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G30 Program, Nagoya University

Linear Algebra II

⓪ Review LAI & Motivation

What did we do in LAI ?

- Solve linear systems $\longleftrightarrow$ $Ax=b$
- Matrices & vectors $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$

for these we had some operations

$\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}3\\4\end{pmatrix}=\begin{pmatrix}4\\6\end{pmatrix}$ · Addition $x+y\in\mathbb{R}^n$ $\quad x,y\in\mathbb{R}^n$

$3\cdot\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}3\\6\end{pmatrix}$ · Scalar multiplication $\lambda x\in\mathbb{R}^n$ $\quad \lambda\in\mathbb{R}$

- Linear maps $F:\mathbb{R}^n\longrightarrow\mathbb{R}^m$

i) $F(x+y)=F(x)+F(y)$

ii) $F(\lambda x)=\lambda F(x)$

- Subspaces $U\subset\mathbb{R}^n$ $\quad 0=\begin{pmatrix}0\\\vdots\\0\end{pmatrix}\in\mathbb{R}^n$

i) $0\in U$

ii) $x,y\in U \Rightarrow x+y\in U$

iii) $x\in U\Rightarrow \lambda x\in U$

- Image & Kernel of linear maps

$$\begin{array}{ccc} F: \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ \cup & & \cup \\ \ker F & & \operatorname{im} F \end{array}$$

- Linear independency & Basis

[In "real life" there are a lot of other examples of objects with "Addition" and "Scalar multiplication" for which these concepts are interesting.]

Example 1

Let $\mathcal{F}(\mathbb{R}, \mathbb{R})$ be the set of all functions $\mathbb{R} \to \mathbb{R}$.

$e^x \in \quad \sin(x) \in \quad x^2+3 \in$

This is a really big set!

$\mathcal{F}(\mathbb{R},\mathbb{R})$ $\in \quad \ni$

For $f, g \in \mathcal{F}(\mathbb{R},\mathbb{R})$ we can also define $f+g$ and λf $(\lambda \in \mathbb{R})$, by

$$(f+g)(x) = f(x) + g(x) \quad \text{and} \quad (\lambda f)(x) = \lambda \cdot f(x).$$

Now consider the subset $U = \{ f \in \mathcal{F}(\mathbb{R},\mathbb{R}) \mid f'' = f \}$.

We have:

i) The function $n(x) = 0$ is an element of U.

ii) If $f, g \in U$, i.e $f'' = f$ and $g'' = g$ then

$$(f+g)'' = f'' + g'' = f + g \implies f+g \in U.$$

iii) $\lambda \in \mathbb{R},\ f \in U \implies \lambda f \in U.$

Therefore: U can be seen as a <u>subspace</u> of $\mathcal{F}(\mathbb{R},\mathbb{R})$.

(The "zero vector" $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ gets replaced by the "zero function" $n(x) = 0$ $\forall x \in \mathbb{R}$).

One can show that $f_1(x) = e^x$, $f_2(x) = e^{-x}$ and their linear combinations $\lambda_1 f_1 + \lambda_2 f_2$ are the only elements in U. $(\lambda_1, \lambda_2 \in \mathbb{R})$ $(U = \operatorname{span}\{f_1, f_2\})$

Moreover: f_1 and f_2 are linearly independent, i.e

$$\lambda_1 f_1(x) + \lambda_2 f_2(x) = 0 \ \ \forall x \in \mathbb{R} \implies \lambda_1 = \lambda_2 = 0.$$

$$(\lambda_1 f_1 + \lambda_2 f_2 = n)$$

$\implies (f_1, f_2)$ is a basis of $U \implies \dim U = 2$.

① Vector spaces

Spaces for which "Addition" and "Scalar multiplication" are defined in such a way that they satisfy the same computation rules as vectors are called vector spaces.

