, but in a different order in general.

4.3. BZ-model

For the BZ-model, we consider the duality $\zeta_q^{\mathrm{BZ}}(\mathbf{k}) = \zeta_q^{\mathrm{BZ}}(\mathbf{k}^\vee)$ for admissible indices $\mathbf{k}$ and compare the coefficients of q^N in this relation.

Write therefore $\mathbf{k}$ in the form

$$\mathbf{k} = (k_1+1, \{1\}^{d_1-1}, \dots, k_r+1, \{1\}^{d_r-1})$$

with $k_1, \dots, k_r, d_1, \dots, d_r \geq 1$ and compute

$$\begin{aligned}\zeta_q^{\mathrm{BZ}}(\mathbf{k}) = &\sum_{\substack{m_1>n_1>\cdots>n_{d_1-1}>m_2>\dots\\ >m_r>\cdots>n_{d_1+\cdots+d_r-r}>0}} \frac{q^{m_1k_1}}{(1-q^{m_1})^{k_1+1}}\frac{1}{1-q^{n_1}}\cdots\frac{1}{1-q^{n_{d_1-1}}}\\ &\times\cdots\frac{q^{m_rk_r}}{(1-q^{m_r})^{k_r+1}}\frac{1}{1-q^{n_{d_1+\cdots+d_{r-1}-(r-1)}}}\cdots\frac{1}{1-q^{n_{d_1+\cdots+d_r-r}}}\\ = &\sum_{\substack{m_1>n_1>\cdots>n_{d_1-1}>m_2>\dots\\ >m_r>\cdots>n_{d_1+\cdots+d_r-r}>0\\ j_1,i_1,\dots,i_{d_1-1},j_2\\ \dots,i_{d_1+\cdots+d_r-r}\geq 0}} \binom{j_1}{k_1}\cdots\binom{j_r}{k_r} q^{m_1j_1+\cdots+m_rj_r+n_1i_1+\cdots+n_{d_1+\cdots+d_r-r}i_{d_1+\cdots+d_r-r}}.\end{aligned}$$

An analogous computation for $\zeta_q^{\mathrm{BZ}}(\mathbf{k}^\vee)$ gives now together with the duality relation the following statement which is equivalent to BZ-duality (proof: two generating series are equal iff their coefficients are equal):

Theorem 4.18. *For all $r \geq 1$, $k_1, \dots, k_r, d_1, \dots, d_r \geq 1$ and $N \geq 1$ we have:*

$$\begin{aligned}&\sum_{\substack{m_1>n_1>\cdots>n_{d_1-1}>m_2>\cdots>m_r>\cdots>n_{d_1+\cdots+d_r-r}>0\\ j_1,i_1,\dots,i_{d_1-1},j_2,\dots,i_{d_1+\cdots+d_r-r}\geq 0\\ m_1j_1+\cdots+m_rj_r+n_1i_1+\cdots+n_{d_1+\cdots+d_r-r}i_{d_1+\cdots+d_r-r}=N}} \binom{j_1}{k_1}\cdots\binom{j_r}{k_r}\\ = &\sum_{\substack{m_1>n_1>\cdots>n_{k_r-1}>m_2>\cdots>m_r>\cdots>n_{k_r+\cdots+k_1-r}>0\\ j_1,i_1,\dots,i_{k_r-1},j_2,\dots,i_{k_r+\cdots+k_1-r}\geq 0\\ m_1j_1+\cdots+m_rj_r+n_1i_1+\cdots+n_{k_r+\cdots+k_1-r}i_{k_r+\cdots+k_1-r}=N}} \binom{j_1}{d_r}\cdots\binom{j_r}{d_1}\end{aligned}$$

Moreover, this is equivalent to the duality of Bradley-Zhao qMZVs. □

This theorem is an excellent consequence of the duality since we get a pure combinatorial result.

It turns out that we can interpret both sides as the number of marked partitions of N: The left-hand sum is the number of partitions of N, where some (exactly $j_1 \geq 0$) rows of the 1st, some (exactly $j_2 \geq 0$) of the (d_1+1)-th part in the Young diagram are marked and so on.

When considering the Young diagrams of these partitions, it is useful to avoid the partitions where some $j_i = 0$ for clearness. Hence, we introduce also markings of the columns of the Young diagrams. We do this in the following way:

Proposition 4.19. *The LHS of Theorem 4.18 corresponds to the number of partitions of N, where the corresponding Young diagram is split up into r sub-Young diagrams with at most $d_1, \dots, d_r$ parts each. We mark k_j rows in the first part of sub-Young diagram j for each $1 \leq j \leq r$. Furthermore, we mark all columns containing corners and some of the others such that the number of colored columns, only belonging to sub-Young diagram j, in total is d_j for each $1 \leq j \leq r$.*

Proof. We obtain the split up into r sub-Young diagrams by the first line of the sum index of the LHS of Theorem 4.18. Also, the row markings are self-explaining when looking at the summand of our LHS-sum in Theorem 4.18.

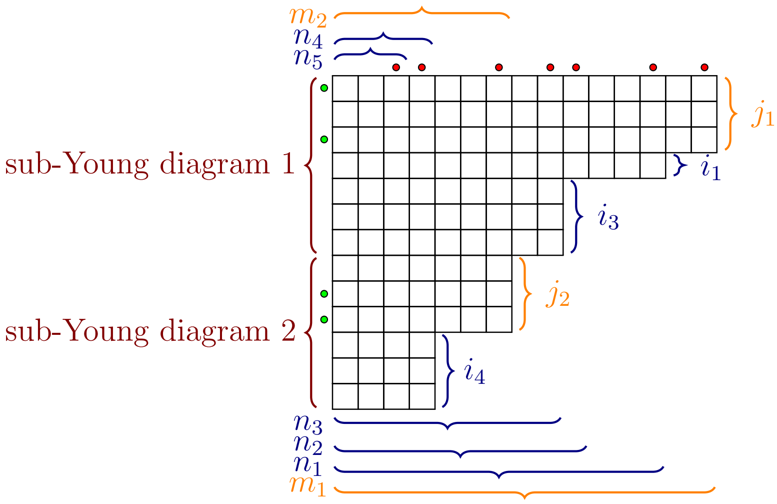
The marked columns represent the indices (from right to left) of shape j_l and i_l. If a marked column is not the rightmost one of a part, this corresponds to whether the corresponding multiplicity i_l is zero. In this case, there's no n_l-part in the partition, which is, on the one hand, the reason for having exactly d_j marked columns that belong only to sub-Young diagram j for each $1 \leq j \leq r$ and on the other hand, it is the reason why we have at most (and not exact) d_i parts in sub-Young diagram i. □

coloring the columns with corners gives us no additional information but is done due to explaining Theorem 4.18 on level of marked partitions in a nice way: Theorem 4.18 tells us that the number, P_N, of such considered partitions of N is the same as when substituting $(k_1, \dots, k_r, d_1, \dots, d_r)$ with $(d_r, \dots, d_1, k_r, \dots, k_1)$.

We need an example of how occurring marked partitions look like:

Example 4.20. The following marked partition of $N = 118$ has $r = 2$ sub-Young diagrams and is assigned to the index $\mathbf{k} = (3, 1, 1, 1, 3, 1, 1)$, i.e.

$$k_1 = k_2 = 2,\ d_1 = 4,\ d_2 = 3 :$$



Remark that n_3 occurs with multiplicity $i_2 = 0$, such as n_5 with multiplicity $i_5 = 0$ which is the reason that the columns corresponding to n_3 and n_5 respectively are marked but contain no corner of the Young diagram.

Denote the number of partitions of N mentioned in Proposition 4.19 as the left hand side of Theorem 4.18 by

$$\mathcal{Q}_N(k_1, \ldots, k_r; d_1, \ldots, d_r).$$

Definition 4.21. Write $\mathcal{Q}_N(k_1, \ldots, k_r; d_1, \ldots, d_r)$ for the number of triples (λ, R, C) where

- λ is a partition of N such that the corresponding representation of λ in Stanley coordinates, $(\mathbf{m}, \mathbf{n})$, has length $\leq d_1 + \cdots + d_r$,
- $R \subset \{1, \ldots, \lambda_h\}$ is a set of colored rows of λ with

  $$R_j := R \cap \{n_1 + \cdots + n_{d_{j-1}-1} + 1, \ldots, n_1 + \cdots + n_{d_j - 1}\}$$

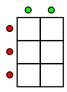
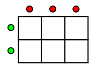
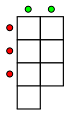
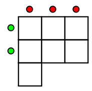
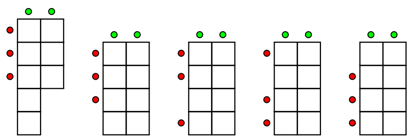
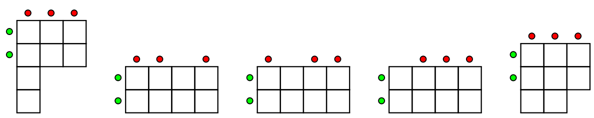
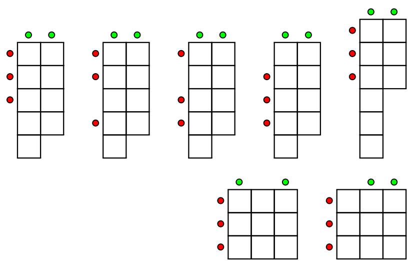
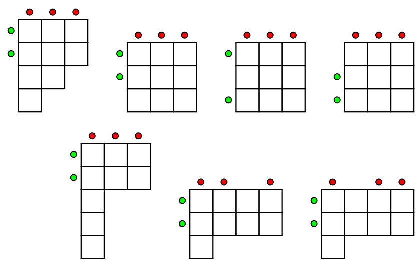
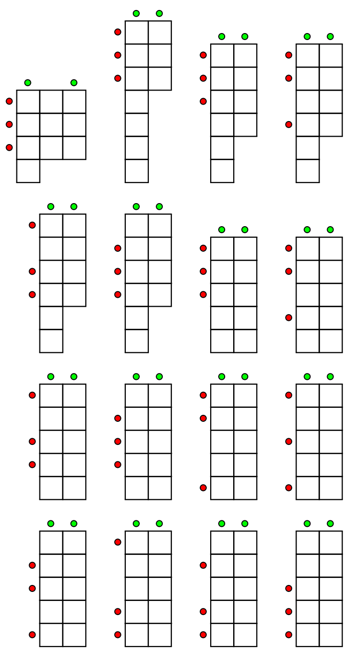
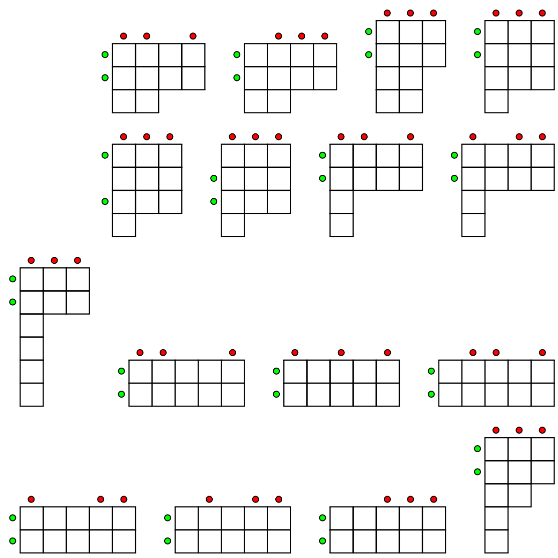
  has the property $|R_j| = k_j$ for all $1 \leq j \leq r$.
- $C \subset \{1, \ldots, \lambda_1'\}$ is a set of colored columns of λ with

  $$C_j := C \cap \{m_{j-1} + 1, \ldots m_j\}$$

  has the properties $|C_j| = d_j$ and $m_j \in C_j$ for all $1 \leq j \leq r$.

For illustration, we give another example:

Example 4.22. Consider the $r = 1$ case with $k_1 = 3$ and $d_1 = 2$ and small N (for $N < 6$ we have 0 partitions):

N	$\mathcal{Q}_N(3;2)$	$\mathcal{Q}_N(2;3)$
6		
7		
8		
9		
10		

One open question is how to construct, in general, a bijection between the partitions of N of both sides.

Lemma 4.23. *From the $r = 1$ case of BZ-duality we get for all $N \in \mathbb{N}$*

$$\sum_{m,n,r>0} \binom{n}{k}\binom{m-r}{d} a_r(N;m,n) = \sum_{m,n,r>0} \binom{n}{d}\binom{m-r}{k} a_r(N;m,n)$$

for arbitrary $k, d \geq 1$, where we denote by $a_r(N;m,n)$ the number of partitions of N with exactly r parts such that the first part is a $(n \times m)$-block.

Proof. Follows from Theorem 3.10 by setting $r = 1$ and comparing there the coefficient of $X^d Y^k$. □

Remark 4.24. Remark that the LHS denotes the number of partitions of a natural number N where we color in the corresponding Young-diagram k of the rows of the first part and d of the columns containing no corner. Lemma 4.23 says that the number of such partitions is invariant under interchanging k and d, which is a non-trivial statement.

Remark 4.25. In contrast to Remark 4.17(iii) and Lemma 4.13, the products of binomial coefficients of the LHS in Theorem 4.18 are in general not the same as the one of the RHS. That makes Theorem 4.18 into a highly non-trivial combinatorial result.

For example, Theorem 4.18 states for $r = 2$, $N = 7$, $d_1 = 1$, $d_2 = 2$, $k_1 = 1$, $k_2 = 1$

$$\sum_{\substack{m_1>m_2>n_1>0\\ j_1,j_2,i_1\geq 0\\ m_1j_1+m_2j_2+n_1i_1=7}} \binom{j_1}{1}\binom{j_2}{1} = \underbrace{\binom{1}{1}\binom{2}{1} + \binom{1}{1}\binom{1}{1}}_{m_1=3} + \underbrace{2\binom{1}{1}\binom{1}{1}}_{m_1=4} + \underbrace{\binom{1}{1}\binom{1}{1}}_{m_1=5}$$

$$= 7 = \underbrace{\binom{3}{2}\binom{1}{1}}_{m_1=2} + \underbrace{\binom{2}{2}\binom{3}{1}}_{m_1=3} = \sum_{\substack{m_1>m_2>0\\ j_1,j_2\geq 0\\ m_1j_1+m_2j_2=7}} \binom{j_1}{2}\binom{j_2}{1}.$$

4.4. Partition function & SZ-qMZVs

When studying SZ-qMZVs and special values of them, we get a connection to the partition numbers:

Lemma 4.26. *Let be p_N the number of partitions of N, then*

$$\sum_{r\geq 1} \zeta_q^{SZ}(\{1\}^r) = \sum_{r\geq 1} \mathrm{g}\binom{\{1\}^r}{\{0\}^r} = \sum_{N\geq 1} p_N q^N.$$

Proof. The first equality is clear by the definition of bi-brackets. We consider now the left side first:

$$\sum_{r\geq 0} \zeta_q^{\mathrm{SZ}}(\{1\}^r) = \sum_{r\geq 0} \sum_{m_1>\cdots>m_r>0} \frac{q^{m_1}}{1-q^{m_1}} \cdots \frac{q^{m_r}}{1-q^{m_r}} = \sum_{r\geq 0} \sum_{\substack{m_1>\cdots>m_r>0\\ n_1,\dots,n_r>0}} q^{m_1n_1+\cdots+m_rn_r}$$

The coefficient of some q^N here is the sum over all $r \in \mathbb{N}_0$, where we sum the number of all partitions of N with exact r different parts, i.e. the number of partitions of N, p_N. □

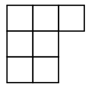
4.5. Number of conjugacy classes of $\mathcal{S}_n$

The partition function also occurs in other contexts than qMZVs, namely when considering equivalence classes of the symmetric group $\mathcal{S}_n$. For more details, we refer to [FH, §4].

Lemma 4.27. *Partitions of $n \in \mathbb{N}$ and conjugacy classes of $\mathcal{S}_n$ are in 1:1-correspondence. In particular, the number of conjugacy classes of $\mathcal{S}_n$ is p_n.*

Proof. Write every $\sigma \in \mathcal{S}_n$ as union of cycles. The length of the cycles form a partition of n. Since a conjugacy class $[\sigma]$ of $\mathcal{S}_n$ is uniquely determined by the lengths of cycles of σ - conjugacy means only to rename the elements $1, \dots, n$, but not to change the structure of σ - the claim follows. □

Example 4.28. The conjugacy class of $\sigma = (1\,4\,3)(2\,6)(5\,7) \in \mathcal{S}_7$ corresponds to the partition

 of $3 + 2 + 2 = 7$.

Remark 4.29. Lemmas 4.26 and 4.27 give a remarkable connection between the number of conjugacy classes of $\mathcal{S}_n$ and SZ-qMZVs. More precisely, fixing r and n, the coefficient of q^n in $\zeta_q^{\mathrm{SZ}}(\{1\}^r)$ is the number of conjugacy classes of $\mathcal{S}_n$ with cycles of exactly r different lengths.

4.6. Number of conjugacy classes of GL(n,K)

The last remark should be the motivation for this section since the main point of this section is the following:

Theorem 4.30. *Let K be a finite field with c elements. Then it is*

$$\mathrm{G}_K := \sum_{n \geq 0} a_{n,K} q^n = \sum_{r \geq 0} (c-1)^r \zeta_q^{OOZ}(\{1\}^r),$$

where $a_{n,K}$ is the number of conjugacy classes of $GL(n, K)$ and where we set $a_{0,K} := 1$ for every field K.

For the proof, we count the conjugacy classes according to the theory of the Frobenius normal form as taught in most of Linear Algebra courses (cf. [Bos, §6], [Fis, §2.5.3]). This will lead to marked partitions, where every column in a partition is colored with one of c resp. $(c-1)$ colors. We will get in this way three different representations of $a_{n,K}$ (Prop. 4.32, 4.36, 4.37). The last one then allows us to prove Theorem 4.30.

In general, if we consider a square matrix $M \in \mathrm{Mat}(n \times n, K)$ over some field K (we will restrict to K a finite field), its conjugacy class has a unique representant of the form

$$M_{g_1,\ldots,g_r} := \begin{pmatrix} B_{g_1} & & 0 \\ & \ddots & \\ 0 & & B_{g_r} \end{pmatrix}$$

for some $r \in \mathbb{N}$, where $g_1, \ldots, g_r \in K[X]$ are monic polynomials with $g_i | g_{i+1}$ for all $1 \leq i < r$ and B_g for some polynomial $g(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_0$ is the $(m \times m)$-square matrix

$$B_g := \begin{pmatrix} 0 & & & -a_0 \\ 1 & & & -a_1 \\ & \ddots & & \vdots \\ & & 1 & -a_{m-1} \end{pmatrix}.$$

In particular, M is non-singular if and only if all the g_i have non-vanishing constant term. Because B_g is a $(\deg(g) \times \deg(g))$-matrix, we obtain for every $M_{g_1,\ldots,g_r}$ that

$$\sum_{i=1}^{r} \deg(g_i) = n. \tag{4.2}$$

In what now follows, we refer to [FF] (can be also found in [Mac]). Because of the divisor condition on the g_i, these tuples of $(g_1, \ldots, g_r)$ are in 1:1-correspondence to the tuples $(h_1, \ldots, h_r)$ of monic polynomials $h_i \in K[X]$ with

$$\sum_{i=1}^{r} i \deg(h_i) = n \tag{4.3}$$

via $h_i := \frac{g_{r+1-i}}{g_{r-i}}$ and $g_i := \prod_{j=1}^{i} h_{r+1-j}$ respectively.

With the restriction to $\mathrm{GL}(n, K)$ (i.e. all g_i have non-vanishing constant term), the mentioned correspondence restricts to the set of the same tuples of polynomials $(h_1, \ldots, h_r)$ with the additional condition that all h_i also have non-vanishing constant terms:

Define the equivalence relation $A \sim B :\Leftrightarrow$ (A and B are conjugate) on $\mathrm{GL}(n, K)$.

Proposition 4.31 (Theory of normal forms, [Bos, §6], [Fis, §2.5.3]). *For all $n \in \mathbb{N}$ we have*

$$\begin{aligned}\mathrm{GL}(n,K)/\sim \;&\overset{1:1}{\longleftrightarrow} \{(g_1,\dots,g_r) \in K[X]^r \textit{ monic}: g_1|\dots|g_r,\ g_1(0),\dots,g_r(0) \neq 0, \\ &\qquad\qquad \sum_{i=1}^r \deg(g_i) = n,\ r \geq 1\} \\ &\overset{1:1}{\longleftrightarrow} \{(h_1,\dots,h_r) \in K[X]^r \textit{ monic}: h_1(0),\dots,h_r(0) \neq 0, \\ &\qquad\qquad \sum_{i=1}^r i \deg(h_i) = n,\ r \geq 1\}.\end{aligned}$$

□

The last identification is obtained via $h_i := \frac{g_{r+1-i}}{g_{r-i}}$ and $g_i := \prod_{j=1}^{i} h_{r+1-j}$ respectively.

At this point, we remark that (4.3) gives that every $(\deg(h_i))_{1\leq i\leq r}$ induces a partition of n with $\leq r$ parts where $\deg(h_i)$ is the multiplicity of i. The coefficients of $X^1,\dots,X^{\deg(h_i)-1}$ can be any element in K (independent of each other), the one of X^0 too with the condition that it is not zero if $\deg(h_i) > 0$, since else h_i is constant and has to be constant 1 because the h_i are monic. I.e. if K has c elements, this means that we have $c^{m-1}(c-1)$ possible choices of h_i when m denotes the (fixed) degree of h (cf. [FF]) for $m > 0$ and exactly 1 if $m = 0$. In particular, the number of conjugacy classes $a_{n,K}$ of $\mathrm{GL}(n,K)$ is the sum over all partitions of n, every one occurring with multiplicity $c^{j_1-1+\dots+j_r-1}(c-1)^r$, where r is the number of parts:

Proposition 4.32. *Let be K a finite field with c elements. Then we have for every $n \in \mathbb{N}$*

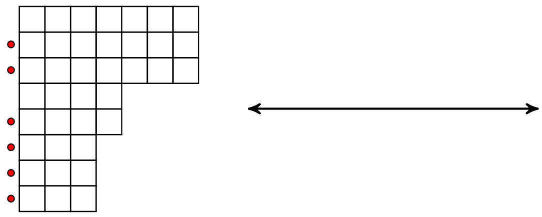
$$a_{n,K} = \sum_{r\geq 1} \sum_{\substack{m_1>\dots>m_r>0 \\ j_1,\dots,j_r\geq 1 \\ m_1j_1+\dots+m_rj_r=n}} c^{j_1+\dots+j_r-r}(c-1)^r.$$

□

We translate what we have done into the language of marked partitions: Take a set of pairwise different colors $\{0,\dots,c-1\}$, call the color 0 'white'. $a_{n,K}$ is then the number of all marked partitions of n such that rows containing corners can be colored arbitrary but not white, all other rows arbitrary. In particular: Given a partition $((m_1,\dots,m_r),(j_1,\dots,j_r))$ of n, we color the jth bottom-most row of part i with color l if and only if the coefficient of X^j in $h_i(X)$ is l. Remark that rows containing corners can not be white since the marking of the row in part i with corner corresponds to the constant coefficient of $h_i(X)$ that can't be 0.

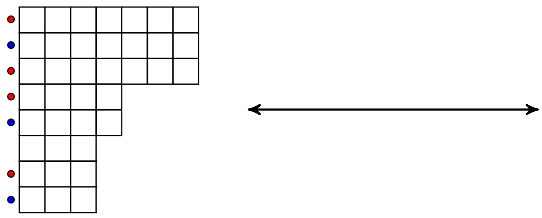
Lemma 4.33. *This 1:1-correspondence between the considered marked partitions of n and tuples of monic polynomials $(h_i)_{1\le i\le r} \in K[X]^r$ with non-vanishing constant coefficient and $\sum_{i=1}^{r} i\deg(h_i) = n$ gives because of Proposition 4.31 a 1:1-correspondence between the considered marked partitions and conjugacy classes of $\mathrm{GL}(n,K)$.* □

Example 4.34. In $K = \mathbb{F}_2$ we color with white (corresponds to 0) and some other color, say red (corresponds to 1). Then for example the following marked partition of $n = 38$ corresponds to the tuple of polynomials $(h_i)_{1\le i\le 7}$ on the right side:



$$\begin{array}{ccl} h_7(X) & = & X^3+X+1 \\ h_4(X) & = & X^2+1 \\ h_3(X) & = & X^3+X^2+X+1 \\ h_1(X) = & h_2(X) & = h_5(X) = h_6(X) = 1 \end{array}$$

Example 4.35. In $K = \mathbb{F}_3$ we color with white (corresponds to 0) and two more colors, say red (for 1) and blue (for 2). Then for example the following marked partition of $n = 38$ corresponds to the tuple of polynomials $(h_i)_{1\le i\le 7}$ on the right side:



$$\begin{array}{ccl} h_7(X) & = & X^3+X^2+2X+1 \\ h_4(X) & = & X^2+X+2 \\ h_3(X) & = & X^3+X+2 \\ h_1(X) = & h_2(X) & = h_5(X) = h_6(X) = 1 \end{array}$$

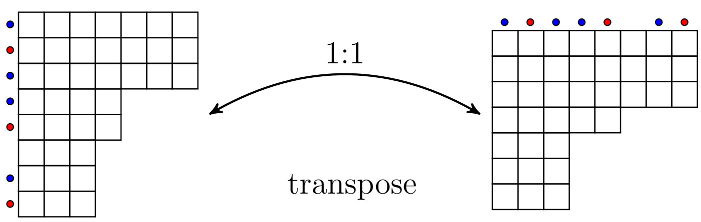
Now, ρ, the map on the set of partitions mapping a partition to the one with transposed Young-diagram is an involution on $\mathcal{P}(n)$ for every $n \in \mathbb{N}$. Therefore, the considered marked partitions are in 1:1-correspondence with the marked partitions with column markings, we get when transposing the Young diagram together with the markings.

I.e. conjugacy classes of $\mathrm{GL}(n,K)$ are in 1:1-correspondence to marked partitions of n such that

(i) columns are colored with one of $c := |K|$ colors $\{0,\dots,c-1\}$,

(ii) columns containing corners are not colored white.

The following picture illustrates this step of transposing the marked partitions:



This gives a reformulation of Proposition 4.32:

Proposition 4.36. *Let be K a finite field with c elements. Then for every $n \in \mathbb{N}$ the number of conjugacy classes of* $\mathrm{GL}(n, K)$ *is*

$$a_{n,K} = \sum_{r\geq 1} \sum_{\substack{m_1>\cdots>m_r>0\\ j_1,\ldots,j_r\geq 1\\ m_1j_1+\cdots+m_rj_r=n}} c^{m_1-r}(c-1)^r.$$

□

Now, count the partitions differently: So far, we sorted partitions by the number of parts r. But now we will sort them by the number of columns that are *not* white colored. If a column contains a corner, it is always not allowed to be colored white, so this is no extra information. If a column, say the m-th left most one in the Young-diagram, contains no corner and is one of the not-white colored columns, we can say that our given partition consists of some extra part of length m with multiplicity 0. Since we can use for the coloring of this m-th column all colors but not white, we get some multiplicity $(c-1)$ for this modified partition. In other words:

Proposition 4.37. *Let be K a finite field with c elements. Then we have for every $n \in \mathbb{N}$*

$$a_{n,K} = \sum_{r\geq 1} \sum_{\substack{m_1>\cdots>m_r>0\\ j_1>0,j_2,\ldots,j_r\geq 0\\ m_1j_1+\cdots+m_rj_r=n}} (c-1)^r.$$

□

Proposition 4.37 allows us now to prove Theorem 4.30:

Proof of Thm. 4.30. Denote by $\delta_\bullet$ the usual Kronecker delta, which is 1 if and only if the condition $\bullet$ in the index holds and 0 else. From Proposition 4.37 we obtain for G_K the formula

$$\begin{aligned}
\mathrm{G}_K &= 1 + \sum_{n\geq 1} a_{n,K} q^n \\
&= 1 + \sum_{n\geq 1} \left(\sum_{r\geq 1} \sum_{\substack{m_1>\cdots>m_r>0\\ j_1>0,j_2,\ldots,j_r\geq 0\\ m_1j_1+\cdots+m_rj_r=n}} (c-1)^r \right) q^n \\
&= 1 + \sum_{r\geq 1} \sum_{\substack{m_1>\cdots>m_r>0\\ j_1>0,j_2,\ldots,j_r\geq 0}} (c-1)^r q^{m_1j_1+\cdots+m_rj_r}
\end{aligned}$$

$$= 1 + \sum_{r \geq 1} (c-1)^r \sum_{m_1 > \cdots > m_r > 0} \prod_{l=1}^{r} \left(\delta_{l \neq 1} \sum_{j_l \geq 1} q^{m_l j_l} \right)$$
$$= 1 + \sum_{r \geq 1} (c-1)^r \sum_{m_1 > \cdots > m_r > 0} \frac{q^{m_1}}{1-q^{m_1}} \frac{1}{1-q^{m_2}} \cdots \frac{1}{1-q^{m_r}}$$
$$= \sum_{r \geq 0} (c-1)^r \zeta_q^{\mathrm{OOZ}}(\{1\}^r),$$

which is exactly the statement of Theorem 4.30. □

At this point we should remark that G_K (and hence $\sum_{r \geq 0} (c-1)^r \zeta_q^{\mathrm{OOZ}}(\{1\}^r)$) for a finite field K not only has three quite nice representations as infinite sum, but that there is furthermore a representation as infinite product:

Lemma 4.38 ([FF], [Mac]). *Let be K a finite field with c elements. Then it is*

$$\mathrm{G}_K = \prod_{k \geq 1} \frac{1-q^k}{1-cq^k}.$$

Proof. The proof follows the lines of [FF, Mac] and uses Proposition 4.32 for some direct calculation in compliance with Cauchy products ([Apo, Kno, MZZ]) and geometric series:

$$\mathrm{G}_K \overset{Prop.4.36}{=} 1 + \sum_{k \geq 1} \sum_{\substack{j_1, \ldots, j_k > 0 \\ m_1 > \cdots > m_k > 0}} q^{m_1 j_1 + \cdots + m_k j_k} \cdot c^{j_1 + \cdots + j_k - k} (c-1)^k$$
$$= \prod_{k \geq 1} \left(1 + (c-1) \sum_{l \geq 1} c^{l-1} q^{kl} \right)$$
$$= \prod_{k \geq 1} \frac{1-q^k}{1-cq^k}.$$

□

Remark 4.39. (i) We defined c to be the number of elements in some finite field. Hence, c is always the power of a prime, and in particular, c can not take all natural number values.

(ii) Nevertheless, all of our computations are correct under the assumption that we want to compute the number respective the generating series of the numbers of the considered marked partitions with c colors for arbitrary $c \in \mathbb{N}$. This means that for all $c \in \mathbb{N}$ we have (either for $c \geq 2$ only, or with $0^0 := 1$ for the $(c = 1)$-case)

$$\sum_{r \geq 0} (c-1)^r \zeta_q^{\mathrm{OOZ}}(\{1\}^r) = \prod_{k \geq 1} \frac{1-q^k}{1-cq^k}$$

$$= 1 + \sum_{r \geq 1} \sum_{\substack{m_1 > \cdots > m_r > 0 \\ j_1, \ldots, j_r \geq 1 \\ m_1 j_1 + \cdots + m_r j_r = n}} c^{j_1 + \cdots + j_r - r} (c-1)^r = 1 + \sum_{r \geq 1} \sum_{\substack{m_1 > \cdots > m_r > 0 \\ j_1, \ldots, j_r \geq 1 \\ m_1 j_1 + \cdots + m_r j_r = n}} c^{m_1 - r} (c-1)^r$$

$$= 1 + \sum_{r \geq 1} \sum_{\substack{m_1 > \cdots > m_r > 0 \\ j_1 > 0, j_2, \ldots, j_r \geq 0 \\ m_1 j_1 + \cdots + m_r j_r = n}} (c-1)^r.$$

Outlook

This chapter briefly summarises the open questions arising from the thesis, some of which could be exciting for future research. Some of these have already been raised in the thesis itself, while others have not yet been.

We have considered various models of qMZVs in chapter 2 and chapter 3, particularly regarding existing duality relations. On the one hand, this raises the question of whether duality relations also exist in other models. On the other hand, one can ask whether there are other relations in the models - both considered and not considered - which can be classified as duality relations. We ask ourselves whether there are other relations in the models that come from a 'nice' anti-automorphism on the underlying algebra. In particular, we are interested in those that do not come by translating into the BZ or SZ-model, applying the known duality there, and translating back, as we have seen for the OOZ-model (Thm. 3.24).

In addition to the dualities in different models considered so far, the question of whether there is a duality of qMZVs independent of the model suggests itself. More precisely, we mean by this whether we can find for each qMZV

$$\zeta_q(k_1, \ldots, k_r; P_1, \ldots, P_r) = \sum_{m_1 > \cdots > m_r > 0} \frac{P_1(q^{m_1})}{(1-q^{m_1})^{k_1}} \cdots \frac{P_r(q^{m_r})}{(1-q^{m_r})^{k_r}}$$

non-negative integers $l_1, \ldots, l_s \geq 0$ and polynomials $Q_1, \ldots, Q_s$ with $\deg(Q_j) \leq l_j$ and $Q_1(0) = 0$ such that

$$\zeta_q(k_1, \ldots, k_r; P_1, \ldots, P_r) = \zeta_q(l_1, \ldots, l_s; Q_1, \ldots, Q_s).$$

Of course, we can rewrite every element of $\mathcal{Z}_q$ in terms of SZ-qMZVs and then apply SZ-duality. However, this would lead to sums of SZ-qMZVs in different depths, which is, in general, not expressable as an $\zeta_q(l_1, \ldots, l_s; Q_1, \ldots, Q_s)$.

It would be of interest to give $l_1, \ldots, l_s$ explicit and in dependence of $k_1, \ldots, k_r$ such as $Q_1, \ldots, Q_s$ in dependence of $P_1, \ldots, P_r$. Furthermore, after finding such a duality relation, it would be interesting to find closed subspaces of $\mathcal{Z}_q$ of this duality in comparison to $\mathcal{Z}_{q,1}$ and BZ-duality.

Back to the models and their dualities: Here, too, questions arise which have not yet been clarified. Thus, it remains an as yet unachieved goal to prove BZ duality via generating series of BZ-qMZVs, in contrast to SZ duality and the partition relation of bi-brackets. If, as mentioned above, further duality relations are found in (possibly also other than the considered) models, we expect a rather simple proof via generating series for these as well.

The motivation for TBZ-qMZVs was that the BZ model does not span the whole $\mathcal{Z}_q$, but only a real subspace of it. Thus, the further, still open question is whether BZ-duality can be generalised to the TBZ model, i.e. we are looking for anti-automorphisms

$$\tau_{TBZ} : \mathfrak{h}^{TBZ} \longrightarrow \mathfrak{h}^{TBZ}$$

with $\mathfrak{h}^{TBZ} := \mathbb{Q}\langle z_{\bar{1}}, z_1, z_2, \dots \rangle$ such that $\zeta_q^{\mathrm{TBZ}} \circ \tau_{TBZ} = \zeta_q^{\mathrm{TBZ}}$ and

$$\tau_{TBZ}(z_{\mathbf{k}}) = z_{\mathbf{k}^\vee}$$

for all admissible indices $\mathbf{k}$. More general, we are looking for extensions of the BZ-model to $\mathcal{Z}_q$ together with a duality relation that extends BZ-duality.

At this point, we remark that duality relations among qMZVs are not only interesting as self-purpose but also because they give pure combinatorial results: Every qMZV is a generating series of sums of products of binomial coefficients. A duality relation leads to equalities of such sums by comparing coefficients. Exactly, these results we obtained for SZ-duality (Lem. 4.13 and for BZ-duality (Thm. 4.18).

In chapter 4 were concerned with the new concept of marked partitions, where we were allowed to color rows and columns of Young diagrams of partitions. In this way, it was possible to prove SZ-duality vividly via those marked partitions (Prop. 4.14, Lem. 4.13, Rem. 4.17). Also, BZ-qMZVs could be expressed as generating series of certainly marked partitions (Prop. 4.19). But we were not able to find the map between those marked partitions of a BZ-qMZV and those of its dual BZ-qMZV. Hence, finding this map could be a research project for the future.

Of course, the question arises where else the concept of marked partitions can be used. We have already considered one such example in §4.6, namely we could assign a particularly marked partition to each conjugacy class of $GL(n, K)$, where K is a finite field.

We end this outlook with a conjecture by Bachmann (not published):

Conjecture 4.1 ([Ba7])**.** *We have*

$$\begin{aligned} &\langle \zeta_q^{BZ}(k_1, \dots, k_r) : \ r \geq 0, \, k_1 \geq 2, k_2, \dots, k_r \geq 1 \rangle_{\mathbb{Q}} \\ =&\langle \zeta_q^{BZ}(k_1, \dots, k_r) : \ r \geq 0, \, k_1, \dots, k_r \geq 2 \rangle_{\mathbb{Q}}. \end{aligned}$$

One idea to prove this conjecture is using connected sums. However, it seems to be a more difficult problem and hence is postponed to the future.

A. A unified approach to qMZVs

This appendix gives a self-contained introduction to q-analogues of multiple zeta values (qMZVs) and was written in the context of the 'Vorbereitungsprojekt', a part of the master's programme. We first recall knowledge about multiple zeta values and their connection to quasi-shuffle algebras before we introduce qMZVs in the second section (see [Zh2] for an overview). There, we consider after some general definition (cf. [Ba6], [Zh3]) most common models of qMZVs and the space $\mathcal{Z}_q$ containing all qMZVs we consider. We will see that one model, the one by Bradley and Zhao, gives not whole $\mathcal{Z}_q$, but an important subalgebra because this model is closely related to MZVs. Another model, bi-brackets, introduced by Bachmann, allows defining a filtration nicely by weight and depth of $\mathcal{Z}_q$. For the associated graded subspaces of $\mathcal{Z}_q$, Bachmann and Kühn gave dimension conjectures ([BK2]) similar to the Broadhurst-Kreimer conjecture that will be considered at the end of the appendix, together with similar observations deduced from the model of q-analogues introduced by Okounkov.

A.1. Multiple Zeta Values

In the following, we introduce multiple zeta values (MZVs), such as famous theorems and conjectures about MZVs, to indicate why they are interesting objects. We refer to [Ba6], [BF], [Wal] and [Zu1] for several overviews of MZVs.

A.1.1. Basics

First of all, we have to clarify the term of a multiple zeta value and some related notation:

Definition A.1 (*Multiple Zeta Value (MZV)*)**.** For an admissible index $\mathbf{k} = (k_1, \ldots, k_r)$ its multiple zeta value is defined as

$$\zeta(\mathbf{k}) := \zeta(k_1, \ldots, k_r) := \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1}} \cdots \frac{1}{m_r^{k_r}},$$

for $r > 0$ and $\zeta(\emptyset) := 1$, where

(i) for $r \geq 0$, a tuple $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{N}_0^r$ is called *index*; for $r = 0$ we write $\mathbf{k} = \emptyset$,

(ii) and an index $\mathbf{k} = (k_1, \ldots, k_r)$ with either $r = 0$ or $k_1 \geq 2$, $k_2, \ldots, k_r \geq 1$ is called an *admissible index.*

Furthermore, we say that

$$\begin{aligned}\mathrm{wt}(\mathbf{k}) &:= k_1 + \cdots + k_r \text{ is the } \textit{weight} \text{ of } \mathbf{k} \text{ and}\\ \mathrm{depth}(\mathbf{k}) &:= r \text{ is the } \textit{depth} \text{ of } \mathbf{k}.\end{aligned}$$

Remark A.2.

(i) The pure definition of MZVs as iterated sums gives that we can rewrite products of single zeta values: For all $k_1,\, k_2 \geq 2$ we have

$$\zeta(k_1)\zeta(k_2) = \zeta(k_1 + k_2) + \zeta(k_1, k_2) + \zeta(k_2, k_1).$$

(ii) Sometimes, it is more convenient to have the sum index not strictly ordered. The corresponding sum is called *multiple zeta star value (MZSV)*: For every admissible index $\mathbf{k} = (k_1, \ldots, k_r)$ the multiple zeta star value of $\mathbf{k}$ is defined as

$$\zeta^\star(\mathbf{k}) := \zeta^\star(k_1, \ldots, k_r) := \sum_{m_1 \geq \cdots \geq m_r > 0} \frac{1}{m_1^{k_1}} \cdots \frac{1}{m_r^{k_r}}$$

for $r > 0$ and $\zeta(\emptyset) := 1$.

In particular, every multiple zeta star value is a finite sum of multiple zeta values.

We focus mainly on MZVs and not on MZSVs, (q-analogues of) MZSVs will be needed when considering the Ohno-Okuda-Zudilin model of qMZVs (§A.2.5). A more detailed description of MZSVs can be found e.g. in [OZ], [LZ] or [EMS].

(iii) A generalized version of MZSVs are *Schur MZVs*. They are interesting objects since they are, via their definition, closely related to partitions and combinatorics. For more details on Schur MZVs we refer to [NPY], [BY], [BC].

The definition of MZVs makes sense as the following proposition shows:

Proposition A.3 ([Ba6, Prop. 1.4])**.** *For every admissible index* $\mathbf{k}$ *the defining sum of the MZV* $\zeta(\mathbf{k})$ *converges absolutely.*

Proof. We have for $\mathbf{k} = (k_1, \ldots, k_r)$:

$$\begin{aligned}\zeta(\mathbf{k}) &= \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1}} \cdots \frac{1}{m_r^{k_r}} \leq \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^2 m_2 \ldots m_r}\\ &= \sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{m > m_2 \cdots > m_r > 0} \frac{1}{m_2 \ldots m_r} \leq \sum_{m=1}^{\infty} \frac{1}{m^2}(1 + \log(m))^{r-1}\end{aligned}$$

and since $(1 + \log(m))^{r-1} = \mathrm{o}(\sqrt{m})$ for every $r \geq 1$, the sum converges absolutely. □

As in the introduction mentioned, we are interested in $\mathbb{Q}$-linear relations among multiple zeta values since with them we can state much about algebraic relations among single zeta values. In (0.1) we got in touch with the so-called *stuffle product* (this will get important at a later point when studying word algebras describing MZVs). By considering the *shuffle product*, we get first linear relations among MZVs:

Proposition A.4. *For every $k_1, k_2 \geq 2$ holds*

$$\zeta(k_1)\zeta(k_2) = \sum_{j=2}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) \zeta(j, k_1+k_2-j). \tag{A.1}$$

□

Proposition A.5. *For $k_1, k_2 \geq 2$ the (finite) double shuffle relations are valid, looking for example like*

$$\zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1+k_2) = \sum_{j=2}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) \zeta(j, k_1+k_2-j).$$

□

Another - non-trivial - relation between multiple zeta values is for example for every $n \geq 1$

$$\zeta(\{2,1\}^n) = \zeta(\{3\}^n), \tag{A.2}$$

where the notation $\{\mathbf{k}\}^n$ means that the index $\mathbf{k}$ is repeated n-times.

For $n = 1$ this can be proven elementary via

$$\begin{aligned}
2\zeta(2,1) &= \sum_{m,n>0} \frac{1}{m(m+n)^2} + \sum_{m,n>0} \frac{1}{n(m+n)^2} = \sum_{m,n>0} \left(\frac{1}{m} + \frac{1}{n} \right) \frac{1}{(m+n)^2} \\
&= \sum_{m,n>0} \frac{1}{mn(m+n)} = \sum_{m,n>0} \frac{1}{n^2} \left(\frac{1}{m} - \frac{1}{m+n} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{m=1}^{n} \frac{1}{m} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^3} + \sum_{n>m>0} \frac{1}{n^2 m} = \zeta(3) + \zeta(2,1).
\end{aligned}$$

To understand the algebraic structure of MZVs better, we introduce

$$\mathcal{Z} := \langle \zeta(\mathbf{k}) \mid \mathbf{k} \text{ admissible} \rangle_{\mathbb{Q}},$$

the $\mathbb{Q}$-vector space of MZVs and for every $k \geq 0$ we define

$$\mathcal{Z}_k := \langle \zeta(\mathbf{k}) \mid \mathbf{k} \text{ admissible of weight } k \rangle_{\mathbb{Q}}.$$

One obtains that $\mathcal{Z}$ is a $\mathbb{Q}$-algebra with the usual multiplication.

Now, $\mathcal{Z} = \sum_{k \geq 0} \mathcal{Z}_k$ is obvious, but there is a much more stronger conjecture about the algebraic structure of MZVs:

Conjecture A.6. *$\mathcal{Z}$ is graded by weight, i.e.*

$$\mathcal{Z} = \bigoplus_{k\geq 0} \mathcal{Z}_k.$$

A consequence of this conjecture would be that every $\mathbb{Q}$-linear relation among MZVs splits up into corresponding relations, where each MZV has the same weight. Examples for such relations we give e.g. in Propositions A.4, A.5, in (A.2) and in theorem A.20 too. In particular, we would get that $\zeta(3)$ is transcendental (we will see later $\zeta(3) = \zeta(2, 1)$ and so, there is only one MZV in weight 3).

Define now the sequence $(d_k)_{k\geq 0}$ via

$$\sum_{k\geq 0} d_k X^k = \frac{1}{1 - X^2 - X^3}.$$

Then another conjecture of Zagier states that the dimension of $\mathcal{Z}_k$ is exactly d_k:

Conjecture A.7 (Zagier)**.** *For every $k \geq 0$*

$$\dim_{\mathbb{Q}}(\mathcal{Z}_k) = d_k.$$

Considering the recursive relations among the d_k, this conjecture goes hand in hand with a conjecture by Hoffman about an explicit basis of $\mathcal{Z}_k$:

Conjecture A.8 (Hoffman)**.** *For every $k \geq 0$ a basis of $\mathcal{Z}_k$ is given by*

$$\{\zeta(k_1, \dots, k_r) \mid r \geq 0,\ k_i \in \{2, 3\},\ k_1 + \cdots + k_r = k\}.$$

Theorem A.9 (Terasoma (2002) [Ter], Deligne-Goncharov (2005) [DG])**.**
For $k \geq 0$ we have $\dim_{\mathbb{Q}}(\mathcal{Z}_k) \leq d_k$. □

Brown gave a refinement of Theorem A.9:

Theorem A.10 (Brown [Bro])**.** *Each MZV is a linear combination of MZVs with only 2s and 3s as arguments, i.e.*

$$\{\zeta(k_1, \dots, k_r) \mid r \geq 0,\ k_i \in \{2, 3\},\ k_1 + \cdots + k_r = k\}$$

generates $\mathcal{Z}_k$. □

We can extend the double shuffle relations to relations in the polynomial ring $\mathcal{Z}[T]$, which gives expressions like '$\zeta(1, k_2, \dots, k_r)$' some sense. These so-called *extended double shuffle relations* give conjecturally all $\mathbb{Q}$-linear relations among MZVs. A more detailed description of them is given in Proposition A.24.

Now, roughly spoken, there are two ways to consider multiple zeta values: On the one hand as iterated sums and on the other as iterated integrals. Of course, they overlap in many points, but let us explain the main ideas of both ways.

MZVs are defined as iterated sums as done above, so let us focus first on the second way:

Viewing MZVs as iterated integrals is done via the following theorem that claims the so-called Kontsevich integral representation of MZVs:

Theorem A.11. *For every $r \geq 1$ and $k_1 \geq 2$, $k_2, \ldots, k_r \geq 1$, we have:*

$$\zeta(k_1, \ldots, k_r) = \int\limits_{1>t_1>\cdots>t_k>0} \omega_1(t_1) \ldots \omega_k(t_k),$$

where $k := k_1 + \cdots + k_r$ and

$$\omega_i(t) := \begin{cases} \frac{dt}{1-t}, & \text{if } i \in \{k_1, k_1 + k_2, \ldots, k_1 + \cdots + k_r\} \\ \frac{dt}{t}, & \text{else} \end{cases}.$$

Example A.12. Consider $r = 2$ with $k_1 = 3$, $k_2 = 1$:

$$\int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \int_0^{t_2} \frac{dt_3}{1-t_3} \int_0^{t_3} \frac{dt_4}{1-t_4} = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \int_0^{t_2} \frac{dt_3}{1-t_3} \int_0^{t_3} \sum_{n\geq 0} t_4^n \, dt_4$$

$$= \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \int_0^{t_2} \frac{dt_3}{1-t_3} \sum_{n>0} \frac{1}{n} t_3^n = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \int_0^{t_2} \sum_{n'\geq 0} t_3^{n'} \sum_{n>0} \frac{1}{n} t_3^n \, dt_3$$

$$= \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \sum_{n>0, n'\geq 0} \frac{1}{n} \frac{1}{n+n'+1} t_2^{n+n'+1} = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \sum_{m_1>m_2>0} \frac{1}{m_1 m_2} t_2^{m_1}$$

$$= \int_0^1 \frac{dt_1}{t_1} \sum_{m_1>m_2>0} \frac{1}{m_1 m_2} \int_0^{t_1} t_2^{m_1-1} \, dt_2 = \int_0^1 \frac{dt_1}{t_1} \sum_{m_1>m_2>0} \frac{1}{m_1^2 m_2} t_1^{m_1}$$

$$= \sum_{m_1>m_2>0} \frac{1}{m_1^3 m_2} = \zeta(3, 1).$$

This is exactly what Thm. A.11 states for $r = 2$ and $k_1 = 3$, $k_2 = 1$.

With the same procedure using induction (on length and weight) and geometric series expansion for expressions of the form $\frac{1}{1-t}$ gives the proof of Theorem A.11 in general. □

A.1.2. Quasi-shuffle algebras

We can view both representations of MZVs, as iterated sums resp. integrals, a bit more abstract on so-called *quasi-shuffle algebras*, introduced by Hoffman ([Ho2]), which are specific word algebras. Especially the stuffle and shuffle product can be described concretely using quasi-shuffle algebras; it will turn out that both are just distinguished quasi-shuffle products.

We can give expressions of the form '$\zeta(1, k_2, \ldots, k_r)$' some sense as an element in the polynomial ring $\mathcal{Z}[T]$ using the concept of quasi-shuffle algebras. Hence, we can extend the double shuffle relations. Conjecturally, these extended set of relations among MZVs gives all $\mathbb{Q}$-linear relations.

Definition A.13 (*Quasi-shuffle algebras*, [Ho2]). Let be A a set and $\diamond$ an associative and commutative product on $\mathbb{Q}A$. Then the quasi-shuffle product $*_\diamond$ on $\mathbb{Q}\langle A\rangle$ deduced from $\diamond$ is defined via

$$\begin{aligned}&\mathbf{1} *_\diamond w = w *_\diamond \mathbf{1} := w,\\ &au *_\diamond bv := a(u *_\diamond bv) + b(au *_\diamond v) + (a \diamond b)(u *_\diamond v)\end{aligned}$$

for any $a, b \in \mathbb{Q}A$ and words $u, v, w \in \mathbb{Q}\langle A\rangle$. The elements in A are called *letters*. We call $(\mathbb{Q}\langle A\rangle, *_\diamond)$ *quasi-shuffle algebra*.