Definition 1.1 A (real) vector space is a set V together with two functions

$$+ : V \times V \longrightarrow V \qquad \cdot : \mathbb{R} \times V \longrightarrow V$$
$$(u, v) \longmapsto u + v \qquad (\lambda, v) \longmapsto \lambda \cdot v = \lambda v$$

Addition — Scalar multiplication

satisfying:

Properties for the addition:

(A.1) $\forall u, v, w \in V: \quad (u+v)+w = u+(v+w)$ (Associativity)

(A.2) $\forall u, v \in V: \quad u + v = v + u$ (Commutativity)

(A.3) There exist an element $n \in V$ (Identity/neutral element), such that $\forall u \in V: \quad n + u = u$.

(A.4) Every element $u \in V$ has an additive Inverse, i.e there exist a $v \in V$ with $u + v = n$.

Compatibility of addition and scalar multiplication:

(C.1) $\forall u, v \in V, \lambda \in \mathbb{R}: \lambda(u+v) = \lambda u + \lambda v$

(C.2) $\forall u \in V, \lambda, \mu \in \mathbb{R}: (\lambda + \mu) u = \lambda u + \mu u$

(C.3) $\forall u \in V, \lambda, \mu \in \mathbb{R} \quad \lambda(\mu u) = (\lambda \mu) u$

(C.4) $\forall u \in V: \quad 1u = u$.

Proposition 1.2 Let V be a vector space and $u \in V$

i) $u + n = u$ (n the element in (A.3))

ii) If $n, \tilde{n} \in V$ both satisfy (A.3) then $n = \tilde{n}$. (we write $n = 0$)

iii) If for a fixed $u \in V$, $v, \tilde{v} \in V$ both satisfy (A.4) then $v = \tilde{v}$.

iv) $u + (-1)u = n$

Proof: i) $u + n \overset{(A.2)}{=} n + u = u$

ii) $n \overset{(A.3)}{=} \tilde{n} + n \overset{i)}{=} \tilde{n}$

iii)
$$\begin{array}{ccc} \tilde{v} + (u+v) & = & \tilde{v} + n = \tilde{v} \\ \| \,(A.1) & & \\ (\tilde{v}+u)+v & = & n + v = v \end{array}$$

$n = 0u + n = 0u$

$\Uparrow$ adding $(-0u)$

$(0u = (0+0)u = 0u + 0u)$

iv) Try yourself. First show $0u = n$ and then
$u + (-1)u \overset{(C.2)}{=} (1-1)u = 0u = n$.

Remarks 1) The operations "+" and "·" must be defined when you specify a vector space.

2) By (A.1) $u + v + w$ is well defined

$$\begin{array}{c} u+v+w \\ /\!/ \quad \backslash\!\backslash \\ (u+v)+w = u+(v+w) \end{array}$$

3) By Prop 1.2 ii) the element n is unique and we write 0 "the zero vector" for it.

4) By Prop. 1.2 iii) the inverse of an element $u \in V$ is unique and we write $v =: -u$. By Prop 1.2 iv) we have
$(-1)u = -u$.

Example 2 i) $V = \mathbb{R}^n$ with usual operations

$$0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad -u = \begin{pmatrix} -u_1 \\ \vdots \\ -u_n \end{pmatrix}$$

ii) $\mathcal{F}(\mathbb{R}, \mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \}$

$$(f+g)(x) = f(x) + g(x), \quad (\lambda f)(x) = \lambda \cdot f(x) \quad \forall x \in \mathbb{R}$$

$$n(x) = 0 \quad \forall x, \quad (-f)(x) = -f(x).$$

$$\mathcal{C}^0(\mathbb{R}, \mathbb{R}) = \{ f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \mid f \text{ is continuous} \}.$$

iii) Infinite sequences $(a_n)_{n \geq 1} = (a_1, a_2, \dots)$

$\mathcal{J}$ = set of all infinite sequences.

iv) Matrices : $V = \mathbb{R}^{m \times n}$ with usual addition & scalar mult.

$$0 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

— In the following V always denotes a vector space —

Definition 1.3 A subset $U \subset V$ is a subspace (of V) if

i) $0 \in U$

ii) $\forall u, v \in U : u + v \in U$

iii) $\forall u \in U, \lambda \in \mathbb{R} : \lambda u \in U$.