Theorem A.14 ([Ho2, Thm. 2.1]). *Let be A a set and $\diamond$ an associative and commutative product on $\mathbb{Q}A$. Then the induced quasi-shuffle product $*_\diamond$ makes $(\mathbb{Q}\langle A\rangle, *_\diamond)$ to a commutative graded $\mathbb{Q}$-algebra.* □

For the theory of MZVs and their q-analogues, we have to work with the following algebras :

Definition A.15. Define the free non-commutative algebra of two letters,

$$\mathfrak{h} := \mathbb{Q}\langle x_0, x_1\rangle.$$

Monomials in the two *letters* x_0, x_1 are called *words*. They form a $\mathbb{Q}$-basis of $\mathfrak{h}$. The empty word is denoted by $\mathbf{1}$ and is the unit of $\mathfrak{h}$.
Also, define the subalgebras

$$\mathfrak{h}^1 := \mathbb{Q}\mathbf{1} \oplus \mathfrak{h}x_1,$$

consisting of all words ending in x_1 and

$$\mathfrak{h}^0 := \mathbb{Q}\mathbf{1} \oplus x_0\mathfrak{h}x_1,$$

consisting of all words starting in x_0 and ending in x_1.

Remark A.16.

(i) We have $\mathfrak{h}^0 \subset \mathfrak{h}^1 \subset \mathfrak{h}$.

(ii) $\mathfrak{h}^1$ is generated by words in $z_k := x_0^{k-1}x_1$, $k \geq 1$, i.e. $\mathfrak{h}^1 = \mathbb{Q}\langle z_1, z_2, \ldots\rangle$.

(iii) $\mathfrak{h}^0$ is generated by the words $z_{k_1} \ldots z_{k_r}$ with $k_1 \geq 2,\ k_i \geq 1$, $r \geq 0$.

(iv) By identifying words $z_{k_1} \ldots z_{k_r}$ with indices $(k_1, \ldots, k_r)$, words in $\mathfrak{h}^0$ correspond to admissible indices.

For us, two special quasi-shuffle products are important:

Definition A.17 (*Stuffle product*, [Ba6, Def. 2.11])**.** For $A = \{z_k = x_0^{k-1}x_1 \mid , k \geq 1\}$, i.e. $\mathbb{Q}\langle A\rangle = \mathfrak{h}^1$, and the diamond product $z_m \diamond z_n := z_{m+n}$, the resulting quasi-shuffle product $*$ on $\mathfrak{h}^1$ is called *stuffle product*.

Definition A.18 (*Shuffle product*, [Ba6, Def. 2.8])**.** For $A = \{x_0, x_1\}$, i.e. $\mathbb{Q}\langle A\rangle = \mathfrak{h}$, and the diamond constant 0, the induced quasi-shuffle product $\sqcup\!\sqcup$ on $\mathfrak{h}$ is called the *shuffle product*.

Remark A.19. There are inclusions of $\mathbb{Q}$-algebras:

(i) $(\mathfrak{h}^0, *) \subset (\mathfrak{h}^1, *)$.

(ii) $(\mathfrak{h}^0, \sqcup\!\sqcup) \subset (\mathfrak{h}^1, \sqcup\!\sqcup) \subset (\mathfrak{h}, \sqcup\!\sqcup)$.

The importance of those two quasi-shuffle products yields from their connection to MZVs, when viewing MZVs algebraically as we do in the next theorem:

Theorem A.20 ([IKZ, Prop. 1])**.** *The evaluation maps*

$$\begin{aligned} \zeta : (\mathfrak{h}^0, *) &\longrightarrow (\mathbb{R}, \cdot), \\ z_{k_1} \dots z_{k_r} &\longmapsto \zeta(k_1, \dots, k_r) \end{aligned}$$

and

$$\begin{aligned} \zeta : (\mathfrak{h}^0, \sqcup\!\sqcup) &\longrightarrow (\mathbb{R}, \cdot), \\ z_{k_1} \dots z_{k_r} &\longmapsto \zeta(k_1, \dots, k_r), \end{aligned}$$

both defined via $\mathbb{Q}$-linearity and $\mathbf{1} \mapsto 1$*, are algebra homorphisms.* □

Remark A.21. The (finite) double shuffle relations considered in Proposition A.5 are now easily obtained from $\zeta(u * v - u \sqcup\!\sqcup w) = 0$ for all $u, v \in \mathfrak{h}^0$.

We can extend now the evaluation map ζ to $(\mathfrak{h}^1, *)$ resp. $(\mathfrak{h}^1, \sqcup\!\sqcup)$ in compliance with the next theorem. It states that every word $u \in \mathfrak{h}^1$ can be written as a polynomial in $T = z_1$ with coefficients in $\mathfrak{h}^0$ where the multiplication can be $*$ or $\sqcup\!\sqcup$. This is called *regularization*.

Example A.22. We can write $z_1 z_3 \in \mathfrak{h}^1$ as

(i) $z_1 z_3 = z_3 * z_1 - (z_3 z_1 + z_4)$,

(ii) $z_1 z_3 = z_3 \sqcup\!\sqcup z_1 - z_3 z_1$.

In this way we can define either $\zeta(1,3) := \zeta(3)T - (\zeta(3,1) + \zeta(4)) \in \mathcal{Z}[T]$ or $\zeta(1,3) := \zeta(3)T - \zeta(3,1) \in \mathcal{Z}[T]$, depending on the product we refer to.

Theorem A.23 ([Ho1, Thm. 3.1])**.** *There are isomorphisms*

$$(\mathfrak{h}^1, *) \simeq (\mathfrak{h}^0[T], *), \quad (\mathfrak{h}^1, ⧢) \simeq (\mathfrak{h}^0[T], ⧢),$$

where T is a formal variable. Both isomorphisms are given through fixing $\mathfrak{h}^0$ and mapping $z_1 \mapsto T$. □

Under this identifications we extend now ζ to $(\mathfrak{h}^1, *)$ resp. $(\mathfrak{h}^1, ⧢)$:

Proposition A.24 ([IKZ, Prop. 1])**.** *For $\bullet \in \{*, ⧢\}$ we have algebra homomorphisms*

$$\zeta^\bullet : (\mathfrak{h}^1, \bullet) \longrightarrow \mathcal{Z}[T],$$

uniquely determined by the properties that $\zeta^\bullet$ extends the evaluation map ζ and maps $z_1 \mapsto T$. □

The *extended double shuffle relations* are now the relations obtained from

$$\zeta^\bullet(u * v - u ⧢ v) = 0$$

for $\bullet \in \{*, ⧢\}$ and $u \in \mathfrak{h}^1, v \in \mathfrak{h}^0$ by comparing coefficients of the monomials in T.
They are conjecturally already given by demanding $u \in \{z_1, z_2, z_2z_1, z_3\}$ ([KKT, Conj. 4.1]).
The extended double shuffle relations get their importance by the next conjecture:

Conjecture A.25. *The extended double shuffle relations give all $\mathbb{Q}$-linear relations among MZVs.*

A more detailed description of this conjecture and further results the authors give in [IKZ].

Remark A.26. There are linear relations among MZVs, called *duality relations* (see Thm. 3.2 in the authors master thesis). By Conjecture A.25 they should be implied by extended double shuffle relations. However, a proof of this statement is not known so far.

A.2. qMZVs

We give now an introduction to q-analogues of multiple zeta values. They are important in the theory of MZVs on the one hand because of their algebraic structure and, on the other hand, since they can be viewed as holomorphic functions in the unit disc giving connections to quasi-modular forms.
We begin with considering general modified q-analogues of MZVs, such as their well-definedness and connection to MZVs. Furthermore, we get here a first time in contact with $\mathcal{Z}_q$, the $\mathbb{Q}$-algebra of all qMZVs. A natural question is which elements generate

this algebra, which leads to different models of modified qMZVs. Every model of qMZVs contains at least one algebraic aspect of MZVs: Schlesinger-Zudilin's model inherits the stuffle product, and Bradley-Zhao's model the duality of MZVs. Also interesting is Bachmann's model since it gives a deep connection to modular forms that play an important role in the theory of MZVs as already Gangl, Kaneko and Zagier (in [GKZ]) and Broadhurst and Kreimer (in [BK]) have shown. For more details about the various models, we refer to the original works [Bra], [Zh1], [Sch], [Zu2], [Ba2], [Ta1], [OOZ], [Oko], such as to [Zh2], where the author gives an overview of the models and their history.

In general, a q-analogue of an object is a modified object in an additional variable q (often a series in (complex) q with $|q| < 1$) that returns the original object in the limit $q \to 1$, taken on the real axis from the left.

For example, a q-analogue of a natural number n is

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}.$$

Modified q-analogues of MZVs are q-series that return a multiple zeta value if we multiply the q-series first with a power of $(1-q)$ (most times, $(1-q)^{\mathrm{wt}(\mathbf{k})}$) and then take the limit $q \to 1$. This is convenient since we can avoid the additional power-of-$(1-q)$-factors without losing the structure we want to consider; furthermore, the spaces spanned by these objects become $\mathbb{Q}$-algebras as we will see, while the spaces spanned by non-modified objects become $\mathbb{Q}(1-q)$-algebras. Most times, we will not distinguish by language between modified and not modified since we focus on *modified* q-analogues of MZVs.

The definition of modified qMZVs we work with is as follows. We omit in the following the word *modified* since we only work with such qMZVs here:

Definition A.27 (*qMZV*, [BK2])**.** (i) Define for $r \geq 0$, $k_1, \ldots, k_r \geq 0$ and polynomials $Q_1 \in X\mathbb{Q}[X]$, $Q_2, \ldots, Q_r \in \mathbb{Q}[X]$ with $\deg Q_j \leq k_j$ for all j

$$\zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r) := \sum_{m_1 > \cdots > m_r > 0} \frac{Q_1(q^{m_1})}{(1-q^{m_1})^{k_1}} \cdots \frac{Q_r(q^{m_r})}{(1-q^{m_r})^{k_r}} \in \mathbb{Q}[\![q]\!],$$

with $\zeta_q(\emptyset, \emptyset) := 1$, where q is a formal variable.

(ii) Furthermore, we define the space

$$\mathcal{Z}_q := \langle \zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r) \,|\, r \geq 0,\, k_1, \ldots, k_r \geq 0,\, Q_1 \in X\mathbb{Q}[X],\, \deg Q_j \leq k_j \rangle_{\mathbb{Q}}.$$

Remark A.28. (i) In Definition A.27 we had to define the empty qMZV $\zeta(\emptyset, \emptyset)$ by hand. That makes sense and is necessary when considering qMZVs on quasi-shuffle algebras.

But $\zeta(\emptyset,\emptyset) := 1$ makes also sense only using the convention that the empty product is 1. Namely, the neat definition of general qMZVs is

$$\zeta_q(k_1,\dots,k_r;Q_1,\dots,Q_r) := \sum_{m_1>\dots>m_r>m_{r+1}=0} \prod_{l=1}^{r} \frac{Q_l(q^{m_l})}{(1-q^{m_l})^{k_l}}.$$

With the mentioned convention of the empty product, the empty qMZV (given through $r=0$) is already defined via this definition and is

$$\zeta(\emptyset,\emptyset) = \sum_{m_1=0} \prod_{l=1}^{0} \frac{Q_l(q^{m_l})}{(1-q^{m_l})^{k_l}} = 1,$$

because we sum one times the empty product. Here, the necessity of the additional variable m_{r+1} turns out since without it, we would have an empty sum which is (by convention) 0.

For $r>0$, this general definition matches with Definition A.27.

This remark transfers directly to the different models of qMZVs we consider. For brevity, we work most times without the additional variable m_{r+1} and define by hand the empty qMZV to be 1.

(ii) The condition $Q_1 \in X\mathbb{Q}[X]$ is necessary for well-definedness and cannot be removed.

(iii) Remark that $\mathcal{Z}_q$ does *not* contain all modified q-analogues of MZVs inevitably. For example, it is not clear yet whether the modified qMZVs introduced by Shen and Qin ([SQ]) are in $\mathcal{Z}_q$.

We use notation from [BK2], where the authors introduce important subspaces of $\mathcal{Z}_q$:

Definition A.29. Define for $d \geq 0$

$$\mathcal{Z}_{q,d} := \langle \zeta_q(k_1,\dots,k_r;Q_1,\dots,Q_r) \in \mathcal{Z}_q |\, r\geq 0,\, k_1,\dots,k_r \geq 1, \deg(Q_j) \leq k_j - d\rangle_{\mathbb{Q}}$$

and

$$\mathcal{Z}^\circ_{q,d} := \langle \zeta_q(k_1,\dots,k_r;Q_1,\dots,Q_r) \in \mathcal{Z}_{q,d} |\, r\geq 0,\, k_1,\dots,k_r \geq 1, Q_j \in X\mathbb{Q}[X]\rangle_{\mathbb{Q}}$$

with the abbreviation $\mathcal{Z}^\circ_q := \mathcal{Z}^\circ_{q,0}$.

Remark A.30. Naively, we could think of $k_1+\dots+k_r$ as the 'weight' of the qMZV $\zeta_q(k_1,\dots,k_r;Q_1,\dots,Q_r)$ in accordance with the definition of weight for MZVs. But this is not well-defined since for example

$$\zeta_q(k_1,\dots,k_r;Q_1,\dots,Q_r) = \zeta_q(k_1+1,\dots,k_r;(1-X)Q_1,Q_2\dots,Q_r).$$

Hence, we need another notion of weight. Such one we will consider when talking about bi-brackets (see Def. A.60).

Remark A.31. (i) The spaces $\mathcal{Z}_q$ and $\mathcal{Z}_q^\circ$ are closed under the operator $q\frac{d}{dq}$ ([Ba4], [BK1, Prop. 3.14]).

(ii) The spaces $\mathcal{Z}_{q,1}$ and $\mathcal{Z}_{q,1}^\circ$ are only conjecturally closed under $q\frac{d}{dq}$ ([Ba7], [Oko, Conj. 1]).

Proposition A.32. *(i) For $q \in \mathbb{C}$, $|q| < 1$, every qMZV converges and can be viewed as holomorphic function inside the unit disk or upper half plane if $q = e^{2\pi i\tau}$ with $\tau \in \mathbb{H}$.*

(ii) For $k_1 \geq 2$, $k_2, \ldots, k_r \geq 1$, the limit

$$\lim_{q\to 1}(1-q)^{k_1+\cdots+k_r}\zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r) = \zeta_q(k_1, \ldots, k_r)\prod_{j=1}^{r} Q_j(1),$$

taken on $\mathbb{R}$ from the left, exists and is obtained by interchanging limit and summation due to absolute convergence on the interval $[0, 1)$.

Proof. For (i) we show the convergence of

$$(1-q)^{k_1+\cdots+k_r}\zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r)$$

for all $k_1 \geq 1$, $k_2, \ldots, k_r \geq 0$ and for $|q| < 1$ which is equivalent to (i) because of $q \neq 1$.

Therefore, fix some q with $|q| < 1$, $r \geq 1$ (for $r = 0$ there is nothing to do), $k_1 \geq 1$, $k_2, \ldots, k_r \geq 0$ and polynomials $Q_1, \ldots, Q_r$ in accordance to the definition of qMZVs.

We can write every Q_j in the form

$$Q_j = Q_j^+ - Q_j^-,$$

where Q_j^+ and Q_j^- are polynomials having coefficients ≥ 0 and the same properties as Q_j (i.e. $\deg \leq k_j$, $Q_1^+, Q_1^- \in X\mathbb{Q}[X]$). I.e. $\zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r)$ is a finite rational linear combination of qMZVs where the corresponding polynomials have non-negative coefficients, which is the reason why we can assume in the following w.l.o.g. that the Q_j all have non-negative coefficients.

Then it is

$$0 \leq |Q_j(q^m)| \leq Q_j(0) + Q_j(1)|q|^m$$

for all $|q| < 1$ and $m \in \mathbb{N}$.

For $j \neq 1$, we use $|q| < 1$ and that the coefficients of Q_j are all non-negative which gives us furthermore

$$0 \leq |Q_j(q^m)| \leq Q_j(0) + Q_j(1) =: c_j.$$

Since $Q_1(0) = 0$, we have

$$0 \leq |Q_1(q^m)| \leq Q_1(1)|q|^m$$

for all $m \in \mathbb{N}$.

Now, since $\left|\frac{(1-q)^k}{(1-q^m)^k}\right| = \left|\frac{1}{(1+q+\cdots+q^{m-1})^k}\right| \leq C_{q,k}$ for all m with an appropriate constant $C_{q,m} \in \mathbb{R}$ (note that $\lim_{m\to\infty}\left|\frac{(1-q)^k}{(1-q^m)^k}\right| = |(1-q)|^k$ exists since $|q| < 1$), we get for all tuples $(m_1, \ldots, m_r)$ of natural numbers

$$0 \leq |1-q|^{k_1+\cdots+k_r} \prod_{j=1}^{r} \frac{|Q_j(q^{m_j})|}{|1-q^{m_j}|^{k_j}} \leq C \cdot |q|^{m_1}$$

with $C := Q_1(1)c_2 \cdots c_r C_{q,k_1} \cdots C_{q,k_r}$.

For fixed $0 \leq s < 1$, the map $x \mapsto s^x$ decreases with exponential growth in x. Hence, for all $n \in \mathbb{N}$, there is a constant M_n (depends on $|q|$) such that

$$|q|^m \leq M_n \frac{1}{m^n}$$

for all $m \in \mathbb{N}$. In particular, for $n = r+1$, we obtain

$$0 \leq |1-q|^{k_1+\cdots+k_r} |\zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r)| \leq CM_{r+1} \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{r+1}}$$

$$= CM_{r+1} \sum_{m_1 > 0} \binom{m_1 - 1}{r-1} \frac{1}{m_1^{r+1}} \overset{r \geq 1}{\leq} CM_{r+1} \sum_{m>0} \frac{1}{m^2} < \infty$$

since $\binom{m-1}{r-1} = \frac{(m-1)\cdot(m-r+1)}{(r-1)!} \leq m^{r-1}$ and because of the well-known fact that the Basler sum converges.

In particular, the defining sum of $(1-q)^{k_1+\cdots+k_r}\zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r)$ converges for every $|q| < 1$, which proves (i).

For the proof of (ii), we remark that on the intervall $[0, 1)$ our sum converges absolutely since there, we can choose $C_{q,k} = 1$ for all q and $k \geq 1$, i.e. independent of q. Due to this uniform convergence, limit and summation can be interchanged. Now, for all $k \geq 0$, $m \in \mathbb{N}$ and $1 \leq j \leq r$,

$$\lim_{q\to 1} Q_j(q^m) = Q_j(1), \quad \lim_{q\to 1} \frac{(1-q)^k}{(1-q^m)^k} = \lim_{q\to 1}\left(\frac{1}{1+q+\cdots+q^{m-1}}\right)^k = \frac{1}{m^k},$$

which is why we get for fixed $k_1 \geq 2$, $k_2, \ldots, k_r \geq 1$

$$\zeta_q(k_1, \ldots, k_r) \prod_{j=1}^{r} Q_j(1) = \sum_{m_1 > \cdots > m_r > 0} \lim_{q\to 1} \prod_{j=1}^{r} \frac{(1-q)^{k_j} Q_j(q^{m_j})}{(1-q^{m_j})^{k_j}}$$

$$\overset{(i)}{=} \lim_{q\to 1} (1-q)^{k_1+\cdots+k_r} \zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r).$$

□

A similar proof for the proposition with some stronger assumption for (i) is given in [BK1, Lemma 6.6].

Often, we talk about $\mathcal{Z}_q$ as algebra. We justify this in the following as we see that we can give $\mathcal{Z}_q$ a structure such that it becomes a quasi-shuffle algebra.

Definition A.33. Consider the alphabet

$$A_Z := \{\tbinom{k}{Q} : Q \in \mathbb{Q}[X],\, k \in \mathbb{N},\, \deg(Q) \leq k\}.$$

Define on $\mathbb{Q}A_Z$ the commutative and associative product $\diamond$ by $w \diamond \mathbf{1} := \mathbf{1} \diamond w := w$ for all $w \in \mathbb{Q}A_Z$, $\mathbb{Q}$-bilinearity and by

$$\begin{aligned} \diamond : \mathbb{Q}A_Z \otimes \mathbb{Q}A_Z &\longrightarrow \mathbb{Q}A_Z, \\ \left(\binom{k_1}{Q_1}, \binom{k_2}{Q_2}\right) &\longmapsto \binom{k_1 + k_2}{Q_1 \cdot Q_2}. \end{aligned}$$

Proposition A.34. *Let be $*$ the induced quasi-shuffle product on $M := A_Z^\circ \mathbb{Q}\langle A_Z \rangle + \mathbb{Q}$, where $A_Z^\circ := \{\tbinom{k}{Q} : Q \in X\mathbb{Q}[X],\, k \in \mathbb{N},\, \deg(Q) \leq k\}$. Then the evaluation map*

$$\begin{aligned} \zeta_q : M &\longrightarrow \mathcal{Z}_q, \\ \binom{k_1}{Q_1} \cdots \binom{k_r}{Q_r} &\longmapsto \zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r) \end{aligned}$$

extended to M by $\mathbb{Q}$-linearity, is an algebra homomorphism, i.e. for all $u, v \in M$ we have

$$\zeta_q(u * v) = \zeta_q(u)\zeta_q(v).$$

Proof. This follows from multiplication of the iterated sums, about which the $\zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r)$ were defined. □

Remark A.35 ([BK2])**.** The subspaces $\mathcal{Z}_{q,d}, \mathcal{Z}_{q,d}^\circ \subseteq \mathcal{Z}_q$ are for all $d \geq 0$ subalgebras of $\mathcal{Z}_q$ by restricting the above defined quasi-shuffle product to the responding subspace.

We need for the definition of quasi-shuffle products in different models of qMZVs the notion of some particular free non-commutative algebras:

Definition A.36. Define two free non-commutative algebras of two letters,

$$\mathfrak{h} := \mathbb{Q}\langle x_0, x_1 \rangle, \quad \mathfrak{R} := \mathbb{Q}\langle p, y \rangle$$

and subalgebras

$$\mathfrak{h}^0 := \mathbb{Q}\mathbf{1} \oplus x_0 \mathfrak{h} x_1, \quad \mathfrak{h}^1 := \mathbb{Q}\mathbf{1} \oplus \mathfrak{h} x_1, \quad \mathfrak{R}^1 := \mathbb{Q}\mathbf{1} \oplus p\mathfrak{R}y, \quad \mathfrak{R}^3 := p\mathbb{Q}\langle p, py \rangle py \oplus \mathbb{Q}\mathbf{1}.$$

Monomials in the two *letters* x_0, x_1 resp. p, y are called *words*. They form a $\mathbb{Q}$-basis of $\mathfrak{h}$ resp. $\mathfrak{R}$. $\mathbf{1}$ is the empty word and hence the unit of $\mathfrak{h}$ resp. $\mathfrak{R}$.

A.2.1. Schlesinger-Zudilin model

One of the most natural questions is how bases or at least generating systems of the $\mathbb{Q}$-vector space $\mathcal{Z}_q$ look like and whether there are interesting subspaces we should consider.

The probably most natural looking generating system when writing the elements of $\mathcal{Z}_q$ in the shape of Definition A.27 is

$$\{\zeta_q(k_1,\dots,k_r;X^{k_1},\dots,X^{k_r}):\ r\geq 0,\ k_1\geq 1,\ k_2,\dots,k_r\geq 0\}$$

since it corresponds to $Q_i(X):=X^{k_i}$. We will see in Proposition A.39 that these indeed generate $\mathcal{Z}_q$.

These generators are named *Schlesinger-Zudilin qMZVs* by Schlesinger (2001, [Sch]) and Zudilin (2003, [Zu1]) who introduced them independently:

Definition A.37 (Schlesinger-Zudilin qMZVs)**.**

(i) Call an index $\mathbf{k}=(k_1,\dots,k_r)\in\mathbb{N}_0^r$ *SZ-admissible index* if $r\geq 0$ and $k_1\geq 1,\ k_i\geq 0$ for all i or $\mathbf{k}=\emptyset$.

(ii) Define for every SZ-admissible index $\mathbf{k}$ the Schlesinger-Zudilin qMZV as

$$\begin{aligned}\zeta_q^{\mathrm{SZ}}(\mathbf{k}):=\zeta_q^{\mathrm{SZ}}(k_1,\dots,k_r)&:=\zeta_q(k_1,\dots,k_r;X^{k_1},\dots,X^{k_r})\\&=\sum_{m_1>\dots>m_r>0}\frac{q^{m_1k_1}}{(1-q^{m_1})^{k_1}}\cdots\frac{q^{m_rk_r}}{(1-q^{m_r})^{k_r}}\end{aligned}$$

with $\zeta_q^{\mathrm{SZ}}(\emptyset):=1$.

We introduced an extended version due to Ebrahimi-Fard, Manchon and Singer (cf. [EMS]) since in the original model, due to Schlesinger and Zudilin, only indices with $k_i\geq 1$ were allowed.

Remark A.38.

(i) It is to note that if one of the indices is 0 in an SZ-qMZV, say $k_j=0$ for some j, then the summand is independent of m_j. Therefore, it is often useful to distinguish between zero and non-zero indices.

(ii) An index $\mathbf{k}$ is SZ-admissible iff $\mathbf{k}+\mathbf{1}$ (every argument of $\mathbf{k}$ is increased by 1) is admissible in sense of Definition A.1(ii).

(iii) Why the name of the SZ-model is attributed not only to Schlesinger, although his publication ([Sch]) was two years before Zudilin's ([Zu1]), is due to the fact that Schlesinger introduced the qMZVs we call SZ-qMZVs in a slightly modified way: He considered

$$\zeta_q^{SZ'}(k_1,\dots,k_r):=\sum_{m_1>\dots>m_r>0}\frac{1}{(1-q^{m_1})^{k_1}\dots(1-q^{m_r})^{k_r}}$$

with $|q| > 1$ instead of $\zeta_q^{\mathrm{SZ}}(k_1, \ldots, k_r)$ (with $|q| < 1$). The latter is today the usual definition and also Zudilin introduced it this way.

On closer inspection, we see that this is almost $\zeta_q^{\mathrm{SZ}}(k_1, \ldots, k_r)$. Namely, it is

$$\zeta_{q^{-1}}^{SZ'}(k_1, \ldots, k_r) = (-1)^{k_1+\cdots+k_r}\zeta_q^{\mathrm{SZ}}(k_1, \ldots, k_r).$$

Further details of the history of SZ-qMZVs can be found, e.g. in [Zh1].

(iv) For some applications as translation/duality in the OOZ-model (Thm. A.90), it is useful to have no strictly ordered index in the defining sum of SZ-qMZVs. Hence, SZ-star-qMZVs are defined as

$$\zeta_q^{\mathrm{SZ},\star}(\mathbf{k}) := \zeta_q^{\mathrm{SZ},\star}(k_1, \ldots, k_r) := \sum_{m_1 \geq \cdots \geq m_r > 0} \frac{q^{m_1 k_1}}{(1-q^{m_1})^{k_1}} \cdots \frac{q^{m_r k_r}}{(1-q^{m_r})^{k_r}}.$$

As for MZVs and MZSVs, every SZ-qMZSV is a finite sum of SZ-qMZVs.

Proposition A.39. *SZ-qMZVs span $\mathcal{Z}_q$, i.e.*

$$\mathcal{Z}_q = \langle \zeta_q^{SZ}(k_1, \ldots, k_r) \,|\, r \geq 0,\, k_1 \geq 1,\, k_i \geq 0 \rangle_{\mathbb{Q}}.$$

Proof. The proof is obtained from the fact that every expression $\frac{X^n}{(1-X)^s}$ for $0 \leq n \leq s$ is a finite $\mathbb{Q}$-linear combination of terms $\frac{X^k}{(1-X)^k}$ for $k \geq 0$. Specifically applies

$$\frac{X^n}{(1-X)^s} = \sum_{p=n}^{s} \binom{s-n}{p-n} \frac{X^p}{(1-X)^p} \tag{A.3}$$

for every $s \in \mathbb{N}_0$ and every $0 \leq n \leq s$. □

In particular, the SZ-model is closed under the operator $q\frac{d}{dq}$ by Remark A.31(i).

SZ-qMZVs satisfy a similar stuffle product as MZVs and a similar duality relation. Both combined give the shuffle product of MZVs, which is a very nice result and application of SZ-qMZVs. This result is mentioned in [EMS] and [Sin].

Definition A.40 (SZ-stuffle product)**.**

(i) Define $u_k := p^k y \in \mathfrak{K}$ for all $k \geq 0$.

(ii) Consider on $\mathfrak{K}$ the usual stuffle product, i.e. define recursively the product $*_{\mathrm{SZ}} : \mathfrak{K} \times \mathfrak{K} \to \mathfrak{K}$ via distributivity and

a) $\mathbf{1} *_{\mathrm{SZ}} w = w *_{\mathrm{SZ}} \mathbf{1} := w$,

b) $u_s v *_{\mathrm{SZ}} u_t w := u_s(v *_{\mathrm{SZ}} u_t v) + u_t(u_s v *_{\mathrm{SZ}} w) + u_{s+t}(v *_{\mathrm{SZ}} w)$

for all words $v, w \in \mathfrak{K}$ and $s, t \geq 0$.

Remark at this point that $\mathfrak{K}^1$ is generated by the words starting in some u_k, $k \geq 1$ and that $\mathfrak{K}^1$ is closed under $*_{\mathrm{SZ}}$.

Definition A.41. Identify the word $u_{k_1} \dots u_{k_r} \in \mathfrak{K}^1$ with the index $(k_1, \dots, k_r)$, especially we define

$$\begin{aligned} \zeta_q^{\mathrm{SZ}} \,:\, \mathfrak{K}^1 &\to \mathbb{Q}[\![q]\!], \\ u_{k_1} \dots u_{k_r} &\mapsto \zeta_q^{\mathrm{SZ}}(k_1, \dots, k_r) \end{aligned}$$

and extend the map ζ_q^{SZ} to $\mathfrak{K}^1$ linearly.

Theorem A.42. *The map ζ_q^{SZ} is an algebra homomorphism on $(\mathfrak{K}^1, *_{SZ})$, i.e. for all words $v, w \in \mathfrak{K}^1$, we have*

$$\zeta_q^{SZ}(v)\zeta_q^{SZ}(w) = \zeta_q^{SZ}(v *_{SZ} w).$$

Proof. The statement follows directly from the definition of SZ-qMZVs as iterated sums. □

We can consider SZ-qMZVs also in another way on $\mathfrak{K}$:

Definition A.43.

(i) Define recursively and by distributivity the SZ-$q$shuffle product $ш_{\mathrm{SZ}} : \mathfrak{K} \times \mathfrak{K} \to \mathfrak{K}$ via

a) $\mathbf{1} ш_{\mathrm{SZ}} w = w ш_{\mathrm{SZ}} \mathbf{1} := w$,

b) $yu ш_{\mathrm{SZ}} v = u ш_{\mathrm{SZ}} yv := y(u ш_{\mathrm{SZ}} v)$,

c) $pu ш_{\mathrm{SZ}} pv := p(u ш_{\mathrm{SZ}} pv) + p(pu ш_{\mathrm{SZ}} v) + p(u ш_{\mathrm{SZ}} v)$

and $\mathbb{Q}$-bilinearity for all $u, v, w \in \mathfrak{K}$.

(ii) Identify an SZ-admissible index $\mathbf{k} = (k_1, \dots, k_r)$ with the word $p^{k_1}y \dots p^{k_r}y \in \mathfrak{K}^1$. Then we can define ζ_q^{SZ} as the more general map

$$\begin{aligned} \zeta_q^{\mathrm{SZ}} : \mathfrak{K}^1 &\longrightarrow \mathcal{Z}_q, \\ p^{k_1}y \dots p^{k_r}y &\longmapsto \zeta_q^{\mathrm{SZ}}(k_1, \dots, k_r), \end{aligned}$$

extended to $\mathfrak{K}^1$ by $\mathbb{Q}$-linearity.

Singer proved that ζ_q^{SZ} is an algebra homomorphism on $(\mathfrak{K}^1, ш_{SZ})$:

Theorem A.44 ([Sin, Thm. 6.2])**.** *The map ζ_q^{SZ} is an algebra homomorphism on $(\mathfrak{K}^1, ш_{SZ})$, i.e. for all words $u, v \in \mathfrak{K}^1$ we have:*

$$\zeta_q^{SZ}(u)\zeta_q^{SZ}(v) = \zeta_q^{SZ}(u ш_{SZ} v).$$

□

Often - as for an elegant proof of SZ-duality (Thm. A.46 below) - it is helpful to consider the generating series of e.g. SZ-qMZVs since they contain all information about the objects in a compact written term:

Theorem A.45. *Define for every $r \geq 1$*

$$\mathfrak{s}\binom{X_1, \ldots, X_r}{Y_1, \ldots, Y_r} := \sum_{\substack{k_1,\ldots,k_r \geq 1 \\ d_1,\ldots,d_r \geq 0}} \zeta_q^{SZ}(k_1, \{0\}^{d_1}, \ldots, k_r, \{0\}^{d_r}) \cdot X_1^{k_1-1} \ldots X_r^{k_r-1} Y_1^{d_1} \ldots Y_r^{d_r}.$$

Then, for every $r \geq 1$ we have:

$$\mathfrak{s}\binom{X_1, \ldots, X_r}{Y_1, \ldots, Y_r} = \sum_{\substack{m_1 > \cdots > m_r > 0 \\ n_1,\ldots,n_r \geq 1}} \prod_{j=1}^{r} (1+X_j)^{n_j-1}(1+Y_j)^{m_j-m_{j+1}-1} q^{m_j n_j},$$

where we set $m_{r+1} := 0$.

Proof. What we need, is the binomial theorem and

$$|\{n_1, \ldots, n_l \in \mathbb{N} : \; m_1 > n_1 > \cdots > n_l > m_2\}| = \binom{m_1 - m_2 - 1}{l}.$$

With this, we get

$$\begin{aligned}
\mathfrak{s}\binom{X_1, \ldots, X_r}{Y_1, \ldots, Y_r} &= \sum_{\substack{k_1,\ldots,k_r \geq 1 \\ d_1,\ldots,d_r \geq 0}} \zeta_q^{\mathrm{SZ}}(k_1, \{0\}^{d_1}, \ldots, k_r, \{0\}^{d_r}) \cdot X_1^{k_1-1} \ldots X_r^{k_r-1} Y_1^{d_1} \ldots Y_r^{d_r} \\
&= \sum_{\substack{k_1,\ldots,k_r \geq 1 \\ d_1,\ldots,d_r \geq 0}} \sum_{\substack{m_1 > n_1 > \cdots > n_{d_1} > m_2 \\ > \cdots > 0}} \prod_{j=1}^{r} \frac{q^{m_j k_j}}{(1-q^{m_j})^{k_j}} X_1^{k_1-1} \ldots X_r^{k_r-1} Y_1^{d_1} \ldots Y_r^{d_l} \\
&= \sum_{\substack{k_1,\ldots,k_r \geq 1 \\ d_1,\ldots,d_r \geq 0}} \sum_{m_1 > \cdots > m_r > 0} \prod_{j=1}^{r} \binom{m_j - m_{j+1} - 1}{d_j} \frac{q^{m_j k_j}}{(1-q^{m_j})^{k_j}} X_1^{k_1-1} \ldots X_r^{k_r-1} Y_1^{d_1} \ldots Y_r^{d_r} \\
&= \sum_{\substack{m_1 > \cdots > m_r > 0 \\ n_1,\ldots,n_r > 0}} \prod_{j=1}^{r} \left[\sum_{k_j \geq 1, d_j \geq 0} \binom{m_j - m_{j+1} - 1}{d_j} \binom{n_j - 1}{k_j - 1} X_j^{k_j-1} Y_j^{d_j} q^{m_j n_j} \right] \\
&= \sum_{\substack{m_1 > \cdots > m_r > 0 \\ n_1,\ldots,n_r > 0}} \prod_{j=1}^{r} (1+X_j)^{n_j-1}(1+Y_j)^{m_j-m_{j+1}-1} q^{m_j n_j}.
\end{aligned}$$

□

SZ-qMZVs satisfy a duality relation, similar to the one of MZVs, which is, together with some related statements, why they are interesting objects.

Theorem A.46 (SZ-Duality; Zhao [Zh2, Thm. 8.3])**.** *Let be $\tilde{\tau} : \mathfrak{K} \to \mathfrak{K}$ the anti-automorphism w.r.t. concatenation, induced by $\tilde{\tau}(p) := y$, $\tilde{\tau}(y) := p$. On $\mathfrak{K}^1$ we have*

$$\zeta_q^{SZ} \circ \tilde{\tau} = \zeta_q^{SZ}.$$

□

Remark A.47 (Comparison)**.** The SZ-model is a very elegant model of qMZVs since it satisfies the q-analogue of the stuffle product of MZVs, and is suitable to handle because of the definition that numerator and denominator have the same polynomial degree. That was also the reason why we got a generating series we can work with. But there is more: The stuffle product for the SZ-model induces together with SZ-duality the shuffle product of MZVs (cf. [Sin], for details see Thm. 3.46, Thm. 3.52).

SZ-shuffle on Rota-Baxter algebras

Most quasi-shuffle products in different models can be defined on so-called *Rota-Baxter algebras* such that qMZVs in the responding model is just a special value of iterated such operators. Exemplary, we do this here for the SZ-model. For other models, see [Sin].

Definition A.48 (*Rota-Baxter operator*)**.** Let C be a ring, $\lambda \in C$, and $\mathcal{A}$ a C-algebra. A *Rota-Baxter operator* (RBO) R of weight λ on $\mathcal{A}$ over C is a C-module endomorphism of $\mathcal{A}$ such that

$$R(x)R(y) = R(xR(y)) + R(R(x)y) + \lambda R(xy) \tag{A.4}$$

for all $x, y \in \mathcal{A}$.
Furhermore, a *Rota-Baxter $\mathcal{C}$-algebra* (RBA) is a pair $(\mathcal{A}, R)$ with a C-algebra $\mathcal{A}$ and a RBO (of some weight λ) on $\mathcal{A}$ over C.

Example A.49. Consider $t\mathbb{Q}[\![t,q]\!]$, the vector space of formal power series $f = \sum\limits_{n \geq 0} a_n t^n$ with $a_n \in \mathbb{Q}[\![q]\!]$ and strictly positive evaluation $\inf\{n \in \mathbb{N} \,|\, a_n \neq 0\}$. It can be viewed as a $\mathbb{Q}[\![q]\!]$-algebra. We will denote it by $\mathcal{A}_q$. Then the operator $P_q : \mathcal{A}_q \to \mathcal{A}_q$, defined via

$$P_q[f](t) := \sum_{n>0} f(q^n t)$$

is a RBO of weight 1.

In the following, let be $T : \mathcal{A}_q \to \mathcal{A}_q$ the operator $t \mapsto \frac{t}{1-t}$. Then we see that every SZ-qMZV is just the value at $t = 1$ of a concatination of Operators P_q and T:

Proposition A.50 ([Sin, Prop. 5.2])**.** *For every $r \geq 1$ and $k_1 \geq 1, k_2, \ldots, k_r \geq 0$ we have:*

$$P_q^{k_1}[TP_q^{k_2}[T \ldots P_q^{k_r}[T] \ldots]](t) = \sum_{m_1 > \cdots > m_r > 0} t^{m_1} \frac{q^{m_1 k_1}}{(1-q^{m_1})^{k_1}} \cdots \frac{q^{m_r k_r}}{(1-q^{m_r})^{k_r}},$$

i.e. in particular $P_q^{k_1}[TP_q^{k_2}[T \ldots P_q^{k_r}[T] \ldots]](1) = \zeta_q^{SZ}(k_1, \ldots, k_r)$. □

That ζ_q^{SZ} is an algebra homomorphism on $(\mathfrak{K}^1, ш_{SZ})$ follows now by the more general statement, saying that

$$\begin{aligned}\phi_q : \mathfrak{K} &\longrightarrow \mathcal{A}_q,\\ p^{k_1}y\cdots p^{k_r}y &\longmapsto P_q^{k_1}[TP_q^{k_2}[T\cdots P_q^{k_r}[T]\cdots]]\end{aligned}$$

is an algebra homomorphism. This follows from obtaining that the letter p corresponds to P_q and y to T. Then (ii) of the defining property of $ш_{\mathrm{SZ}}$ (if y is in one of both sides of the product leading, then we may pull it out) corresponds to the fact that by multiplication in $\mathcal{A}_q$ we may pull out some factor (since $\mathcal{A}_q$ is commutative), especially a factor $\frac{t}{1-t}$, corresponding to T. Analogously, we can clarify that defining property (iii) of $ш_{\mathrm{SZ}}$ corresponds to the RHS of the definition of an RBO of weight 1, (A.4). But now, the LHS of (A.4) will give the desired product in $\mathcal{A}_q$, making ϕ_q to an algebra homomorphism.

A.2.2. Bradley-Zhao model

In depth one, the BZ-model of qMZVs was first considered by Kaneko, Kurokawa and Wakayama in 2002, [KKW]. The general model then was introduced by J. Zhao in 2003 ([Zh1]) and independent of Zhao by D. M. Bradley in 2004 ([Bra]). This clarifies its name. BZ-qMZVs satisfy the same duality as MZVs why this model plays an important role in the context of qMZVs.

Definition A.51 (*Bradley-Zhao-qMZVs*)**.** For every admissible index $\mathbf{k} = (k_1, \dots, k_r)$, i.e. $k_1 \geq 2, k_2, \dots, k_r \geq 1$, we define

$$\begin{aligned}\zeta_q^{\mathrm{BZ}}(\mathbf{k}) := \zeta_q^{\mathrm{BZ}}(k_1, \dots, k_r) &:= \zeta_q(k_1, \dots, k_r; X^{k_1-1}, \dots, X^{k_r-1})\\ &= \sum_{m_1 > \cdots > m_r > 0} \frac{q^{m_1(k_1-1)}}{(1-q^{m_1})^{k_1}} \cdots \frac{q^{m_r(k_r-1)}}{(1-q^{m_r})^{k_r}}.\end{aligned}$$

Proposition A.52. *BZ-qMZVs span a proper subspace of $\mathcal{Z}_q$, namely it is*

$$\mathcal{Z}_{q,1} = \langle \zeta_q^{BZ}(k_1, \dots, k_r) \,|\, r \geq 0,\ k_1 \geq 2,\ k_i \geq 1 \rangle_{\mathbb{Q}}.$$

Proof. Every BZ-qMZV is by definition an element of $\mathcal{Z}_{q,1}$. That also every element of $\mathcal{Z}_{q,1}$ can be written as rational linear combination, follows from the identity

$$\frac{X^s}{(1-X)^n} = \frac{X^s}{(1-X)^{s+1}} \left(1 + \frac{X}{1-X}\right)^{n-s+1}$$

for all $0 \leq s < n$ and since on the RHS we get a rational linear combination of expressions of the form $\frac{X^{k-1}}{(1-X)^k}$.

$\zeta_q(1; X) \in \mathcal{Z}_q$, e.g., can not be written in terms of BZ-qMZVs. This fact can be proven with arguments similar to [BK1, Thm. 2.14(ii)]. □

BZ-qMZVs satisfy a quasi-shuffle product, in analogy to the stuffle product of MZVs since it is induced by multiplication of iterated sums:

Definition A.53. (i) We can define ζ_q^{BZ} also as a map

$$\begin{aligned}\zeta_q^{\mathrm{BZ}}:\mathfrak{h}^0&\longrightarrow\mathcal{Z}_q,\\ z_{k_1}\dots z_{k_r}&\longmapsto\zeta_q^{\mathrm{BZ}}(k_1,\dots,k_r)\end{aligned}$$

by $\mathbb{Q}$-linear continuation and $\mathbf{1}\mapsto 1$.

(ii) Define on $\mathbb{Q}\{z_k : k\in\mathbb{N}\}$ the commutative and associative product $\diamond_{BZ}$ via

$$z_{k_1}\diamond_{BZ} z_{k_2}:=z_{k_1+k_2}+z_{k_1+k_2-1}$$

for all $k_1,k_2\geq 1$ and $\mathbf{1}\diamond_{BZ} w := w\diamond_{BZ}\mathbf{1} := w$ for all $w\in\mathfrak{h}^0$. Let be $*_{BZ}$ the induced quasi-shuffle product on $\mathfrak{h}^1$.

Notice that $\mathfrak{h}^0\subset\mathfrak{h}^1$ is closed under $*_{BZ}$. We have the following:

Proposition A.54. *The map ζ_q^{BZ} on $(\mathfrak{h}^0,*_{BZ})$ is a algebra homomorphism, i.e. for all $u,v\in\mathfrak{h}^0$ we have*

$$\zeta_q^{BZ}(u*_{BZ}v)=\zeta_q^{BZ}(u)\zeta_q^{BZ}(v).$$

Proof. This is elementary calculation using the product of iterated sums and the fact

$$\frac{q^{m(k-2)}}{(1-q^m)^k}=\frac{q^{m(k-2)}}{(1-q^m)^{k-1}}+\frac{q^{m(k-1)}}{(1-q^m)^k}.$$

The latter corresponds to the diamond $\diamond_{BZ}$. □

Theorem A.55. *Define for $r\geq 1$ the generating series of BZ-qMZVs,*

$$\begin{aligned}&\mathfrak{b}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r}\\ &:=\sum_{\substack{k_1,\dots,k_r\geq 1\\ d_1,\dots,d_r\geq 1}}\zeta_q^{BZ}(k_1+1,\{1\}^{d_1-1},\dots,k_r+1,\{1\}^{d_r-1})X^{k_1-1}Y_1^{d_1-1}\cdots X_r^{k_r-1}Y_r^{d_r-1}.\end{aligned}$$

Then, we have for every $r\geq 1$

$$\begin{aligned}&\mathfrak{b}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r}\\ &=\sum_{\substack{l_1,\dots,l_r\geq 1\\ \delta_1,\dots,\delta_r\in\{0,1\}}}(-1)^{r-(\delta_1+\dots+\delta_r)}\mathfrak{s}\binom{\delta_1X_1,0,\dots,0,\dots,\delta_rX_r,0,\dots,0}{Y_1,\underbrace{Y_1,\dots,Y_1}_{l_1-1},\dots,Y_r,\underbrace{Y_r,\dots,Y_r}_{l_r-1}}\prod_{j=1}^r(1+\delta_jX_j)Y_j^{l_j-1}.\end{aligned}$$

Proof. We give a proof for clarity reasons only for $r = 1$. The statement for general r can be proven in the same way: So, we have to show that

$$\mathfrak{b}\binom{X}{Y} = \sum_{l\geq 1,\, \delta\in\{0,1\}} (-1)^{1-\delta}\mathfrak{s}\binom{\delta X, 0, \dots, 0}{Y, \underbrace{Y, \dots, Y}_{l-1}}(1+\delta X)Y^l.$$

Using the definition of $\mathfrak{b}$ and $\mathfrak{s}$ as generating series of BZ- resp. SZ-qMZVs, this equality is equivalent to (note that we plug in $X_2 = \cdots = X_l = 0$ in each summand resp. $X_1 = 0$ additionally in the last sum)

$$\begin{aligned}
&\sum_{k,d\geq 1} \zeta_q^{\mathrm{BZ}}(k+1, \{1\}^{d-1})X^kY^d \\
&= \sum_{l\geq 1} \sum_{\substack{k_1\geq 1, k_2=\cdots=k_l=1\\ d_1,\dots,d_l\geq 0}} \zeta_q^{\mathrm{SZ}}(k_1, \{0\}^{d_1}, \dots, k_l, \{0\}^{d_l})X^{k_1-1}Y^{d_1+\cdots+d_l}(1+X)Y^l \\
&\quad - \sum_{l\geq 1} \sum_{\substack{k_1=\cdots=k_l=1\\ d_1,\dots,d_l\geq 0}} \zeta_q^{\mathrm{SZ}}(k_1, \{0\}^{d_1}, \dots, k_l, \{0\}^{d_l})Y^{d_1+\cdots+d_l}Y^l.
\end{aligned}$$

The RHS equals

$$\begin{aligned}
&\sum_{l\geq 1} \sum_{\substack{k_1>1\\ d_1,\dots,d_l\geq 0}} \zeta_q^{\mathrm{SZ}}(k_1, \{0\}^{d_1}, 1, \{0\}^{d_2}, \dots, 1, \{0\}^{d_l})X^{k_1-1}Y^{d_1+\cdots+d_l+l} \\
&\quad + \sum_{l\geq 1} \sum_{\substack{k_1\geq 1\\ d_1,\dots,d_l\geq 0}} \zeta_q^{\mathrm{SZ}}(k_1, \{0\}^{d_1}, 1, \{0\}^{d_2}, \dots, 1, \{0\}^{d_l})X^{k_1}Y^{d_1+\cdots+d_l+l} \\
&= \sum_{l\geq 1} \sum_{\substack{k_1>1\\ d_1,\dots,d_l\geq 0}} \zeta_q^{\mathrm{SZ}}(k_1, \{0\}^{d_1}, 1, \{0\}^{d_2}, \dots, 1, \{0\}^{d_l})X^{k_1-1}Y^{d_1+\cdots+d_l+l} \\
&\quad + \sum_{l\geq 1} \sum_{\substack{k_1>1\\ d_1,\dots,d_l\geq 0}} \zeta_q^{\mathrm{SZ}}(k_1-1, \{0\}^{d_1}, 1, \{0\}^{d_2}, \dots, 1, \{0\}^{d_l})X^{k_1-1}Y^{d_1+\cdots+d_l+l} \\
&= \sum_{d\geq 0} \sum_{\varepsilon_1,\dots,\varepsilon_d\in\{0,1\}} \sum_{k_1>1} \zeta_q^{\mathrm{SZ}}(k_1, \varepsilon_1, \dots, \varepsilon_d)X^{k_1-1}Y^{d+1} \\
&\quad - \sum_{d\geq 0} \sum_{\varepsilon_1,\dots,\varepsilon_d\in\{0,1\}} \sum_{k_1>1} \zeta_q^{\mathrm{SZ}}(k_1-1, \varepsilon_1, \dots, \varepsilon_d)X^{k_1-1}Y^{d+1} \\
&= \sum_{k_1>1,\, d\geq 0} \zeta_q^{\mathrm{BZ}}(k_1, \{1\}^d)X^{k_1-1}Y^{d+1} \\
&= \sum_{k,d\geq 1} \zeta_q^{\mathrm{BZ}}(k+1, \{1\}^{d-1})X^kY^d.
\end{aligned}$$

In the second last step we used an explicit translation of BZ-qMZVs into SZ-qMZVs that can be found e.g. in Proposition 2.23. Furthermore, the statement for general $r \geq 1$ follows by an analogous calculation. □

Remark A.56. We can formulate the theorem also more understandable:
For $r \geq 1$ we have

$$\mathfrak{b}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} \equiv \sum_{l_1,\dots,l_r\geq 1} \mathfrak{s}\binom{X_1,0,\dots,0,\dots,X_r,0,\dots,0}{Y_1,\underbrace{Y_1,\dots,Y_1}_{l_1-1},\dots,Y_r,\underbrace{Y_r,\dots,Y_r}_{l_r-1}} \prod_{j=1}^{r}(1+\delta_j X_j)Y_j^{l_j-1}$$

modulo terms not divisible by $\prod_{j=1}^{r} X_j Y_j$.