Example 3 i) $\mathcal{P} = \{ f : \mathbb{R} \to \mathbb{R} \mid f$ is a polynomial fct.$\}$

e.g. $f(x) = 2x^3 - x^2 + 1$

$$\mathcal{P} \underset{\text{subspace}}{\subset} \mathcal{C}^0(\mathbb{R}, \mathbb{R}) \subset \mathcal{F}(\mathbb{R}, \mathbb{R})$$

$$\cup$$

$$U \text{ (from Example 1)}$$

ii) Subspaces of $\mathbb{R}^n$

iii) $\{ (a_n)_{n \geq 1} \in \mathcal{J} \mid \lim_{n \to \infty} a_n \text{ exists} \} \subset \mathcal{J}$

e.g: $a_n = \frac{1}{n}$

iv) $\{\text{invertible matrices}\} \subset \mathbb{R}^{m \times n}$ ~~are~~ is Not a subspace

<u>Proposition 1.4</u> If $U \subset V$ is a subspace then U is also a vector space with the operations inherited from V.

<u>Proof</u>: Exercise (follows directly from the definition).

<u>Definition 1.5</u> V vector space, $v_1, \dots, v_n \in V$

i) <u>Span</u> of $v_1 \dots v_n$: $\operatorname{span}\{v_1, \dots, v_n\} = \{ \underbrace{\sum_{i=1}^{n} \lambda_i v_i}_{\text{linear combination of } v_1 \dots v_n} \mid \lambda_1, \dots, \lambda_n \in \mathbb{R} \}$

ii) $v_1, \dots, v_n$ <u>span</u> (or <u>generate</u>) V if $\operatorname{span}\{v_1 \dots v_n\} = V$.

iii) V is <u>finitely generated</u> if there exist $v_1, \dots, v_n$ s.th $V = \operatorname{span}\{v_1 \dots v_n\}$

iv) $v_1 \dots v_n$ are linearly independent if $\lambda_1 v_1 + \dots + \lambda_n v_n = 0 \Rightarrow \lambda_1 = \dots = \lambda_n = 0$

v) $B = (v_1, \dots, v_n)$ is a <u>basis</u> of V if $v_1, \dots, v_n$ are lin. ind. and $\operatorname{span}\{v_1, \dots, v_n\} = V$.

Convention: $\{0\} = \operatorname{span}\{\emptyset\}$ (empty set) is also fin. gen.

<u>Example 4</u>

i) $\mathbb{R}^n = \operatorname{span}\{e_1, \dots, e_n\}$ is fin. gen.

ii) $\mathcal{P} = \{ f : \mathbb{R} \to \mathbb{R} \mid f(x) = \sum_{i=1}^{m} a_i x^i \quad a_1, \dots, a_m \in \mathbb{R} \}$

is <u>not</u> fin. gen.:

If $f_1, \dots, f_m \in \mathcal{P}$ would generate $\mathcal{P}$, then $g(x) = x^{d+1}$ would not be in $\operatorname{span}\{f_1 \dots f_m\}$ if $d = \max\{\deg(f_i) \mid i = 1, \dots, m\}$.

iii) Set $\mathcal{P}_n = \{ f \in \mathcal{P} \mid \deg(f) \leq n \}$. $\mathcal{P}_n$ is finitely generated with basis $B = (f_0, \dots, f_n)$, where $f_j(x) = x^j$.

The following Propositions, Lemmas & Theorems were already proven in LA I for the case $V = \mathbb{R}^n$. Almost all of their proofs are exactly the same.

Proposition 1.6 $v_1, \dots, v_n \in V$. The following statements are equivalent.

i) $v_1, \dots, v_n$ are lin. dependent (i.e. $\exists \lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \neq 0$ with $\sum_{i=1}^{n} \lambda_i v_i = 0$)

ii) $\exists j: \quad v_j \in \operatorname{span}\{v_1, \dots, \cancel{v_j}, \dots, v_n\}$

iii) $\exists j: \quad \operatorname{span}\{v_1, \dots, v_n\} = \operatorname{span}\{v_1, \dots, \cancel{v_j}, \dots, v_n\}$

Proof: Same as for $V = \mathbb{R}^n$

Lemma 1.7 If $v_1, \dots, v_\ell \in V$ are lin. indep. and $V = \operatorname{span}\{w_1, \dots, w_m\}$ then $\ell \leq m$

Proof: We will show that if $\ell > m$, then $v_1, \dots, v_\ell$ are lin. dependent.