One of the main reasons why BZ-qMZVs are of interest is that they satisfy the same duality as MZVs:

Theorem A.57 (BZ-Duality; Bradley [Bra, Thm. 5], Seki-Yamamoto [SY, Thm. 1.2])**.** *Let be $\tau : \mathfrak{h} \to \mathfrak{h}$ the anti-automorphism w.r.t. concatenation, induced by $\tau(x_0) := x_1$, $\tau(x_1) := x_0$. Then, we have on $\mathfrak{h}^0$*

$$\zeta_q^{BZ} \circ \tau = \zeta_q^{BZ}.$$

□

Besides this duality, there is a bigger class of relations, the q-Ohno relations. BZ-duality is a special case of them:

Theorem A.58 (q-Ohno relations; Okuda-Takeyama [OT, Thm. 1])**.** *For any admissible index $\mathbf{k} = (k_1,\dots,k_r)$ and any $c \in \mathbb{N}_0$ we have*

$$\sum_{|\mathbf{c}|=c} \zeta_q^{BZ}(\mathbf{k}+\mathbf{c}) = \sum_{|\mathbf{c}|=c} \zeta_q^{BZ}(\mathbf{k}^\vee+\mathbf{c}),$$

where we sum over all $\mathbf{c} = (c_1,\dots,c_r) \in \mathbb{N}_0^r$ with $|\mathbf{c}| := c_1 + \cdots + c_r = c$. □

Remark A.59 (Comparison)**.** The indisputable advantage of this model is that it satisfies the same duality as MZVs. In particular, the latter follows from BZ-duality by taking the limit $q \to 1$ after multiplication with $(1-q)^{k_1+\cdots+k_r}$.

On the other hand, it is a bit challenging to handle, like finding a 'good' generating series. For these things, the SZ-model is most times superior to the BZ-model.

A.2.3. Bi-brackets

Another interesting model of q-analogues are so-called brackets (introduced in Bachmann's master thesis [Ba1], further investigated in [BK1]) and their generalization, bi-brackets, introduced by Bachmann in his PhD thesis ([Ba2]).

The motivation for introducing bi-brackets comes from examining the Fourier expansion of Eisenstein series and their generalization, Multiple Eisenstein series, such as their derivatives (studied in [Ba4]). From there, the original definition is justified:

Definition/Proposition A.60 ([Ba4])**.**

(i) Define for $r \geq 0$, $k_1, \dots, k_r \geq 1$, $d_1, \dots, d_r \geq 0$ the bi-bracket

$$\mathrm{g}\binom{k_1, \dots, k_r}{d_1, \dots, d_r} := \sum_{\substack{m_1 > \dots > m_r > 0 \\ n_1, \dots, n_r > 0}} \frac{m_1^{d_1}}{d_1!} \cdots \frac{m_r^{d_r}}{d_r!} \frac{n_1^{k_1-1} \dots n_r^{k_r-1}}{(k_1-1)! \dots (k_r-1)!} q^{m_1 n_1 + \dots + m_r n_r}$$
$$= \sum_{m_1 > \dots > m_r > 0} \frac{m_1^{d_1}}{d_1!} \cdots \frac{m_r^{d_r}}{d_r!} \frac{P_{k_1}(q^{m_1})}{(1-q^{m_1})^{k_1}} \cdots \frac{P_{k_r}(q^{m_r})}{(1-q^{m_r})^{k_r}},$$

where P_k is the k-th *Eulerian polynomial*, defined via

$$\frac{P_k(X)}{(1-X)^k} := \sum_{n>0} \frac{n^{k-1}}{(k-1)!} X^n.$$

Additionally, we set $\mathrm{g}\binom{\emptyset}{\emptyset} := 1$ as usually. Furthermore, denote by $k_1 + \dots + k_r$ the *weight* and by r the *depth* of the bi-bracket.

(ii) For any $r \geq 0$ and any index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{N}^r$ the *bracket* of $\mathbf{k}$ is defined as

$$\mathrm{g}(\mathbf{k}) := \mathrm{g}(k_1, \dots, k_r) := \zeta_q(k_1, \dots, k_r; P_{k_1}, \dots, P_{k_r})$$
$$= \sum_{m_1 > \dots > m_r > 0} \frac{P_{k_1}(q^{m_1})}{(1-q^{m_1})^{k_1}} \cdots \frac{P_{k_r}(q^{m_r})}{(1-q^{m_r})^{k_r}}.$$

We set for $r = 0$ again $\mathrm{g}(\emptyset) := 1$

Remark A.61. (i) An explicit expression of Eulerian polynomials is

$$P_k(X) = \frac{1}{(k-1)!} \sum_{n=1}^{k} \left(\sum_{j=0}^{n-1} (-1)^j \binom{k}{j} (n-j)^{k-1} \right) X^n.$$

(ii) The name *(bi-)bracket* comes from the original notation where (bi-) brackets were denoted by $[\dots]$ instead of $\mathrm{g}(\dots)$.

(iii) Every bracket $\mathrm{g}(k_1, \dots, k_r)$ is a bi-bracket since

$$\mathrm{g}(k_1, \dots, k_r) = \mathrm{g}\binom{k_1, \dots, k_r}{0, \dots, 0}.$$

They generalize Eisenstein series since for even k, $\mathrm{g}\binom{k}{0}$ is the usual Eisenstein series of weight k, G_k, minus the constant term, i.e.

$$G_k = -\frac{B_k}{2k!} + \mathrm{g}\binom{k}{0}.$$

Furthermore, for every $d > 0$, it is

$$\left(q\frac{d}{dq}\right)^d G_k = \frac{(k+d-1)!d!}{(k-1)!}\, \mathrm{g}\binom{k+d}{d}.$$

With the observation done before, one can obtain that the space of quasi-modular forms, which is $\mathbb{Q}[G_2, G_4, G_6]$, is a proper subspace of $\mathcal{Z}_q$. In this way, we get a connection to modular forms, which play an important role in the theory of MZVs as considered e.g. in [GKZ].

Bi-brackets and their structure are well known, for more details than in this section we refer to [Ba4], [Ba3], [Ba5], [BK2],[BK1], [Zu2].

Theorem A.62 ([BK2, Thm. 2.3])**.** *Bi-brackets span the space $\mathcal{Z}_q$, i.e.*

$$\mathcal{Z}_q = \left\langle \mathrm{g}\binom{k_1, \dots, k_r}{d_1, \dots, d_r} \,\middle|\, r \geq 0,\ k_i \geq 1,\ d_i \geq 0 \right\rangle_{\mathbb{Q}}.$$

□

Also, the algebra of bi-brackets can be viewed as a quasi-shuffle algebra:

Definition A.63. Consider the alphabet

$$A_z^{bi} := \{z_{k,d} :\ k, d \in \mathbb{Z},\ k \geq 1,\ d \geq 0\}$$

and define on $\mathbb{Q}A_z^{bi}$ the product $\boxdot$ by

$$\begin{aligned} z_{k_1,d_1} \boxdot z_{k_2,d_2} :=& \binom{d_1+d_2}{d_1} \sum_{1 \leq j \leq k_1} \lambda^j_{k_1,k_2} z_{j,d_1+d_2} + \binom{d_1+d_2}{d_1} \sum_{1 \leq j \leq k_2} \lambda^j_{k_2,k_1} z_{j,d_1+d_2} \\ &+ \binom{d_1+d_2}{d_1} z_{k_1+k_2,d_1+d_2} \end{aligned}$$

and $\mathbb{Q}$-bilinear continuation to $\mathbb{Q}A_z^{bi}$. Here, $\lambda^j_{a,b}$ is defined as

$$\lambda^j_{a,b} := (-1)^{b-1}\binom{a+b-j-1}{a-j}\frac{B_{a+b-j}}{(a+b-j)!}.$$

Bachmann showed in his work [Ba4] that this product is associative and commutative. Hence, it induces a quasi-shuffle product $\circledast$:

Theorem A.64 ([Ba4, Thm. 3.6])**.** *(i) ([Ba4, Thm. 3.6]): The evaluation map, defined via*

$$\begin{aligned} \mathrm{g} : (\mathbb{Q}A_z^{bi}, \circledast) &\longrightarrow (\mathcal{Z}_q, \cdot), \\ z_{k_1,d_1} \dots z_{k_r,d_r} &\longmapsto \mathrm{g}\binom{k_1, \dots k_r}{d_1, \dots, d_r}, \end{aligned}$$

$\mathrm{g}(\emptyset) := 1$ *and $\mathbb{Q}$-linear continuation, is an algebra homomorphism.*

(ii) ([Zu2, Thm. 2]): The quasi-shuffle product $\circledast$ implies the stuffle product of MZVs. □

As for SZ-qMZVs, also for bi-brackets it is often convenient to work with their generating series because of the quite compact written term containing all information. Bachmann did this already during his PhD:

Theorem A.65 ([Ba4, Theorem 2.3])**.** *Let be*

$$\mathfrak{g}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} := \sum_{\substack{k_1,\dots,k_r>0\\ d_1,\dots,d_r>0}} g\binom{k_1,\dots,k_r}{d_1-1,\dots,d_r-1} X_1^{k_1-1}\dots X_r^{k_r-1} Y_1^{d_1-1}\dots Y_r^{d_r-1}$$

for every $r \geq 1$. Then we have

$$\mathfrak{g}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} = \sum_{\substack{m_1>\dots>m_r>0\\ n_1,\dots,n_r\geq 1}} \prod_{j=1}^{r} e^{m_jY_j} e^{n_jX_j} q^{m_jn_j}.$$

Proof. We have for every $r \geq 1$

$$\begin{aligned}
&\mathfrak{g}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r}\\
&= \sum_{\substack{k_1,\dots,k_r>0\\ d_1,\dots,d_r>0}} \sum_{\substack{m_1>\dots>m_r>0\\ n_1,\dots,n_r>0}} \left(\prod_{j=1}^{r} \frac{(m_jY_j)^{d_j-1}}{(d_j-1)!}\right)\left(\prod_{j=1}^{r} \frac{(n_jX_j)^{k_j-1}}{(k_j-1)!}\right) q^{m_1n_1+\dots+m_rn_r}\\
&= \sum_{\substack{m_1>\dots>m_r>0\\ n_1,\dots,n_r>0}} \left(\prod_{j=1}^{r}\sum_{d_j=1}^{\infty} \frac{(m_jY_j)^{d_j-1}}{(d_j-1)!}\right)\left(\prod_{j=1}^{r}\sum_{k_j=1}^{\infty} \frac{(n_jX_j)^{k_j-1}}{(k_j-1)!}\right) q^{m_1n_1+\dots+m_rn_r}\\
&= \sum_{\substack{m_1>\dots>m_r>0\\ n_1,\dots,n_r\geq 1}} \prod_{j=1}^{r} e^{m_jY_j} e^{n_jX_j} q^{m_jn_j}.
\end{aligned}$$

□

One application of the generating series of bi-brackets is for proving a nice identity among bi-brackets, the so-called *partition relation*, which can be viewed as a kind of duality in the model of bi-brackets:

Theorem A.66 (Partition relation, [Ba4, Theorem 2.3])**.** *For all $r \geq 1$ we have*

$$\mathfrak{g}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} = \mathfrak{g}\binom{Y_1+\dots+Y_r,\dots,Y_1+Y_2,Y_1}{X_r,X_{r-1}-X_r,\dots,X_1-X_2}. \tag{A.5}$$

□

Remark A.67. The name of this relation comes from the fact that $\mathfrak{g}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r}$ is a sum over all partitions with exactly r distinct parts and the relation itself is obtained by taking the sum over the partitions with transposed Young diagram.

Another application of the generating series of bi-brackets is to give elegant translations between bi-brackets and the SZ-model. That is possible since SZ-qMZVs, as well as bi-brackets, span $\mathcal{Z}_q$ (Prop. A.39, Thm. A.62):

Theorem A.68 (Translation bi-brackets-SZ-model)**.** *(i) For every $r \geq 1$ we have*

$$\prod_{j=1}^{r} e^{X_j} e^{Y_1+\cdots+Y_j} \cdot \mathfrak{s}\begin{pmatrix} e^{X_1}-1,\dots,e^{X_r}-1 \\ e^{Y_1}-1,\dots,e^{Y_1+\cdots+Y_r}-1 \end{pmatrix} = \mathfrak{g}\begin{pmatrix} X_1,\dots,X_r \\ Y_1,\dots,Y_r \end{pmatrix}.$$

(ii) For every $r \geq 1$ we have

$$\mathfrak{s}\begin{pmatrix} X_1,\dots,X_r \\ Y_1,\dots,Y_r \end{pmatrix} = \left(\prod_{j=1}^{r}(1+X_j)(1+Y_1+\cdots+Y_j)\right)^{-1} \mathfrak{g}\begin{pmatrix} \ln(X_1+1),\dots,\ln(X_r+1) \\ \ln(Y_1+1),\dots,\ln(Y_1+\cdots+Y_r+1) \end{pmatrix}.$$

□

Proof. (i) Multiply both sides in Theorem A.45 with $\prod_{j=1}^{r}(1+X_j)(1+Y_j)$ and then substitute

$$X_j \mapsto e^{X_j}-1, \quad Y_j \mapsto e^{Y_1+\cdots+Y_j}-1 \qquad \text{for all } 1 \leq j \leq r.$$

(ii) Substitute in (i)

$$X_j \mapsto \ln(X_j+1), \quad Y_j \mapsto \ln(Y_1+\cdots+Y_j+1) \qquad \text{for all } 1 \leq j \leq r.$$

□

Remark A.69. From Theorem A.68 we obtain a new proof of the well-known fact that bi-brackets and SZ-qMZVs span the same $\mathbb{Q}$-vector space, $\mathcal{Z}_q$.

We obtain a direct, but less elegant, translation of bi-brackets into SZ-qMZVs when using the identity (A.3) and elementary calculations:

Theorem A.70. *For every $r \in \mathbb{N}$, $k_1,\dots,k_r \in \mathbb{N}$, $d_1,\dots,d_r \in \mathbb{N}_0$, we have:*

$$\mathrm{g}\begin{pmatrix} k_1,\dots,k_r \\ d_1,\dots,d_r \end{pmatrix} = \sum_{\substack{1\leq n_j\leq p_j\leq k_j \\ 0\leq f_j\leq d_j \\ 1\leq j\leq r}} c_{\mathbf{k}}^{\mathbf{d}}(\mathbf{n}) \cdot \left[\prod_{j=1}^{r}\binom{d_j}{f_j}\binom{k_j-n_j}{p_j-n_j}\right]$$

$$\times \prod_{j=1}^{r} \sum_{g_j=0}^{F^{\mathbf{f}}_{\mathbf{h},\mathbf{g}}(j)} \binom{F^{\mathbf{f}}_{\mathbf{h},\mathbf{g}}(j)}{g_j} \sum_{l_j=0}^{g_j} \left(\delta_{g_j=0} + \sum_{\substack{s_1+\dots+s_{l_j}=g_j \\ s_i \geq 1}} \binom{g_j}{s_1,\dots,s_{l_j}} \right) \sum_{h_j=0}^{H^{\mathbf{f}}_{\mathbf{h},\mathbf{g}}(j)} \binom{H^{\mathbf{f}}_{\mathbf{h},\mathbf{g}}(j)}{h_j}$$

$$\times \zeta_q^{SZ}(p_1, \{0\}^{l_1}, \dots, p_r, \{0\}^{l_r})$$

$$= \sum_{\substack{1\leq n_j \leq p_j \leq k_j \\ 0 \leq f_j \leq d_j \\ 1 \leq j \leq r}} c^{\mathbf{d}}_{\mathbf{k}}(\mathbf{n}) \left(\prod_{j=1}^{r} \binom{d_j}{f_j} \binom{k_j - n_j}{p_j - n_j} \right)$$

$$\times \sum_{g_1=0}^{f_1} \binom{f_1}{g_1} \sum_{l_1=0}^{g_1} \left[\delta_{g_1=0} + \sum_{\substack{s_1+\dots+s_{l_1}=g_1 \\ s_i \geq 1}} \binom{g_1}{s_1,\dots,s_{l_1}} \right] \sum_{h_1=0}^{f_1-g_1} \binom{f_1 - g_1}{h_1}$$

$$\times \dots$$

$$\times \dots \times \sum_{h_{r-1}=0}^{\substack{f_1+\dots+f_{r-1} \\ -(g_1+\dots+g_{r-1}) \\ -(h_1+\dots+h_{r-2})}} \binom{f_1 + \dots + f_{r-1} - (g_1 + \dots g_{r-1} + h_1 + \dots h_{r-2})}{h_{r-1}}$$

$$\times \sum_{l_r=0}^{\substack{f_1+\dots+f_r \\ -(g_1+\dots+g_{r-1}) \\ -(h_1+\dots+h_{r-1})}} \left[\delta_{\substack{f_1+\dots+f_r \\ -(g_1+\dots+g_{r-1}) \\ -(h_1+\dots+h_{r-1})=0}} + \sum_{\substack{s_1+\dots+s_{l_r} \\ =f_1+\dots+f_r \\ -(g_1+\dots+g_{r-1}) \\ -(h_1+\dots+h_{r-1}) \\ s_i\geq 1}} \binom{\substack{f_1+\dots+f_r-(g_1+\dots+g_{r-1} \\ +h_1+\dots+h_{r-1})}}{s_1,\dots,s_{l_r}} \right]$$

$$\times \zeta_q^{SZ}(p_1, \{0\}^{l_1}, \dots, p_r, \{0\}^{l_r})$$

with $c^{\mathbf{d}}_{\mathbf{k}}(\mathbf{n}) := \prod_{l=1}^{r} \frac{1}{d_l!(k_l-1)!} \left(\sum_{i=0}^{n_l-1} (-1)^i \binom{k_l}{i} (n_l - i)^{k_l-1} \right) \in \mathbb{Q}$ *and*

$$F^{\mathbf{f}}_{\mathbf{h},\mathbf{g}}(j) := f_j + \sum_{i=1}^{j-1}(f_i - g_i - h_i), \quad H^{\mathbf{f}}_{\mathbf{h},\mathbf{g}}(j) := F^{\mathbf{f}}_{\mathbf{h},\mathbf{g}}(j) - g_j = f_j - g_j + \sum_{i=1}^{j-1}(f_i - g_i - h_i)$$

for all $1 \leq j \leq r$.

Remark that in the $(j = r)$-factor all summands with $g_r < F^{\mathbf{f}}_{\mathbf{h},\mathbf{g}}(r)$ are 0 and that is why the sum over g_r consists only of the $(g_r = F^{\mathbf{f}}_{\mathbf{h},\mathbf{g}}(r))$-summand which is $\binom{F^{\mathbf{f}}_{\mathbf{h},\mathbf{g}}(r)}{F^{\mathbf{f}}_{\mathbf{h},\mathbf{g}}(r)} = 1$. But then $F^{\mathbf{f}}_{\mathbf{h},\mathbf{g}}(r) - g_r = 0$, why the sum over h_r consists only of the $(h_r = 0)$-summand which is again 1 why the factor for $j = r$ is only the sum over l_j with $g_r = F^{\mathbf{f}}_{\mathbf{h},\mathbf{g}}(r)$.

Proof. According to the definition of bi-brackets, we get:

$$\mathrm{g}\binom{k_1,\dots,k_r}{d_1,\dots,d_r} = \sum_{m_1>\dots>m_r>0} \frac{m_1^{d_1}}{d_1!}\cdot\ldots\cdot\frac{m_r^{d_r}}{d_r!}\frac{P_{k_1}(q^{m_1})}{(1-q^{m_1})^{k_1}}\cdot\ldots\cdot\frac{P_{k_r}(q^{m_r})}{(1-q^{m_r})^{k_r}}$$

$$= \sum_{m_1>\dots>m_r>0}\prod_{j=1}^{r}\frac{m_j^{d_j}}{d_j!}\frac{\frac{1}{(k_j-1)!}\sum\limits_{n_j=1}^{k_j}\left(\sum\limits_{i=0}^{n_j-1}(-1)^i\binom{k_j}{i}(n_j-i)^{k_j-1}\right)q^{m_jn_j}}{(1-q^{m_j})^{k_j}}$$

$$= \sum_{\substack{m_1>\dots>m_r>0\\ 1\le n_j\le k_j\\ 1\le j\le r}} c_{\mathbf{k}}^{\mathbf{d}}(\mathbf{n})\prod_{j=1}^{r}m_j^{d_j}\frac{q^{m_jn_j}}{(1-q^{m_j})^{k_j}}$$

$$\overset{(A.3)}{=} \sum_{\substack{1\le n_j\le k_j\\ 1\le j\le r}} c_{\mathbf{k}}^{\mathbf{d}}(\mathbf{n})\sum_{m_1>\dots>m_r>0}\prod_{j=1}^{r}\left(\sum_{f_j=0}^{d_j}\binom{d_j}{f_j}(m_j-1)^{f_j}\sum_{p_j=n_j}^{k_j}\binom{k_j-n_j}{p_j-n_j}\frac{q^{m_jp_j}}{(1-q^{m_j})^{p_j}}\right)$$

$$= \sum_{\substack{1\le n_j\le p_j\le k_j\\ 0\le f_j\le d_j\\ 1\le j\le r}} c_{\mathbf{k}}^{\mathbf{d}}(\mathbf{n})\left(\prod_{j=1}^{r}\binom{d_j}{f_j}\binom{k_j-n_j}{p_j-n_j}\right)\sum_{m_1>\dots>m_r>0}\prod_{j=1}^{r}(m_j-1)^{f_j}\frac{q^{m_jp_j}}{(1-q^{m_j})^{p_j}}.$$

For the second last '=', we used the binomial theorem in the form

$$m_l^{d_l} = ((m_l-1)+1)^{d_l} = \sum_{j=0}^{d_l}\binom{d_l}{j}(m_l-1)^j.$$

To avoid confusion about the above big sum, we will concentrate on the last sum, which will give the stated theorem via the following combinatorial argument:

We can interpret the factor $(m_j-1)^{f_j}$ as the same sum without this factor and additional summands $b_1^j,\dots,b_{f_j}^j$ (for better understanding, the second index, j, is in the exponent; this is also for a_i^j below the case) where they range between 0 and m_j, i.e. $m_j > b_1^j,\dots,b_{f_j}^j > 0$.

Since these additional summands aren't ordered, we have to make them ordered via we distinguish between

- how many of the b_i^j can have the property $m_t > b_i^j > m_{t+1}$ for $1\le t\le r-1$,
- how many of such b_i^j can coincide,
- how many of the b_i^j can be equal m_t for some $2\le t\le r$.

This is done inductive:

- The property $m_1 > b_i^j > m_2$ can have 0 upto f_1 of the b_i^j. For every such number, g_1, we have $\binom{f_1}{g_1}$ choices, which g_1 of the possible b_i^j lie between m_1 and m_2. This gives the $\sum_{g_1=0}^{F_{\mathbf{h},\mathbf{g}}^{\mathbf{f}}(1)}$-sum (remark $F_{\mathbf{h},\mathbf{g}}^{\mathbf{f}}(1) = f_1$).

- Independent of the above choice, the g_1 chosen b_i^j can coincide. The ways, how this is possible, are: they coincide all ($l_1 = 1$), they can coincide in two different ways ($l_1 = 2$) and so on or they are all different $l_1 = g_1$. The number of ways how they can coincide gives the multinomial coefficient. If $g_1 = 0$, nothing coincides ($l_1 = 0$) and there is exact one possibility, which clarifies the Kronecker-delta $\delta_{g_1=0}$ and so the $\sum_{l_2=0}^{g_1}$-sum is declared.

- Now, there are $f_1 - g_1$ of the b_i^j of the first item left, i.e. they are all $\leq m_2$. Hence, 0 upto $f_1 - g_1$ of them can equal m_2. For every such number, h_1, there are $\binom{f_1-g_1}{h_1}$ possibilities, which of them are equal m_2. This clarifies the $\sum_{h_1=0}^{H_{\mathbf{h},\mathbf{g}}^{\mathbf{f}}(1)}$-sum (remark $H_{\mathbf{h},\mathbf{g}}^{\mathbf{f}}(1) = f_1 - g_1$).

- Now, $f_1 - g_1 - h_1$ of the b_i^j of the first item are left and all are $< m_2$.

- So we can do every of the above three steps (i)-(iii) again, now with m_2 and m_3 instead of m_1 and m_2 (and $F_{\mathbf{h},\mathbf{g}}^{\mathbf{f}}(2) = f_2 + (f_1 - g_1 - h_1)$ instead of $F_{\mathbf{h},\mathbf{g}}^{\mathbf{f}}(1)$ for the number of b_i^j's less than m_2).

- We are now done since we get as remaining part of the formula (after ordering the indices) the sum

$$\sum_{\substack{m_j > a_1^j > \cdots > a_{l_j}^j > m_{j+1} \\ 1 \leq j \leq r}} \frac{q^{m_1 p_1}}{(1-q^{m_1})^{p_1}} \cdots \frac{q^{m_r p_r}}{(1-q^{m_r})^{p_r}} = \zeta_q^{\mathrm{SZ}}(p_1, \{0\}^{l_1}, \ldots, p_r, \{0\}^{l_r}),$$

where we set $m_{r+1} := 0$ as usual.

□

Example A.71. Consider g $\binom{3,2}{0,2}$ and compute without using Theorem A.70

$$\mathrm{g}\binom{3,2}{0,2}$$
$$= \sum_{m_1 > m_2 > 0} \frac{m_1^0}{0!} \frac{m_2^2}{2!} \frac{P_3(q^{m_1})}{(1-q^{m_1})^3} \frac{P_2(q^{m_2})}{(1-q^{m_2})^2}$$

$$
\begin{aligned}
&= \frac{1}{2} \sum_{m_1>m_2>0} m_2^2 \frac{\frac{1}{2}q^{m_1}(q^{m_1}+1)}{(1-q^{m_1})^3} \frac{q^{m_2}}{(1-q^{m_2})^2} \\
&= \frac{1}{4} \sum_{m_1>m_2>0} m_2^2 \left(\frac{q^{m_1}}{1-q^{m_1}} + 3\frac{q^{2m_1}}{(1-q^{m_1})^2} + 2\frac{q^{3m_1}}{(1-q^{m_1})^3} \right) \left(\frac{q^{m_2}}{1-q^{m_2}} + \frac{q^{2m_2}}{(1-q^{m_2})^2} \right) \\
&= \frac{1}{4} \sum_{m_1>m_2>0} ((m_2-1)^2 + 2(m_2-1)+1) \left(\frac{q^{m_1}}{1-q^{m_1}} + 3\frac{q^{2m_1}}{(1-q^{m_1})^2} + 2\frac{q^{3m_1}}{(1-q^{m_1})^3} \right) \\
&\qquad \times \left(\frac{q^{m_2}}{1-q^{m_2}} + \frac{q^{2m_2}}{(1-q^{m_2})^2} \right) \\
&= \frac{1}{4} \sum_{m_1>m_2>b_1,\, b_2>0} \left(\frac{q^{m_1}}{1-q^{m_1}} + 3\frac{q^{2m_1}}{(1-q^{m_1})^2} + 2\frac{q^{3m_1}}{(1-q^{m_1})^3} \right) \left(\frac{q^{m_2}}{1-q^{m_2}} + \frac{q^{2m_2}}{(1-q^{m_2})^2} \right) \\
&\quad + \frac{1}{2} \sum_{m_1>m_2>b>0} \left(\frac{q^{m_1}}{1-q^{m_1}} + 3\frac{q^{2m_1}}{(1-q^{m_1})^2} + 2\frac{q^{3m_1}}{(1-q^{m_1})^3} \right) \left(\frac{q^{m_2}}{1-q^{m_2}} + \frac{q^{2m_2}}{(1-q^{m_2})^2} \right) \\
&\quad + \frac{1}{4} \sum_{m_1>m_2>0} \left(\frac{q^{m_1}}{1-q^{m_1}} + 3\frac{q^{2m_1}}{(1-q^{m_1})^2} + 2\frac{q^{3m_1}}{(1-q^{m_1})^3} \right) \left(\frac{q^{m_2}}{1-q^{m_2}} + \frac{q^{2m_2}}{(1-q^{m_2})^2} \right) \\
&= \frac{1}{4} \big(2\zeta_q^{\mathrm{SZ}}(1,1,0,0) + 2\zeta_q^{\mathrm{SZ}}(1,2,0,0) + 2\cdot 3\zeta_q^{\mathrm{SZ}}(2,1,0,0) + 2\cdot 3\zeta_q^{\mathrm{SZ}}(2,2,0,0) \\
&\qquad + 2\cdot 2\zeta_q^{\mathrm{SZ}}(3,1,0,0) + 2\cdot 2\zeta_q^{\mathrm{SZ}}(3,2,0,0) + \zeta_q^{\mathrm{SZ}}(1,1,0) + \zeta_q^{\mathrm{SZ}}(1,2,0) + 3\zeta_q^{\mathrm{SZ}}(2,1,0) \\
&\qquad + 3\zeta_q^{\mathrm{SZ}}(2,2,0) + 2\zeta_q^{\mathrm{SZ}}(3,1,0) + 2\zeta_q^{\mathrm{SZ}}(3,2,0) \big) \\
&\quad + \frac{1}{2} \big(\zeta_q^{\mathrm{SZ}}(1,1,0) + \zeta_q^{\mathrm{SZ}}(1,2,0) + 3\zeta_q^{\mathrm{SZ}}(2,1,0) + 3\zeta_q^{\mathrm{SZ}}(2,2,0) + 2\zeta_q^{\mathrm{SZ}}(3,1,0) \\
&\qquad + 2\zeta_q^{\mathrm{SZ}}(3,2,0) \big) \\
&\quad + \frac{1}{4} \big(\zeta_q^{\mathrm{SZ}}(1,1) + \zeta_q^{\mathrm{SZ}}(1,2) + 3\zeta_q^{\mathrm{SZ}}(2,1) + 3\zeta_q^{\mathrm{SZ}}(2,2) + 2\zeta_q^{\mathrm{SZ}}(3,1) + 2\zeta_q^{\mathrm{SZ}}(3,2) \big) \\
&= \frac{1}{2} \big(\zeta_q^{\mathrm{SZ}}(1,1,0,0) + \zeta_q^{\mathrm{SZ}}(1,2,0,0) + 3\zeta_q^{\mathrm{SZ}}(2,1,0,0) + 3\zeta_q^{\mathrm{SZ}}(2,2,0,0) + 2\zeta_q^{\mathrm{SZ}}(3,1,0,0) \\
&\qquad + 2\zeta_q^{\mathrm{SZ}}(3,2,0,0) \big) \\
&\quad + \frac{1}{4} \big(3\zeta_q^{\mathrm{SZ}}(1,1,0) + 3\zeta_q^{\mathrm{SZ}}(1,2,0) + 9\zeta_q^{\mathrm{SZ}}(2,1,0) + 9\zeta_q^{\mathrm{SZ}}(2,2,0) + 6\zeta_q^{\mathrm{SZ}}(3,1,0) \\
&\qquad + 6\zeta_q^{\mathrm{SZ}}(3,2,0) \big) \\
&\quad + \frac{1}{4} \big(\zeta_q^{\mathrm{SZ}}(1,1) + \zeta_q^{\mathrm{SZ}}(1,2) + 3\zeta_q^{\mathrm{SZ}}(2,1) + 3\zeta_q^{\mathrm{SZ}}(2,2) + 2\zeta_q^{\mathrm{SZ}}(3,1) + 2\zeta_q^{\mathrm{SZ}}(3,2) \big).
\end{aligned}
$$

On the other hand, when applying our Theorem A.70, we first should give the explicit expression of the used Eulerian Polynomials P_3 and P_2:

It is $P_3(X) = \frac{1}{2}X(X+1)$ and

$$
\frac{1}{(3-1)!} \sum_{n=1}^{3} \left(\sum_{i=0}^{n-1} (-1)^i \binom{3}{i} (n-i)^{3-1} \right) X^n
$$

$$
\begin{aligned}
=&\frac{1}{2}\left((-1)^0\binom{3}{0}(1-0)^2\right)X^1+\frac{1}{2}\left((-1)^0\binom{3}{0}(2-0)^2+(-1)^1\binom{3}{1}(2-1)^2\right)X^2\\
&+\frac{1}{2}\left((-1)^0\binom{3}{0}(3-0)^2+(-1)^1\binom{3}{1}(3-1)^2(-1)^2\binom{3}{2}(3-2)^2\right)X^3\\
=&\frac{1}{2}X+\frac{1}{2}X^2=P_3(X).
\end{aligned}
$$

Analogously, we find for $P_2(X)=X$

$$
\begin{aligned}
&\frac{1}{(2-1)!}\sum_{n=1}^{2}\left(\sum_{i=0}^{n-1}(-1)^i\binom{2}{i}(n-i)^{2-1}\right)X^n\\
=&\left((-1)^0\binom{2}{0}(1-0)^1\right)X+\left((-1)^0\binom{2}{0}(2-0)^1+(-1)^1\binom{2}{1}(2-1)^1\right)X^2\\
=&X=P_2(X).
\end{aligned}
$$

Remark that we have in this example $r=2$ and $d_1=0$ why we can have only $f_1=0$ for f_1. Then also $g_1=0$ is the only possible value for g_1, why $l_1=0$ is the only one for l_1 and $h_1=0$ is the only possible for h_1. In particular, the sums over g_1, l_1, h_1 reduces in this example just to a factor 1.

Therefore, Theorem A.70 says in this example

$$
\mathrm{g}\binom{3,2}{0,2}=\sum_{\substack{1\le n_1\le 3,\,1\le n_2\le 2\\ n_1\le p_1\le 3,\,n_2\le p_2\le 2\\ 0\le f_2\le 2}} c_{3,2}^{0,2}(n_1,n_2)\binom{3-n_1}{p_1-n_1}\binom{2}{f_2}\binom{2-n_2}{p_2-n_2}
$$

$$
\times\sum_{l_2=0}^{f_2}\left(\delta_{f_2=0}+\sum_{\substack{s_1+\cdots+s_{l_2}=f_2\\ s_i\ge 1}}\binom{f_2}{s_1,\ldots,s_{l_2}}\right)\zeta_q^{\mathrm{SZ}}(p_1,p_2,\{0\}^{l_2}).
$$

Now, a simple computation gives $c_{3,2}^{0,2}(3,n_2)=c_{3,2}^{0,2}(n_1,2)=0$ for all $n_1,n_2\in\mathbb{N}$ since

$$
\sum_{i=0}^{3-1}(-1)^i\binom{3}{i}(3-1)^{3-1}=\sum_{i=0}^{2-1}(-1)^i\binom{2}{i}(2-1)^{2-1}=0
$$

and $c_{3,2}^{0,2}(n_1,n_2)=\frac{1}{4}$ for $n_1\le 2$, $n_2\le 1$.

Hence, in the above sum it is enough to consider the sum over $1\le n_1\le 2$ and $1\le n_2\le 1$, i.e. in every summand we have $n_2=1$, why we get

$$
\mathrm{g}\binom{3,2}{0,2}=\frac{1}{4}\sum_{\substack{1\le n_1\le 2,\\ n_1\le p_1\le 3,\,n_2\le p_2\le 2\\ 0\le f_2\le 2}}\binom{3-n_1}{p_1-n_1}\binom{2}{f_2}\binom{1}{p_2-1}
$$

$$\times \sum_{l_2=0}^{f_2} \left(\delta_{f_2=0} + \sum_{\substack{s_1+\cdots+s_{l_2}=f_2 \\ s_i \geq 1}} \binom{f_2}{s_1, \ldots, s_{l_2}} \right) \zeta_q^{\mathrm{SZ}}(p_1, p_2, \{0\}^{l_2}).$$

Now, $\binom{1}{p_1-1} = 1$ holds independent of $p_1 \in \{1,2\}$ and in the $(f_2 = 0)$-sum, the sum over l_2 reduces to $\delta_{f_2=0}\zeta_q^{\mathrm{SZ}}(p_1,p_2) = \zeta_q^{\mathrm{SZ}}(p_1,p_2)$. In the $(f_2 = 1)$-sum, the sum over l_2 reduces to $\zeta_q^{\mathrm{SZ}}(p_1,p_2,0)$ since the only summand that occurs is the $l_2 = 1$ one and because of the observation

$$\sum_{\substack{s_1=1 \\ s_1 \geq 1}} \binom{1}{1} = 1.$$

In the $(f_2 = 2)$-sum, we see that in the l_2-sum only the summands for $l_2 = 1$ and $l_2 = 2$ occur. For $l_2 = 1$, we get the $(f_2 = 1)$-sum (with another pre-factor) since

$$\sum_{\substack{s_1=2 \\ s_1 \geq 1}} \binom{2}{s_1} = 1$$

again.

All in all, we obtain

$$\begin{aligned}
\mathrm{g}\begin{pmatrix}3,2\\0,2\end{pmatrix} &= \frac{1}{4} \sum_{\substack{1\leq n_1\leq 2\\ n_1\leq p_1\leq 3\\ 1\leq p_2\leq 2}} \binom{3-n_1}{p_1-n_1}\binom{2}{0}\zeta_q^{\mathrm{SZ}}(p_1,p_2) + \frac{1}{4} \sum_{\substack{1\leq n_1\leq 2\\ n_1\leq p_1\leq 3\\ 1\leq p_2\leq 2}} \binom{3-n_1}{p_1-n_1}\binom{2}{1}\zeta_q^{\mathrm{SZ}}(p_1,p_2,0)\\
&\quad + \frac{1}{4} \sum_{\substack{1\leq n_1\leq 2\\ n_1\leq p_1\leq 3\\ 1\leq p_2\leq 2}} \binom{3-n_1}{p_1-n_1}\binom{2}{2} \left(\zeta_q^{\mathrm{SZ}}(p_1,p_2,0) + \zeta_q^{\mathrm{SZ}}(p_1,p_2,0,0)\right)\\
&= \frac{1}{4}\big(\zeta_q^{\mathrm{SZ}}(1,1) + \zeta_q^{\mathrm{SZ}}(1,2) + 3\zeta_q^{\mathrm{SZ}}(2,1) + 3\zeta_q^{\mathrm{SZ}}(2,2) + 2\zeta_q^{\mathrm{SZ}}(3,1) + 2\zeta_q^{\mathrm{SZ}}(3,2)\big)\\
&\quad + \frac{3}{4}\big(\zeta_q^{\mathrm{SZ}}(1,1,0) + \zeta_q^{\mathrm{SZ}}(1,2,0) + 3\zeta_q^{\mathrm{SZ}}(2,1,0) + 3\zeta_q^{\mathrm{SZ}}(2,2,0)\\
&\qquad + 2\zeta_q^{\mathrm{SZ}}(3,1,0) + 2\zeta_q^{\mathrm{SZ}}(3,2,0)\big)\\
&\quad + \frac{1}{2}\big(\zeta_q^{\mathrm{SZ}}(1,1,0,0) + \zeta_q^{\mathrm{SZ}}(1,2,0,0) + 3\zeta_q^{\mathrm{SZ}}(2,1,0,0) + 3\zeta_q^{\mathrm{SZ}}(2,2,0,0)\\
&\qquad + 2\zeta_q^{\mathrm{SZ}}(3,1,0,0) + 2\zeta_q^{\mathrm{SZ}}(3,2,0,0)\big),
\end{aligned}$$

which is exactly what we've got in the explicit calculation before, confirming the theorem in this example.

Remark A.72. Zudilin's model of qMZVs, *multiple q-zeta brackets*, is closely related to bi-brackets ([Zu2] for details). They are defined for $k_1, \ldots, k_r, d_1, \ldots, d_r \geq 1$ ($r \geq 0$) as

$$\mathfrak{z}_q\binom{k_1, \ldots, k_r}{d_1, \ldots, d_r} := c \sum_{\substack{m_1,\ldots,m_r>0\\ n_1,\ldots,n_r>0}} m_1^{d_1-1} n_1^{k_1-1} \cdots m_r^{d_r-1} n_r^{k_r-1} q^{(m_1+\cdots+m_r)n_1+\cdots+m_r n_r}$$

with $c := \frac{1}{\prod_{j=1}^{r}(k_j-1)!(d_j-1)!}$.

This model also of interest because of the duality relation,

$$\mathfrak{z}_q\binom{k_1, \ldots, k_r}{d_1, \ldots, d_r} = \mathfrak{z}_q\binom{d_r, \ldots, d_1}{k_r, \ldots, k_1},$$

which is exactly the partition relation for bi-brackets ([Zu2, Prop. 4]).

We consider this model not in more detail since it is 'more or less' Bachmann's model of bi-brackets, and the connection to bi-brackets is also studied very well in [Zu2].

Remark A.73 (Comparison)**.** Bi-brackets are very good for connecting qMZVs direct with modular forms. Furthermore, this model gives with the partition relation another view onto SZ-duality since both are equivalent (cf. Prop. 3.22). However, explicit calculations are usually complicated in this model because the explicit representation of Eulerian polynomials is not very nice.

A.2.4. Takeyama-Bradley-Zhao model

We saw in Proposition A.52 that the $\mathbb{Q}$-span of BZ-qMZVs is a proper subspace of $\mathcal{Z}_q$. But it is for some situations comfortable to extend the BZ-model of qMZVs, especially when the elements of the model should span $\mathcal{Z}_q$.

This extension for the BZ-model is done as follows, as Takeyama did in [Ta1]:

Definition A.74. Denote $\overline{\mathbb{N}} := \{\overline{1}\} \cup \mathbb{N} = \{\overline{1}, 1, 2, 3, \ldots\}$ and define for every $r \geq 1$, $k_1, \ldots, k_r \in \overline{\mathbb{N}}$, $k_1 \neq 1$

$$\zeta_q^{\mathrm{TBZ}}(k_1, \ldots, k_r) := \sum_{m_1 > \cdots > m_r > 0} f(k_1, m_1) \ldots f(k_r, m_r),$$

where $f(\overline{1}, m) := \frac{q^m}{1-q^m}$, $f(k, m) := \frac{q^{(k-1)m}}{(1-q^m)^k}$, for $k \geq 1$. We set $\zeta_q^{\mathrm{TBZ}}(\emptyset) := 1$.

Proposition A.75. *The TBZ-model spans $\mathcal{Z}_q$, i.e.*

$$\mathcal{Z}_q = \langle \zeta_q^{TBZ}(k_1, \ldots, k_r) \,|\, r \geq 0,\, k_i \in \overline{\mathbb{N}},\, k_1 \neq 1 \rangle_{\mathbb{Q}}.$$

Proof. This is a direct consequence of Prop. A.77 and Prop. A.80 below, since we get from there that TBZ-qMZVs span the same space as SZ-qMZVs, i.e. $\mathcal{Z}_q$. □

Hence, TBZ-qMZVs are closed under the operator $q\frac{d}{dq}$.

The extended version of the BZ-model satisfies a quasi-shuffle product that is compatible with the one of the non-extended model:

Definition A.76. (i) Define $\mathfrak{h}^{TBZ} := \mathbb{Q}\langle z_{\overline{1}}, z_1, z_2, \dots\rangle$. Then we can view ζ_q^{TBZ} also as a map

$$\begin{aligned}\zeta_q^{\text{TBZ}} : \mathfrak{h}^{TBZ} &\longrightarrow \mathcal{Z}_q,\\ z_{k_1}\dots z_{k_r} &\longmapsto \zeta_q^{\text{TBZ}}(k_1,\dots,k_r),\end{aligned}$$

extended $\mathbb{Q}$-linearly and sending $\mathbf{1} \mapsto 1$.

(ii) Define now on the alphabet $\mathbb{Q}\{z_k : k \in \overline{\mathbb{N}}\}$ the associative and commutative product $\diamond_{TBZ}$ via

$$\begin{aligned}&z_{k_1} \diamond_{TBZ} z_{k_2} := z_{k_1+k_2} + z_{k_1+k_2-1}, \quad z_k \diamond_{TBZ} z_{\overline{1}} := z_{\overline{1}} \diamond_{TBZ} z_k := z_{k+1},\\ &z_{\overline{1}} \diamond_{TBZ} z_{\overline{1}} := z_2 - z_1\end{aligned}$$

for all $k, k_1, k_2 \in \mathbb{N}$.

Furthermore, let be $*_{TBZ}$ the induced quasi-shuffle product on $\mathbb{Q}\langle z_{\overline{1}}, z_1, z_2, \dots\rangle$.

Some straightforward computation shows that $\diamond_{TBZ}$ is indeed commutative and associative.

Proposition A.77. *The map ζ_q^{TBZ} is an algebra homomorphism, i.e. we have for all $u, v \in \mathbb{Q}\langle z_{\overline{1}}, z_1, z_2, \dots\rangle$*

$$\zeta_q^{TBZ}(u *_{TBZ} v) = \zeta_q^{TBZ}(u)\zeta_q^{TBZ}(v).$$

Proof. The proof is just computation, analogously to the one in the BZ-model (Prop. A.54). □

Remark A.78. For the TBZ-model, no 'good' generating series is known. This means that we don't know a one that could be written nicely or that would give us new results about TBZ-qMZVs.

Proposition A.79 (Duality)**.** *Let the maps U_{TBZ} and V_{TBZ} be defined as in Prop. A.80 resp. A.82 below. Then, we have for all $w \in \mathfrak{h}^{TBZ}$*

$$\zeta_q^{TBZ}(w) = (\zeta_q^{TBZ} \circ \mathrm{V}_{TBZ} \circ \tilde{\tau} \circ \mathrm{U}_{TBZ})(w).$$

Proof. U_{TBZ} and V_{TBZ} are translation maps of TBZ-qMZVs into the SZ-model resp. vice versa. Hence, the proof follows since SZ-duality holds. □

Takeyama calls this duality 'resummation duality' ([Ta1, Thm. 4]).

We give now an explicit translation into the SZ-model and vice versa:

Proposition A.80 (Translation TBZ into SZ)**.**
For every $d_1, \dots, d_r \in \mathbb{N}_0$*,* $k_1, \dots, k_{r-1} \in \mathbb{N}$ *(with* $k_1 \geq 2$ *if* $d_1 = 0$*) we have*

$$\begin{aligned} &\zeta_q^{TBZ}(\{\overline{1}\}^{d_1}, k_1, \dots, k_{r-1}, \{\overline{1}\}^{d_r}) \\ &= \sum_{\substack{\delta_j \in \{0,1\} \\ 1 \leq j \leq r-1}} \zeta_q^{SZ}(\{1\}^{d_1}, k_1 - \delta_1, \dots, \{1\}^{d_{r-1}}, k_{r-1} - \delta_{r-1}, \{1\}^{d_r}). \end{aligned}$$

Denote the corresponding map $\mathfrak{h}^{TBZ} \to \mathfrak{K}^1$ *by* U_{TBZ}*.*

Proof. This is a direct consequence of the definition of SZ-qMZVs and the identity

$$\frac{X^{k-1}}{(1-X)^k} = \frac{X^{k-1}}{(1-X)^{k-1}} + \frac{X^k}{(1-X)^k} \quad \text{for } k \in \mathbb{N}.$$

□

Example A.81. Consider $\zeta_q^{\mathrm{TBZ}}(\overline{1}, 2, 1)$. We have

$$\begin{aligned} \zeta_q^{\mathrm{TBZ}}(\overline{1}, 2, 1) &= \sum_{m_1 > m_2 > m_3 > 0} \frac{q^{m_1}}{1 - q^{m_1}} \frac{q^{m_2}}{(1 - q^{m_2})^2} \frac{1}{1 - q^{m_3}} \\ &= \sum_{m_1 > m_2 > m_3 > 0} \frac{q^{m_1}}{1 - q^{m_1}} \left(\frac{q^{m_2}}{1 - q^{m_2}} + \frac{q^{2m_2}}{(1 - q^{m_2})^2} \right) \left(1 + \frac{q^{m_3}}{1 - q^{m_3}} \right) \\ &= \zeta_q^{\mathrm{SZ}}(1, 2, 1) + \zeta_q^{\mathrm{SZ}}(1, 1, 1) + \zeta_q^{\mathrm{SZ}}(1, 2, 0) + \zeta_q^{\mathrm{SZ}}(1, 1, 0). \end{aligned}$$

Proposition A.82 (Translation SZ into TBZ)**.**
For every $k_1, \dots, k_r \in \mathbb{N}$*,* $d_1, \dots, d_r \in \mathbb{N}_0$ *we have*

$$\begin{aligned} &\zeta_q^{SZ}(k_1, \{0\}^{d_1}, \dots, k_r, \{0\}^{d_r}) \\ &= \sum_{\substack{1 \leq j_i \leq k_i, \\ \varepsilon_i \in \{\overline{1}, 1\}^{d_i} \\ 1 \leq i \leq r}} (-1)^{\sum_{i=1}^{r} k_i - j_i + |\varepsilon_i|} \zeta_q^{TBZ}(j_1 \delta_{j_1 \neq 1} + \overline{1} \delta_{j_1 = 1}, \varepsilon_1, \dots, j_r \delta_{j_r \neq 1} + \overline{1} \delta_{j_r = 1}, \varepsilon_r), \end{aligned}$$

where $|\varepsilon|$ *counts the* $\overline{1}$*'s in* ε*; we denote the corresponding map* $\mathfrak{K}^1 \to \mathfrak{h}^{TBZ}$ *by* V_{TBZ}*.*

Proof. This is a direct consequence of the identity

$$1 = \frac{1-X}{1-X} = \frac{1}{1-X} - \frac{X}{1-X}$$

for the zero-entries and

$$\frac{X^k}{(1-X)^k} = \sum_{1 \leq j \leq k} (-1)^{k-j} \left[\frac{X^{j-1}}{(1-X)^j} \delta_{j \neq 1} + \frac{X}{1-X} \delta_{j=1} \right] \quad \text{for } k \in \mathbb{N}$$

for the non-zero entries and using the definition of SZ- and TBZ-qMZVs.