For $j = 1, \dots, \ell$ write $v_j = \sum_{i=1}^{m} \alpha_{ij} w_i \quad (\alpha_{ij} \in \mathbb{R})$

We have,

$$\sum_{j=1}^{\ell} \lambda_j v_j = \sum_{j=1}^{\ell} \lambda_j \left(\sum_{i=1}^{m} \alpha_{ij} w_i \right) = \sum_{i=1}^{m} \underbrace{\left(\sum_{j=1}^{\ell} \lambda_j \alpha_{ij} \right)}_{=:\beta_i} w_i$$

Set $A = (\alpha_{ij}) = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1\ell} \\ \vdots & & \\ \alpha_{m1} & \cdots & \alpha_{m\ell} \end{pmatrix} \in \mathbb{R}^{m \times \ell}, \quad \lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_\ell \end{pmatrix}$

This gives $A\lambda = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}$.

If $\ell > m$ the equation $A\lambda = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ has infinetly many solution.

In particular $\exists \lambda \neq 0$ with $A\lambda = 0 \Rightarrow \sum_{j=1}^{\ell} \lambda_j v_j = 0 \underset{\lambda \neq 0}{\Rightarrow} v_1, \dots, v_\ell$ lin. dependent

$\square$

<u>Theorem 1.8</u> Let V be finitely generated.

i) V has a (finite) basis

ii) All bases of V have the same number of elements

iii) If $v_1, \dots, v_\ell \in V$ are linearly independent then there exist $v_{\ell+1}, \dots, v_n \in V$ such that $(v_1, \dots, v_n)$ is a basis of V.

iv) If $V = \text{span}\{w_1, \dots, w_m\}$, then there exist a subset $\{u_1, \dots, u_\ell\} \subset \{w_1, \dots, w_m\}$, such that $(u_1, \dots, u_\ell)$ is a basis of V.

<u>Proof</u>: Same as in LA I. (ii) follows from Lemma 1.7, i) follows from iv))

<u>Definition 1.9</u> V fin. gen. with basis $(v_1, \dots, v_n)$. Then $\dim(V) = n$ is the <u>dimension</u> of V.

<u>Corollary 1.10</u> Let $\dim V = n$, $v_1, \dots, v_n \in V$. Then the following statements are equivalent:

i) $v_1, \dots, v_n$ are lin. indep.

ii) $V = \text{span}\{v_1, \dots, v_n\}$

iii) $(v_1, \dots, v_n)$ is a basis of V.

<u>Proposition 1.11</u> Let V be fin. gen. and $U \subset V$ be a subspace. Then U is also finitely generated.

<u>Proof</u>: Can be done by using Prop. 1.6 + Lemma 1.7.

Example 5

i) $\mathbb{R}^{m\times n}$ has a basis $(E_{ij})_{i,j} \Rightarrow \dim \mathbb{R}^{m\times n} = m\cdot n$

$$E_{ij} = \begin{array}{cc} & j \\ & \begin{pmatrix} 0 & & \vdots & & 0 \\ & \ddots & \vdots & & \\ \cdots & \cdots & 1 & \cdots & \cdots \\ & & \vdots & \ddots & \\ 0 & & \vdots & & 0 \end{pmatrix} & i \end{array}$$

ii) $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is not fin. gen. because $\mathcal{P} \subset \mathcal{F}(\mathbb{R}, \mathbb{R})$ is not.

Proposition 1.12 Let $\mathcal{B} = (v_1, \dots, v_n)$ be a basis of V. Then $\forall u \in V \ \exists! \lambda_1, \dots, \lambda_n \in \mathbb{R} : u = \sum_{i=1}^{n} \lambda_i v_i$.

Definition 1.13 Let $\mathcal{B} = (v_1, \dots, v_n)$ be a basis of V.

i) The $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ in Prop. 1.12 are called the coordinates of u in the basis $\mathcal{B}$.

ii) The vector $[u]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \in \mathbb{R}^n$ is the coordinate vector of u