The last identity is proven via direct calculation and some well-known identities among binomial coefficients. For $k \in \mathbb{N}$, we have

$$\begin{aligned}
X^k =& \sum_{1\le l\le k} (-1)^{k-l}\binom{k-1}{k-l} X^l + \sum_{1\le l\le k-1} (-1)^{k-l-1}\binom{k-1}{k-l} X^l \\
=& X\sum_{1\le l\le k} (-1)^{k-l}\binom{k-1}{l-1} X^{l-1} + \sum_{2\le l\le k} (-1)^{k-l}\binom{k-1}{k-l+1} X^{l-1} \\
=& (-1)^{k-1} X \sum_{0\le l\le k-1} (-1)^{l}\binom{k-1}{l} X^l + \sum_{2\le l\le k} (-1)^{k+l}\left[\sum_{2\le j\le l}\binom{k-j}{k-l}\right] X^{l-1} \\
=& (-1)^{k-1} X(1-X)^{k-1} + \sum_{2\le j\le k}\sum_{j\le l\le k} (-1)^{k+l}\binom{k-j}{k-l} X^{l-1} \\
=& (-1)^{k-1} X(1-X)^{k-1} + \sum_{2\le j\le k} (-1)^{k-j} X^{j-1} \sum_{j\le l\le k} (-1)^{l-j}\binom{k-j}{l-j} X^{l-j} \\
=& (-1)^{k-1} X(1-X)^{k-1} + \sum_{2\le j\le k} (-1)^{k-j} X^{j-1} \sum_{0\le l\le k-j} (-1)^{l}\binom{k-j}{l} X^{l} \\
=& (-1)^{k-1} X(1-X)^{k-1} + \sum_{2\le j\le k} (-1)^{k-j} X^{j-1}(1-X)^{k-j} \\
=& \sum_{1\le j\le k} (-1)^{k-1} X(1-X)^{k-1}\delta_{j=1} + (-1)^{k-j} X^{j-1}(1-X)^{k-j}.
\end{aligned}$$

Dividing by $(1-X)^k$ gives the identity. □

Example A.83. We have

$$\begin{aligned}
\zeta_q^{\mathrm{SZ}}(3,0,1) =& \zeta_q^{\mathrm{TBZ}}(\overline{1},1,\overline{1}) - \zeta_q^{\mathrm{TBZ}}(\overline{1},\overline{1},\overline{1}) - \zeta_q^{\mathrm{TBZ}}(2,1,\overline{1}) \\
&+\zeta_q^{\mathrm{TBZ}}(2,\overline{1},\overline{1}) + \zeta_q^{\mathrm{TBZ}}(3,1,\overline{1}) - \zeta_q^{\mathrm{TBZ}}(3,\overline{1},\overline{1}), \\
\zeta_q^{\mathrm{SZ}}(1,2,0,0) =& -\zeta_q^{\mathrm{TBZ}}(\overline{1},\overline{1},1,1) + \zeta_q^{\mathrm{TBZ}}(\overline{1},\overline{1},1,\overline{1}) + \zeta_q^{\mathrm{TBZ}}(\overline{1},\overline{1},\overline{1},1) \\
&-\zeta_q^{\mathrm{TBZ}}(\overline{1},\overline{1},\overline{1},\overline{1}) + \zeta_q^{\mathrm{TBZ}}(\overline{1},2,1,1) - \zeta_q^{\mathrm{TBZ}}(\overline{1},2,1,\overline{1}) \\
&-\zeta_q^{\mathrm{TBZ}}(\overline{1},2,\overline{1},1) + \zeta_q^{\mathrm{TBZ}}(\overline{1},2,\overline{1},\overline{1}).
\end{aligned}$$

By SZ-duality, we know $\zeta_q^{\mathrm{SZ}}(3,0,1) = \zeta_q^{\mathrm{SZ}}(1,2,0,0)$ giving the relation

$$\begin{aligned}
&\zeta_q^{\mathrm{TBZ}}(\overline{1},1,\overline{1}) - \zeta_q^{\mathrm{TBZ}}(\overline{1},\overline{1},\overline{1}) - \zeta_q^{\mathrm{TBZ}}(2,1,\overline{1}) + \zeta_q^{\mathrm{TBZ}}(2,\overline{1},\overline{1}) + \zeta_q^{\mathrm{TBZ}}(3,1,\overline{1}) - \zeta_q^{\mathrm{TBZ}}(3,\overline{1},\overline{1}) \\
=& -\zeta_q^{\mathrm{TBZ}}(\overline{1},\overline{1},1,1) + \zeta_q^{\mathrm{TBZ}}(\overline{1},\overline{1},1,\overline{1}) + \zeta_q^{\mathrm{TBZ}}(\overline{1},\overline{1},\overline{1},1) - \zeta_q^{\mathrm{TBZ}}(\overline{1},\overline{1},\overline{1},\overline{1}) + \zeta_q^{\mathrm{TBZ}}(\overline{1},2,1,1) \\
&-\zeta_q^{\mathrm{TBZ}}(\overline{1},2,1,\overline{1}) - \zeta_q^{\mathrm{TBZ}}(\overline{1},2,\overline{1},1) + \zeta_q^{\mathrm{TBZ}}(\overline{1},2,\overline{1},\overline{1})
\end{aligned}$$

in the TBZ-model.

Remark A.84 (Comparison). (i) Comfortable about the TBZ-model is that its span is $\mathcal{Z}_q$ as for the most common models of qMZVs, which is good when we want to compare results in different models such as (SZ-)duality.

(ii) It can be not comforting to work with $\mathfrak{h}^{TBZ}$ because of the extra letter $\overline{1}$ since it doesn't guarantee that we can switch smoothly to $\mathfrak{h}^0$ or $\mathfrak{K}^1$, which are the underlying word algebras for all the other models we consider.

A.2.5. Ohno-Okuda-Zudilin model

Another model for q-analogues of MZVs is the one first considered in 2012 ([OOZ]) and named after Ohno, Okuda and Zudilin. One application of this model is that some particular sum of OOZ-qMZVs is the generating series of the number of conjugacy classes of $\mathrm{GL}(n, K)$ for a finite field K (cf. §4.6).

Definition A.85. (i) We will work with an extended version of OOZ-qMZVs: Define for $r \geq 1$ and all integers $k_1, \dots, k_r \geq 0$ with $k_1 \geq 1$

$$\begin{aligned}\zeta_q^{\mathrm{OOZ}}(k_1, \dots, k_r) &:= \zeta_q(k_1, \dots, k_r; X, 1 \dots, 1) \\ &= \sum_{m_1 > \dots > m_k > 0} \frac{q^{m_1}}{(1-q^{m_1})^{k_1} \cdots (1-q^{m_r})^{k_r}}\end{aligned}$$

and set as usual $\zeta_q^{\mathrm{OOZ}}(\emptyset) := 1$.

When identifying an SZ-admissible index $(k_1, \dots, k_r)$ with $p^{k_1} y \cdots p^{k_r} y \in \mathfrak{K}^1$, we can define ζ_q^{OOZ} also as the map, uniquely determined through $\mathbf{1} \mapsto 1$, $\mathbb{Q}$-linearity and

$$\begin{aligned}\zeta_q^{\mathrm{OOZ}} : \mathfrak{K}^1 &\longrightarrow \mathcal{Z}_q, \\ p^{k_1} y \dots p^{k_r} y &\longmapsto \zeta_q^{\mathrm{OOZ}}(k_1, \dots, k_r).\end{aligned}$$

(ii) For a connection to the BZ-model, it is sometimes useful to restrict to admissible indices: Hence, we could define ζ_q^{OOZ} also on $\mathfrak{h}^0$ as the map

$$\begin{aligned}\zeta_q^{\mathrm{OOZ}} : \mathfrak{h}^0 &\longrightarrow \mathcal{Z}_q, \\ z_{k_1} \dots z_{k_r} &\longmapsto \zeta_q^{\mathrm{OOZ}}(k_1, \dots, k_r),\end{aligned}$$

extended to $\mathfrak{h}^0$ by $\mathbb{Q}$-linearity and mapping $\mathbf{1} \mapsto 1$.

It is natural to expect and necessary to work with the OOZ-model that its $\mathbb{Q}$-span is $\mathcal{Z}_q$, which is indeed the case:

Proposition A.86. *For the span of the OOZ-model we have*

(i) $\mathcal{Z}_{q,1} = \langle \zeta_q^{OOZ}(k_1, \dots, k_r) \,|\, r \geq 0,\, k_1 \geq 2,\, k_i \geq 1 \rangle_{\mathbb{Q}}$,

(ii) $\mathcal{Z}_q = \langle \zeta_q^{OOZ}(k_1, \ldots, k_r) \,|\, r \geq 0,\, k_1 \geq 1,\, k_i \geq 0 \rangle_{\mathbb{Q}}$.

Proof. A proof can be obtained from Proposition A.92, where we give explicit translations of OOZ-qMZVs into SZ-qMZVs and vice versa (resp. restricted OOZ-qMZVs into BZ-qMZVs), which is the reason why OOZ-qMZVs span the same $\mathbb{Q}$-vector space as SZ-qMZVs (resp. BZ-qMZVs) and that is $\mathcal{Z}_q$ (resp. $\mathcal{Z}_{q,1}$). □

In particular, the OOZ-model we work with is closed under the operator $q\frac{d}{dq}$ (by Rem. A.31(i)), while the restricted model is only conjecturally closed (by Rem. A.31(ii)).

Also, the restricted OOZ-model satisfies a shuffle product:

Definition A.87. Define the map T via $\mathbf{1} \mapsto 1$ and

$$\begin{aligned} T : \mathfrak{h}^0 &\longrightarrow \mathfrak{h}^1, \\ z_n v &\longmapsto z_n v - z_{n-1} v \end{aligned}$$

for all $n \geq 2$, $v \in \mathfrak{h}^1$ and $\mathbb{Q}$-linearity.

Then, the quasi-shuffle product on $\mathfrak{h}^0$ for the OOZ-model, $\sqcup\!\sqcup_{\mathrm{OOZ}}$, is defined as the unique map $\mathfrak{h}^0 \otimes \mathfrak{h}^0 \to \mathfrak{h}^0$ satisfying

$$T(u \sqcup\!\sqcup_{\mathrm{OOZ}} v) := T(u) * T(v)$$

for all $u, v \in \mathfrak{h}^0$.

Theorem A.88 ([CEM, Section 4.5])**.** *The product $\sqcup\!\sqcup_{OOZ}$ is well-defined and the evaluation map*

$$\begin{aligned} \zeta_q^{OOZ} : (\mathfrak{h}^0, \sqcup\!\sqcup_{OOZ}) &\longrightarrow (\mathcal{Z}_q, \cdot), \\ w &\longmapsto \zeta_q^{OOZ}(w) \end{aligned}$$

is an algebra homomorphism, i.e. in particular, we have for all $u, v \in \mathfrak{h}^0$

$$\zeta_q^{OOZ}(u \sqcup\!\sqcup_{OOZ} v) = \zeta_q^{OOZ}(u) \zeta_q^{OOZ}(v).$$

□

Further details to the quasi-shuffle structure, OOZ-qMZVs imply, can be found in [CEM] and [EMS].

As for other models, we consider a generating series for (the extended version of) OOZ-qMZVs:

Proposition A.89. *Define for all $r \geq 0$*

$$\mathfrak{oz}(X_1, \ldots, X_r, Y_1, \ldots, Y_r) := \sum_{\substack{k_1, \ldots, k_r \geq 1 \\ d_1, \ldots, d_r \geq 0}} \zeta_q^{OOZ}(k_1, \{0\}^{d_1}, \ldots, k_r, \{0\}^{d_r}) \frac{X_1^{k_1}}{k_1!} Y_1^{d_1} \cdots \frac{X_r^{k_r}}{k_r!} Y_r^{d_r}.$$

Then, we have for all $r \geq 0$

$$\mathfrak{oz}(X_1, \dots, X_r, Y_1, \dots, Y_r) = \sum_{m_1 > \dots > m_r > 0} q^{m_1} \prod_{j=1}^{r} (1+Y_j)^{m_j - m_{j+1} - 1} \left(e^{\frac{X_j}{1-q^{m_j}}} - 1 \right)$$

with $m_{r+1} := 0$.

Proof. We use the definition of the exponential function, the geometric sum identity and the binomial theorem to obtain

$$\mathfrak{oz}(X_1, \dots, X_r, Y_1, \dots, Y_r) = \sum_{\substack{k_1, \dots, k_r \geq 1 \\ d_1, \dots, d_r \geq 0}} \sum_{\substack{m_1 > n_1 > \dots > n_{d_1} \\ > m_2 > \dots > n_{d_1 + \dots + d_r} > 0}} q^{m_1} \prod_{j=1}^{r} \frac{1}{(1-q^{m_j})^{k_j}} \frac{X_j^{k_j}}{k_j!} Y_j^{d_j}$$

$$= \sum_{m_1 > \dots > m_r > 0} q^{m_1} \prod_{j=1}^{r} \left(\sum_{k_j \geq 1,\, d_j \geq 0} \frac{\left(\frac{X_j}{1+q^{m_j}}\right)^{k_j}}{k_j!} \binom{m_j - m_{j+1} - 1}{d_j} Y_j^{d_j} \right)$$

$$= \sum_{m_1 > \dots > m_r > 0} q^{m_1} \prod_{j=1}^{r} (1+Y_j)^{m_j - m_{j+1} - 1} \left(e^{\frac{X_j}{1-q^{m_j}}} - 1 \right).$$

□

For the OOZ-model, no individual duality relation is known so far. But we can translate into the SZ-model (resp. BZ-model for restricted definition), apply there SZ-duality (resp. BZ-duality) and then translate back into the OOZ-model, giving $\mathbb{Q}$-linear relations among OOZ-qMZVs:

Theorem A.90 (Duality, [EMS, Thm. 5.5, Thm. 5.9])**.** *Let be* U, V *as in Prop. A.92.*

(i) For all $w \in \mathfrak{h}^0$ we have

$$\zeta_q^{OOZ}(w) = (\zeta_q^{OOZ} \circ \mathrm{U}^{-1} \circ \tau \circ \mathrm{U})(w).$$

(ii) For all $w \in \mathfrak{K}^1$ we have

$$\zeta_q^{OOZ}(w) = (\zeta_q^{OOZ} \circ \mathrm{V}^{-1} \circ \tilde{\tau} \circ \mathrm{V})(w).$$

(iii) For any $w \in \mathfrak{K}^1$ we have

$$\zeta_q^{OOZ}(w) = (\zeta_q^{SZ,\star} \circ \tilde{\tau})(w),$$

where $\zeta_q^{SZ,\star}$ is the map of multiple zeta star values, defined via $\mathbf{1} \mapsto 1$, $\mathbb{Q}$-linearity and

$$\zeta_q^{SZ,\star} : \mathfrak{K}^1 \longrightarrow \mathcal{Z}_q,$$
$$p^{k_1} y \cdots p^{k_r} y \longmapsto \sum_{m_1 \geq \dots \geq m_r > 0} \frac{q^{m_1 k_1}}{(1-q^{m_1})^{k_1}} \cdots \frac{q^{m_r k_r}}{(1-q^{m_r})^{k_r}}.$$

□

Example A.91. Consider $w = p^2ypy$, i.e. $\tilde{\tau}(w) = pypyy$. It is

$$\begin{aligned}
\zeta_q^{\mathrm{OOZ}}(w) &= \zeta_q^{\mathrm{OOZ}}(2,1) = \sum_{m_1>m_2>0} \frac{q^{m_1}}{(1-q^{m_1})^2(1-q^{m_2})} \\
&= \sum_{m_1>m_2>0} \left(\frac{q^{m_1}}{1-q^{m_1}} + \frac{q^{2m_1}}{(1-q^{m_1})^2} \right) \left(1 + \frac{q^{m_2}}{1-q^{m_2}} \right) \\
&= \zeta_q^{\mathrm{SZ}}(1,0) + \zeta_q^{\mathrm{SZ}}(1,1) + \zeta_q^{\mathrm{SZ}}(2,0) + \zeta_q^{\mathrm{SZ}}(2,1) \\
&\overset{SZ-duality}{=} \zeta_q^{\mathrm{SZ}}(2) + \zeta_q^{\mathrm{SZ}}(1,1) + \zeta_q^{\mathrm{SZ}}(2,0) + \zeta_q^{\mathrm{SZ}}(1,1,0) \\
&= \left(\sum_{m_1=m_2=m_3>0} + \sum_{m_1>m_2=m_3>0} + \sum_{m_1=m_2>m_3>0} + \sum_{m_1>m_2>m_3>0} \right) \frac{q^{m_1}}{1-q^{m_1}} \frac{q^{m_2}}{1-q^{m_2}} \\
&= \sum_{m_1 \geq m_2 \geq m_3 > 0} \frac{q^{m_1}}{1-q^{m_1}} \frac{q^{m_2}}{1-q^{m_2}} = \zeta_q^{\mathrm{SZ},\star}(1,1,0) = \zeta_q^{\mathrm{SZ},\star}(\tilde{\tau}(w)),
\end{aligned}$$

verifying (iii) of the theorem in this case.

As above noted, we want to give the translation maps from the OOZ-model into the SZ- (resp. BZ-)model. This is what we do now:

Proposition A.92 ([EMS, Prop. 5.7, Rem. 5.8])**.**

(i) Translating the BZ-model into the restricted OOZ-model is done via the map U*, given through*

$$\begin{aligned}
\mathrm{U} : \mathfrak{h}^0 &\longrightarrow \mathfrak{h}^0, \\
z_{k_1} \dots z_{k_r} &\longmapsto \sum_{\substack{2 \leq n_1 \leq k_1 \\ 1 \leq n_j \leq k_j,\, j \geq 2}} \binom{k_1-2}{n_1-2} \binom{k_2-1}{n_2-1} \cdots \binom{k_r-1}{n_r-1} z_{n_1} \dots z_{n_r}
\end{aligned}$$

and $\mathbb{Q}$*-linear continuation.*

Then, for all $w \in \mathfrak{h}^0$ *we have*

$$\zeta_q^{OOZ}(w) = (\zeta_q^{BZ} \circ \mathrm{U})(w).$$

(ii) Analogously, translating the SZ-model into the OOZ-model is done via the map V*, given through*

$$\begin{aligned}
\mathrm{V} : \mathfrak{K}^1 &\longrightarrow \mathfrak{K}^1, \\
p^{k_1}y \dots p^{k_r}y &\longmapsto \sum_{\substack{1 \leq n_1 \leq k_1 \\ 0 \leq n_j \leq k_j,\, j \geq 2}} \binom{k_1-1}{n_1-1} \binom{k_2}{n_2} \cdots \binom{k_r}{n_r} p^{n_1}y \dots p^{n_r}y
\end{aligned}$$

and $\mathbb{Q}$*-linear continuation.*

Then, for all $w \in \mathfrak{K}^1$ *we have*

$$\zeta_q^{OOZ}(w) = (\zeta_q^{SZ} \circ \mathrm{V})(w).$$

(iii) *Furthermore,* U *and* V *are linear isomorphisms. Hence, we get translation in the other direction too:*

$$\zeta_q^{BZ} = \zeta_q^{OOZ} \circ \mathrm{U}^{-1},$$
$$\zeta_q^{SZ} = \zeta_q^{OOZ} \circ \mathrm{V}^{-1}.$$

□

Notice that this proposition proves in particular Proposition A.86.

Remark A.93 (Comparison)**.** (i) The 'duality relations' (i) and (ii) can be indeed viewed as duality in the OOZ-model, the authors in [EMS] do so. Still, we should compare them with the partition relation in the model of bi-brackets: The partition relation is the same as when translating bi-brackets into SZ-qMZVs, applying SZ-duality and then translating back into bi-brackets.

(ii) Duality relation (iii) is no 'real' duality relation, but in [EMS] called so, since $\tilde{\tau}$ gives SZ-duality in the SZ-model. However, this relation is interesting because it is, in truth, the translation map of the OOZ-model into the SZS-model, and this translation is obtained by the 'duality map' $\tilde{\tau}$.

A.2.6. Okounkov qMZVs

In the context of enumerative geometry and Hilbert schemes, a model of qMZVs introduced by Okounkov (see [Oko]) is very important.

Definition A.94 (Okounkov qMZVs)**.** Define for all indices $\mathbf{k} = (k_1, \ldots, k_r)$ $(r \geq 0)$ with every entry ≥ 2 the Okounkov-qMZV

$$\begin{aligned}\zeta_q^{\mathrm{Oko}}(\mathbf{k}) := \zeta_q^{\mathrm{Oko}}(k_1, \ldots, k_r) &:= \zeta_q(k_1, \ldots, k_r; p_{k_1}, \ldots, p_{k_r}) \\ &= \sum_{m_1 > \cdots > m_r > 0} \prod_{j=1}^{r} \frac{p_{k_j}(q^{m_j})}{(1-q^{m_j})^{k_j}},\end{aligned}$$

where

$$p_k(X) := \begin{cases} X^{\frac{k}{2}}, \ k \text{ even}, \\ X^{\frac{k-1}{2}}(1+X), \ k \text{ odd} \end{cases}$$

and $\zeta_q^{\mathrm{Oko}}(\emptyset) := 1$ as usually.

Proposition A.95. *The Okounkov-model spans $\mathcal{Z}_{q,1}^{\circ}$, i.e.*

$$\mathcal{Z}_{q,1}^{\circ} = \langle \zeta_q^{Oko}(k_1, \ldots, k_r) \mid r \geq 0, \, k_i \geq 2 \rangle_{\mathbb{Q}}.$$

Proof. This is stated and proven in [BK2, §2 (iv)]. □

In particular, the span of Okounkov-qMZVs is only conjecturally closed under $q\frac{d}{dq}$ by Remark A.31(ii).

As the other models of qMZVs, also Okounkov-qMZVs satisfy a quasi-shuffle product:

Definition A.96. (i) Consider $\mathfrak{h}^{Oko} := \mathbb{Q}\langle z_2, z_3 \ldots \rangle$. Then we can define ζ_q^{Oko} as map

$$\begin{aligned} \zeta_q^{\mathrm{Oko}} : \mathfrak{h}^{Oko} &\longrightarrow \mathcal{Z}_q, \\ z_{k_1} \ldots z_{k_r} &\longmapsto \zeta_q^{\mathrm{Oko}}(k_1, \ldots, k_r) \end{aligned}$$

by $\mathbb{Q}$-linear continuation and $\mathbf{1} \mapsto 1$.

(ii) Define on the alphabet $A_{Oko} := \mathbb{Q}\{z_2, z_3, \ldots\}$ the following commutative and associative product by $\mathbb{Q}$-bilinearity, $\mathbf{1} \diamond_{Oko} w := w \diamond \mathbf{1}$ for all $w \in A_{Oko}$ and

$$\begin{aligned} \diamond_{Oko} : A_{Oko} \otimes A_{Oko} &\longrightarrow A_{Oko}, \\ (z_{k_1}, z_{k_2}) &\longmapsto \begin{cases} z_{k_1+k_2-1} + z_{k_1+k_2+1}, & \text{if } k_1,\ k_2 \text{ odd}, \\ z_{k_1+k_2}, & \text{else}. \end{cases} \end{aligned}$$

Furthermore, let be $*_{Oko}$ the induced quasi-shuffle product on $\mathfrak{h}^{Oko}$.

Proposition A.97. *The map ζ_q^{Oko} is an algebra homomorphism, i.e. we have for all $w_1, w_2 \in \mathfrak{h}^{Oko}$*

$$\zeta_q^{Oko}(w_1 *_{Oko} w_2) = \zeta_q^{Oko}(w_1)\zeta_q^{Oko}(w_2).$$

Proof. This follows immediately by the definition of Okounkov-qMZVs as iterated sums and $p_{k_1}(X)p_{k_2}(X) = p_{k_1+k_2}(X)$ if at least one of k_1 and k_2 is even and

$$p_{k_1}(X)p_{k_2}(X) = \left(X^{\frac{(k_1+k_2-1)-1}{2}} + X^{\frac{(k_1+k_2+1)-1}{2}}\right)(1+X) = p_{k_1+k_2-1}(X) + p_{k_1+k_2+1}(X)$$

if both are odd. □

Remark A.98. For the Okounkov model, there is no 'good' generating series known so far, i.e. no one that has a nice representation or would lead to further results. However, we can introduce a generating series (for every $r \geq 1$):

$$\begin{aligned} \mathfrak{o}(X_1, \ldots, X_r) &:= \sum_{k_1, \ldots, k_r \geq 2} \zeta_q^{\mathrm{Oko}}(k_1, \ldots, k_r) X_1^{k_1-2} \cdots X_r^{k_r-2} \\ &= \sum_{m_1 > \cdots > m_r > 0} \prod_{j=1}^{r} \frac{q^{m_j}(1 - q^{m_j} + (1+q^{m_j})X_j)}{(1-q^{m_j})((1-q^{m_j})^2 - q^{m_j}X_j)}. \end{aligned}$$

Remark A.99. Since the Okounkov model doesn't span the same space as the SZ- or BZ-model, we cannot apply SZ- or BZ-duality via translation into the respective model and applying SZ- or BZ-duality and translating back as we did in the OOZ-model. Also, no individual duality relation for the Okounkov model is known so far.

Remark A.100 (Comparison)**.** For considering duality relations or related topics, the Okounkov model is not the best choice. But the deep connection to Hilbert schemes of points ([Oko]) makes this model essential and is for this well suited. Also, direct calculations are pretty good to do in this model because of the 'almost' monomial polynomials p_k.

A.3. Subalgebras of $\mathcal{Z}_q$

There are several subalgebras of $\mathcal{Z}_q$ that are of interest. One of the most important is the algebra of quasi-modular forms. Others get their importance by conjectures that they are not only subalgebras but also equal $\mathcal{Z}_q$, which would give - assuming that they are true - a much deeper understanding of the structure of $\mathcal{Z}_q$. They are all verified for small weights, often by computer assistance.

Bachmann for example considered, before he introduced bi-brackets, (mono-)brackets and their algebra

$$\mathcal{MD} := \langle \mathrm{g}(k_1, \dots, k_r) \,|\, r \geq 0,\, k_1 \geq 2,\, k_i \geq 1 \rangle_{\mathbb{Q}}.$$

$\mathcal{MD}$ contains the classical Eisenstein series G_2, G_4, G_6 because of

$$G_2 = -\frac{1}{24} + \mathrm{g}(2), \quad G_4 = \frac{1}{1440} + \mathrm{g}(4), \quad G_6 = -\frac{1}{60480} + \mathrm{g}(6),$$

which is the reason why the ring of quasi-modular forms $\widetilde{M}(\mathrm{SL}_2(\mathbb{Z}))_{\mathbb{Q}}$ is contained in $\mathcal{MD}$,

$$\widetilde{M}(\mathrm{SL}_2(\mathbb{Z}))_{\mathbb{Q}} \overset{\text{[KZ]}}{=} \mathbb{Q}[G_2, G_4, G_6] \subset \mathcal{MD}$$

(denote that this is a proper inclusion since e.g. $\mathrm{g}(2,1)$ is not constant zero and has odd weight; hence, $\mathrm{g}(2,1)$ cannot be written as algebraic expression over $\mathbb{Q}$ in terms of G_2, G_4, G_6).

Remark that we have seen $\mathcal{MD}$ already but with different name:

Proposition A.101 ([BK2, Thm. 2.3(ii)])**.** *It is* $\mathcal{MD} = \mathcal{Z}_q^\circ$. □

Hence, $\mathcal{MD}$ is stable under $q\frac{d}{dq}$ (Rem. A.31(ii)/[BK1, Thm. 1.7]).

After expanding mono-brackets to bi-brackets (their $\mathbb{Q}$-algebra is usually denoted by $\mathcal{BD}$ and it is proven that $\mathcal{BD} = \mathcal{Z}_q$), by comparing dimensions in small weight, Bachmann conjectured that they span the same space:

Conjecture A.102 ([Ba4, Conj. 4.3])**.** *It is $\mathcal{MD} = \mathcal{BD}$. In other words: $\mathcal{Z}_q^\circ = \mathcal{Z}_q$.*

There are more subalgebras of $\mathcal{Z}_q$ that are often considered:

Proposition A.103. *It is*

$$\mathcal{Z}_{q,1}^\circ = \langle \mathrm{g}(\mathbf{k}) \,|\, k_i \geq 2\rangle_{\mathbb{Q}} = \langle \zeta_q^{Oko}(\mathbf{k}) \,|\, k_i \geq 2\rangle_{\mathbb{Q}} = \langle \zeta_q^{BZ}(\mathbf{k}) \,|\, k_i \geq 2\rangle_{\mathbb{Q}}.$$

Proof. For the first equality we refer to [BK2, Thm. 2.3(iii)], $\mathcal{Z}_{q,1}^\circ = \langle \zeta_q^{\mathrm{Oko}}(\mathbf{k}) \,|\, k_i \geq 2\rangle_{\mathbb{Q}}$ is Proposition A.95 and the proof of $\mathcal{Z}_{q,1}^\circ = \langle \zeta_q^{\mathrm{BZ}}(\mathbf{k}) \,|\, k_i \geq 2\rangle_{\mathbb{Q}}$ works analogously as the proof of Proposition A.52. □

We get other important subalgebras of $\mathcal{Z}_q$ when considering bi-brackets again. By defining the weight and depth as done, we get a filtration by weight and depth on $\mathcal{Z}_q$ (resp. on every subalgebra of $\mathcal{Z}_q$):

Definition A.104. Let be A a subalgebra of $\mathcal{Z}_q$ and $r, s \geq 0$. Define

(i) the weight filtration $\mathrm{Fil}_r^W(A) := \left\langle b = \mathrm{g}\binom{k_1,\dots,k_s}{d_1,\dots,d_s} \in A \,\middle|\, 0 \leq s \leq r,\ \mathrm{wt}(b) \leq r \right\rangle_{\mathbb{Q}}$,

(ii) the depth filtration $\mathrm{Fil}_k^D(A) := \left\langle b = \mathrm{g}\binom{k_1,\dots,k_s}{d_1,\dots,d_s} \in A \,\middle|\, 0 \leq s \leq r \right\rangle_{\mathbb{Q}}$,

(iii) $\mathrm{Fil}_{r,s}^{W,D}(A) := \mathrm{Fil}_r^W \mathrm{Fil}_s^D(A)$

and denote by gr_r^W, resp. $\mathrm{gr}_{r,s}^{W,D}$ the associated graded $\mathbb{Q}$-vector spaces.

For the dimensions of the graded parts of $\mathcal{Z}_q$, Bachmann and Kühn give in [BK2] conjectures standing in analogy to the one by Zagier and Broadhurst-Kreimer. Hence, for completeness, we want to state the latter ones first:

We use for this the notation

$$\begin{aligned}\mathcal{Z}_k &:= \langle \zeta(\mathbf{k}) |\ \mathrm{wt}(\mathbf{k}) = k\rangle_{\mathbb{Q}},\\ \mathcal{Z}_n^d &:= \langle \zeta(\mathbf{k}) |\ \mathrm{wt}(\mathbf{k}) = n,\ \mathrm{depth}(\mathbf{k}) = d\rangle_{\mathbb{Q}}\end{aligned}$$

for all $k \geq 2$, $n \geq 2$, $d \geq 0$.

Conjecture A.105. *(i) (Zagier). It is*

$$\sum_{k\geq 0} \dim_{\mathbb{Q}}(\mathcal{Z}_k) X^k = \frac{1}{1 - X^2 - X^3} = \frac{1}{1-x^2} \cdot \frac{1}{1 - O_3(x)}$$

with $O_3(X) := \frac{X^3}{1-X^2}$.

(ii) (Hofmann). More stronger, it is conjectured that the MZVs of indices containing only 2's and 3's build a basis of $\mathcal{Z}$.

Remark that Hofmann's conjecture is in accordance with Brown's theorem, which says that MZVs with indices containing only 2's and 3's generate $\mathcal{Z}$ (Thm. A.10).

Conjecture A.106 (Broadhurst-Kreimer, [BK]). *It is*

$$1+\sum_{n\geq 1,d\geq 1}\dim_{\mathbb{Q}}\left(\mathcal{Z}_n^d\big/\mathcal{Z}_n^{d-1}\right)X^nY^d=\frac{1+E_2(X)Y}{1-O_3(X)Y+S(X)Y^2-S(X)Y^4},$$

where $E_2(X):=\frac{X^2}{1-X^2}$ *and where* $S(X):=\frac{X^{12}}{(1-X^4)(1-X^6)}$ *is the generating series of the dimension of the space of cusp forms.*

Conjecture A.107 ([BK2, Conj. 1.3]). *(i) The dimensions of the weight graded parts of* $\mathcal{Z}_q$ *are given through*

$$\sum_{k\geq 0}\dim_{\mathbb{Q}}\left(\operatorname{gr}_k^W\mathcal{Z}_q\right)X^k=\frac{1}{1-X-X^2-X^3+X^6+X^7+X^8+X^9}$$
$$=\frac{1}{(1-X^2)(1-X^4)(1-X^6)}\cdot\frac{1}{1-D(X)O_1(X)+D(X)(E_4(X))},$$

where we set $D(X):=\frac{1}{1-X^2}$, $O_1(X):=\frac{X}{1-X^2}$, $E_4(X):=\frac{X^4}{1-X^2}$.

(ii) $\mathcal{Z}_q$ *is generated by bi-brackets of indices only containing 1's, 2's and 3's.*

(iii) For the weight and depth graded parts of $\mathcal{Z}_q$*, we have*

$$\sum_{k,l\geq 0}\dim_{\mathbb{Q}}\left(\operatorname{gr}_{k,l}^{W,D}\mathcal{Z}_q\right)X^kY^l$$
$$=\frac{1+D(X)E_2(X)Y+D(X)S(X)Y^2}{1-a_1(X)Y+a_2(X)Y^2-a_3(X)Y^3-a_4(X)Y^4+a_5(X)Y^5}$$

with

$$a_1(X):=D(X)O_1(X),\ a_2(X):=D(X)\sum_{k\geq 1}\dim_{\mathbb{Q}}(M_k(\mathrm{SL}_2(\mathbb{Z})))^2X^k,$$
$$a_4(X)=a_5(X):=O_1(X)S(X),\ a_4(X):=D(X)\sum_{k\geq 1}\dim_{\mathbb{Q}}(S_k(\mathrm{SL}_2(\mathbb{Z})))^2X^k.$$

Okounkov conjectured the following about the structure of qMZVs he introduced and which are named after him:

Conjecture A.108 ([Oko, Conjecture 1]). *(i) We have*

$$\sum_{k\geq 0}\dim_{\mathbb{Q}}\operatorname{gr}_k^w(\mathcal{Z}_q^{Oko})t^k=\frac{1}{1-t^2-t^3-t^4-t^5+t^8+t^9+t^{10}+t^{11}+t^{12}}$$
$$=\frac{1}{(1-t^2)(1-t^4)(1-t^6)}\cdot\frac{1}{1-D(t)O_3(t)+2D(t)S(t)}.$$

(ii) $\mathcal{Z}_q^{Oko}$ *is spanned by the* $\zeta_q^{Oko}(\mathbf{k})$ *with* $2 \leq k_i \leq 5$.

Finally, in the graph below, we give an overview of the diverse, most commonly considered, subalgebras of $\mathcal{Z}_q$.

Remark A.109 (to the overview)**.**

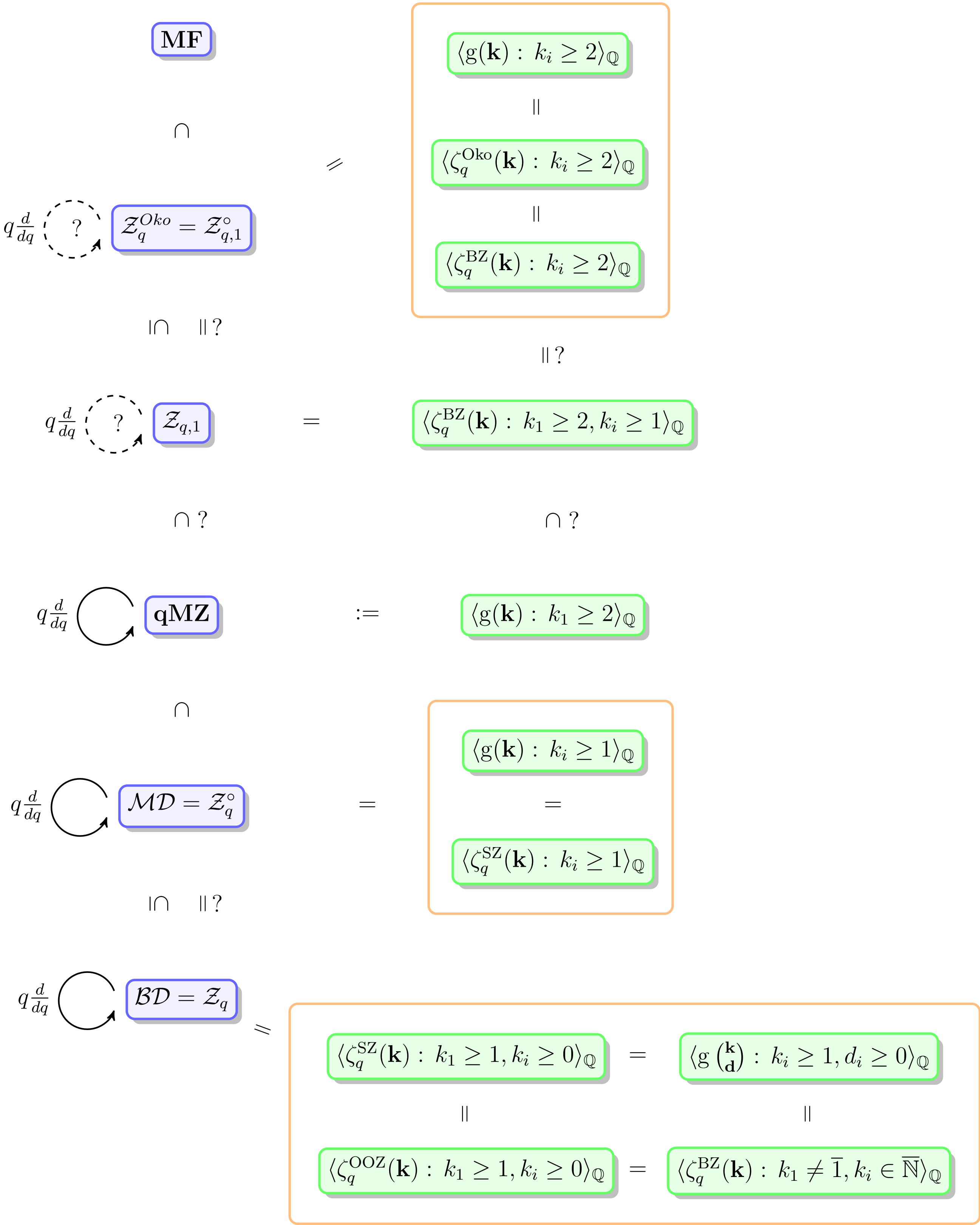
(i) **MF** denotes the algebra of modular forms, where we take a modular form formally via its Fourier expansion in a canonical way as an element of the bigger algebras.

(ii) $q\frac{d}{dq}$ indicates that the respective algebra is closed under $q\frac{d}{dq}$. Dashed arcs with question marks mean that it is not proven yet but conjectured.

(iii) Furthermore, the equality signs with a question mark mean that equality is conjectured but not proven yet.

(iv) Equalities in blue boxes are just different notations from different papers for the same algebra.

(v) In the green boxes, the indices $\mathbf{k}$ can have arbitrary length ≥ 0. For the sake of clarity, this is not written in the overview.



MF
$\cap$
$q\frac{d}{dq}$?
$\mathcal{Z}_q^{Oko} = \mathcal{Z}_{q,1}^{\circ}$
$\langle \mathrm{g}(\mathbf{k}) : k_i \geq 2 \rangle_{\mathbb{Q}}$
$\langle \zeta_q^{\mathrm{Oko}}(\mathbf{k}) : k_i \geq 2 \rangle_{\mathbb{Q}}$
$\langle \zeta_q^{\mathrm{BZ}}(\mathbf{k}) : k_i \geq 2 \rangle_{\mathbb{Q}}$
$\cap$ ‖ ?
‖ ?
$q\frac{d}{dq}$?
$\mathcal{Z}_{q,1}$
$=$
$\langle \zeta_q^{\mathrm{BZ}}(\mathbf{k}) : k_1 \geq 2, k_i \geq 1 \rangle_{\mathbb{Q}}$
$\cap$?
$\cap$?
$q\frac{d}{dq}$
qMZ
$:=$
$\langle \mathrm{g}(\mathbf{k}) : k_1 \geq 2 \rangle_{\mathbb{Q}}$
$\cap$
$q\frac{d}{dq}$
$\mathcal{MD} = \mathcal{Z}_q^{\circ}$
$=$
$\langle \mathrm{g}(\mathbf{k}) : k_i \geq 1 \rangle_{\mathbb{Q}}$
$=$
$\langle \zeta_q^{\mathrm{SZ}}(\mathbf{k}) : k_i \geq 1 \rangle_{\mathbb{Q}}$
$\cap$ ‖ ?
$q\frac{d}{dq}$
$\mathcal{BD} = \mathcal{Z}_q$
$\langle \zeta_q^{\mathrm{SZ}}(\mathbf{k}) : k_1 \geq 1, k_i \geq 0 \rangle_{\mathbb{Q}}$ $=$ $\langle \mathrm{g}\binom{\mathbf{k}}{\mathbf{d}} : k_i \geq 1, d_i \geq 0 \rangle_{\mathbb{Q}}$
$\langle \zeta_q^{\mathrm{OOZ}}(\mathbf{k}) : k_1 \geq 1, k_i \geq 0 \rangle_{\mathbb{Q}}$ $=$ $\langle \zeta_q^{\mathrm{BZ}}(\mathbf{k}) : k_1 \neq \bar{1}, k_i \in \overline{\mathbb{N}} \rangle_{\mathbb{Q}}$

Bibliography

[Apo] T. Apostol: *Mathematical Analysis*, Addison Wesley **2** (1974), p. 204.

[Ba1] H. Bachmann: *Multiple Zeta-Werte und die Verbindung zu Modulformen durch Multiple Eisensteinreihen*, Masterthesis (2012).

[Ba2] H. Bachmann: *Multiple Eisenstein series and q-analogues of multiple zeta values*, PhD Thesis (2015).

[Ba3] H. Bachmann: *Double shuffle relations for q-analogues of Multiple Zeta Values, their derivatives and the connection to multiple Eisenstein series*, RIMS Kôkyûroku **2017** (2015), 22–43.

[Ba4] H. Bachmann: *The algebra of bi-brackets and regularized multiple Eisenstein series*, J. Number Theory, **200** (2019), 260–294.

[Ba5] H. Bachmann: *Multiple Eisenstein series and q-analogues of multiple zeta values*, Periods in Quantum Field Theory and Arithmetic, Springer Proceedings in Mathematics & Statistics **314** (2020), 173–235.

[Ba6] H. Bachmann: Lectures on *Multiple Zeta Values and Modular Forms*, Nagoya University (2020).

[Ba7] H. Bachmann: private communication (2021).

[BC] H. Bachmann, S. Charlton: *Generalized Jacobi-Trudi determinants and evaluations of Schur multiple zeta values*, European J. Combin. **87** (2020), 103–133.

[BK1] H. Bachmann, U. Kühn: *The algebra of generating functions for multiple divisor sums and applications to multiple zeta values*, Ramanujan J. **40** (2016), 605–648.

[BK2] H. Bachmann, U. Kühn: *A dimension conjecture for q-analogues of Multiple Zeta Values*, Periods in Quantum Field Theory and Arithmetic, Springer Proceedings in Mathematics & Statistics **314** (2020), 237–258.

[BvI] H. Bachmann, J.-W. van Ittersum: *Partitions, Multiple Zeta Values and the q-bracket* (in preparation).

[BY] H. Bachmann and Y. Yamasaki: *Checkerboard style Schur multiple zeta values and odd single zeta values*, Math. Z. **290** (3) (2018), 1173–1197.

[BO] Spencer Bloch and Andrei Okounkov: *The character of the infinite wedge representation*, Adv. Math., **149**(1):1–60, 2000.

[Bos] S. Bosch: *Lineare Algebra*, Springer Spektrum, Berlin, Heidelberg (2014).

[Bra] D. M. Bradley: *Multiple q-zeta values*, J. of Algebra **283** (2005), 752–798.

[BK] D. Broadhurst, D. Kreimer: *Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops*, Phys. Lett. B **393** (1997), 403–412.

[Bro] F. Brown: *Mixed Tate motives over* $\mathbb{Z}$, Ann. of Math., **175**(2) (2012), 949–976.

[BF] J. I. Burgos Gil and J. Fresán: *Multiple Zeta Values: From Numbers to Motives*, to appear.

[CGS] S. Carr, H. Gangl, L. Schneps: *On the Broadhurst-Kreimer generating series for multiple zeta values*, Contemp.Math. **648** (2015) 57–78.

[CEM] J. Castillo Medina, K. Ebrahimi-Fard, and D. Manchon: *Unfolding the double shuffle structure of q-multiple zeta values*, Bull. Aust. Math. Soc., **91**(3) (2015), 368–388.

[DG] P. Deligne, A. B. Goncharov: *Groupes fondamentaux motiviques de Tate mixte*: Ann. Sci. École Norm. Sup. (4), **38**(1) (2005), 1–56.

[EG] K. Ebrahimi-Fard, L. Guo: *Mixable shuffles, quasi-shuffles and Hopf algebras*, J. Algebraic Combin. **24** (2006), 83–101.

[EMS] K. Ebrahimi-Fard, D. Manchon, and J. Singer: *Duality and (q-)Multiple Zeta Values*, Adv. Math. **298** (2016), 254–285.

[Eca] J. Ecalle: *The flexion structure and dimorphy: flexion units, singulators, generators, and the enumeration of multizeta irreducibles*, Asymptotics in dynamics, geometry and PDEs, generalized Borel summation. **2** (2011), 27–11.

[FF] W. Feit, N.J. Fine: *Pairs of commuting matrices over a finite field*, Duke Math. J. **27** (1960), 91–94.

[Fis] G. Fischer: *Lernbuch Lineare Algebra und Analytische Geometrie*, Springer Vieweg, Wiesbaden (2012).

[FB] E. Freitag, R. Busam: *Complex Analysis*, Springer, Berlin Heidelberg **2** (2009).

[FH] W. Fulton, J. Harris: *Representation Theory - A First Course*, Graduate Texts in Mathematics **129**, Springer, New York, NY (2004).

[GKZ] H. Gangl, M. Kaneko, D. Zagier: *Double zeta values and modular forms*, Automorphic forms and zeta functions, Tokyo, Japan, World Scientific (2006).

[Ho1] M. E. Hoffman: *The algebra of multiple harmonic series*, J. Algebra **194** (1997), 477–495.

[Ho2] M. E. Hoffman: *Quasi-shuffle products*, J. Algebraic Combin. **11** (2000), 49–68.

[Ho3] M. E. Hoffman: Webpage with over 600 articles on MZVs `https://www.usna.edu/Users/math/meh/biblio.html` (seen in August 2021).

[IKZ] K. Ihara, M. Kaneko and D. Zagier: *Derivation and double shuffle relations for Multiple Zeta Values*, Compositio Math. **142** (2006), 307–338.

[KZ] M. Kaneko, D. Zagier: *A generalized Jacobi theta function and quasimodular forms*, The moduli space of curves, Progr. Math. **129** (1995), 165–172.

[KKW] M. Kaneko, N. Kurokawa and M. Wakayama: *A variation of Eulers approach to values of the Riemann zeta function*, Kyushu J. Math. **57** (2003), 175–192.

[KKT] M. Kaneko, M. Noro, K. Tsurumaki: *On a Conjecture for the Dimension of the Space of the Multiple Zeta Values*, In: Stillman M., Verschelde J., Takayama N. (eds) Software for Algebraic Geometry. The IMA Volumes in Mathematics and its Applications, **148**. Springer, New York, NY (2008), 47–58.

[KT] N. Kawasaki, T. Tanaka: *On the duality and the derivation relations for Multiple Zeta Values*, Ramanujan J. **47** (2018), 651–658.

[Kim] A. Kimura: *The intersection of the duality and the derivation relations for multiple zeta values*, (in preparation).

[Kno] K. Knopp: *Infinite Sequences and Series*, Dover Publications **4** (1956), 146–150.

[Li] Z. Li: *Derivation relations and duality for the sum of Multiple Zeta Values*, Funct. Approx. Comment. Math. **58** (2) (2018), 215–220.

[LZ] E. Linebarger, J. Zhao: *A family of multiple harmonic sum and multiple zeta star value identities*, Mathematika **61** (2015), 63–71.

[Mac] I. G. MacDonald: *Numbers of conjugacy classes in some finite classical groups*, Bulletin Austr. Math. Soc. **23** (1981).

[Maz] D. Mazur: *Combinatorics A Guided Tour*, The Mathematical Association of America (2010), 94.

[MZZ] V. Montesinos Santalucía, P. Zizler, V. Zizler: *An Introduction to Modern Analysis*, Springer International Publishing (2015), 97–99.

[NPY] M. Nakasuji, O. Phuksuwan, Y. Yamasaki: *On Schur multiple zeta functions: A combinatoric generalization of multiple zeta functions*, Adv. Math. **333** (2018), 570–619.

[OZ] Y. Ohno, W. Zudilin: *Zeta Stars*, Number Theory and Physics **2**(2) (2008), 325–347.

[OOZ] Y. Ohno, J. Okuda and W. Zudilin: *Cyclic q-MZSV sum*, J. Number Theory **132** (2012), 144–155.

[Oko] A. Okounkov: *Hilbert schemes and multiple q-zeta values*, Functional Analysis and Its Applications **48** (2014), 138–144.

[OT] J. Okuda, Y. Takeyama: *On relations for the multiple q-zeta values*, Ramanujan J. **14**(3) (2007), 379–387.

[RL] G. Rybicki, A. P. Lightman: *Radiative Processes in Astrophysics*, Wiley-Interscience, New York (1979), p. 25.

[Sch] K.-G. Schlesinger: *Some remarks on q-deformed multiple polylogarithms*, arXiv:math/0111022.

[Sek] S. Seki: *Connectors*, to appear in RIMS Kokyuroku.

[SY] S. Seki, S. Yamamoto: *A new proof of the duality of Multiple Zeta Values and its generalizations*, Int. J. of Number Theory, **15**(6) (2019), 1261–1265.

[SQ] Z. Shen, Z. Qin: *Hilbert schemes of points and quasi-modularity* PAMQ **16** (5) (2020), 1673–1706.

[Sin] J. Singer: *q-Analogues of Multiple Zeta Values and their application in renormalization*, Dissertation, Erlangen-Nürnberg University (2017).

[Sta] R. P. Stanley: *Irreducible symmetric group characters of rectangular shape*, Sem. Lothar. Combin. **50** (2003/2004).

[Ta1] Y. Takeyama: *The algebra of a q-analogue of multiple harmonic series*, SIGMA **9** (2013), 1–15.

[Ta2] Y. Takeyama: *Derivations on the algebra of multiple harmonic q-series and their applications*, Ramanujan J. **52** (2020), 41–65.

[Ter] T. Terasoma: *Mixed Tate motives and multiple zeta values*, Invent. Math. **149**(2) (2002), 339–369.

[vIt] J.-W. van Ittersum: *A symmetric Bloch–Okounkov theorem*, Res. Math. Sci. **8**(2) (2021).

[Wal] M. Waldschmidt: Lectures on Multiple Zeta Values, imsc 2011. `https://webusers.imj-prg.fr/~michel.waldschmidt/articles/pdf/MZV2011IMSc.pdf`, 2012.

[Za1] D. Zagier: *Zetafunktionen und quadratische Körper*, Hochschultext. Springer, Berlin, Heidelberg (1981).

[Za2] D. Zagier: *Values of Zeta Functions and Their Applications*, First European Congress of Mathematics Paris (1994).

[Za3] D. Zagier: *Partitions, quasimodular forms, and the Bloch-Okounkov theorem*, Ramanujan Journal **41** (2016), 345–368.

[Zh1] J. Zhao: *Multiple q-zeta functions and multiple q-polylogarithms*, Ramanujan J., **14**(2) (2007), 189–221.

[Zh2] J. Zhao: *Uniform approach to double shuffle and duality relations of various q-Analogs of Multiple Zeta Values via Rota-Baxter algebras*, In J. I. Burgos Gil, K. Ebrahimi-Fard, H. Gangl (Eds.), Periods in quantum field theory and arithmetic. Cham: Springer (2014), 259–292.

[Zh3] J. Zhao: *Multiple Zeta Functions, Multiple Polylogarithms And Their Special Values*, World Scientific (2016).

[Zu1] W. Zudilin: *Algebraic relations for Multiple Zeta Values*, Russian Math. Surveys **58**(1) (2003), 1–29.

[Zu2] W. Zudilin: *Multiple q-Zeta Brackets*, Mathematics **3**(1) (2015), 119–130.

Eidesstattliche Erklärung

Die vorliegende Arbeit habe ich selbständig verfasst und keine anderen als die angegebenen Hilfsmittel - insbesondere keine im Quellenverzeichnis nicht benannten Internet-Quellen - benutzt.
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