Linear Algebra II

Overview notes

G30 Program, Nagoya University (Spring 2023)

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Lecture notes and exercises are available at: https://www.henrikbachmann.com/la2_2023.html

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These notes serve as a compact overview of the definitions, propositions, lemmas, corollaries, and theorems given in the lectures. This is Version 1 from April 9, 2023. The proofs, examples, and explanations are provided in the handwritten notes/lectures. The reference book for this course is [B], and we will probably cover Chapters 4,6,7 and 9 during this semester.

If you find any typos in this note, please let me know!

Contents

References

[B] O. Bretscher: *Linear Algebra with Applications*, 4th edition, Pearson 2009.

1 Vector spaces

Definition 1.1. *A* **(real) vector space** *is a set V together with two functions*

Addition

$$+ : V \times V \longrightarrow V, \quad (u,v) \longmapsto u+v$$

Scalar multiplication

$$\cdot : \mathbb{R} \times V \longrightarrow V, \quad (\lambda, v) \longmapsto \lambda \cdot v = \lambda v$$

satisfying the following properties:

- *Properties of the addition:*

(*A.1*) $\forall u, v, w \in V$*:* $(u+v)+w = u+(v+w)$. *(Associativity)*

(*A.2*) $\forall u, v \in V$*:* $u+v = v+u$. *(Commutativity)*

(*A.3*) $\exists n \in V, \forall u \in V$*:* $n+u = u$. *(Identity/neutral element of addition)*

(*A.4*) $\forall u \in V, \exists v \in V$*:* $u+v = n$. *(Inverse elements of addition)*

- *Compatibility of addition and scalar multiplication:*

(*C.1*) $\forall u, v \in V, \lambda \in \mathbb{R}$*:* $\lambda \cdot (u+v) = \lambda u + \lambda v$. *(Distributivity I)*

(*C.2*) $\forall u \in V, \lambda, \mu \in \mathbb{R}$*:* $(\lambda + \mu) \cdot u = \lambda u + \mu u$. *(Distributivity II)*

(*C.3*) $\forall u \in V, \lambda, \mu \in \mathbb{R}$*:* $\lambda \cdot (\mu u) = (\lambda \mu) \cdot u$.

(*C.4*) $\forall u \in V$*:* $1 \cdot u = u$.

We write $(V, +, \cdot)$ for the vector space V if we want to emphasize which addition and scalar multiplication we are using.

Proposition 1.2. *Let V be a vector space and $u \in V$.*

i) $u + n = u$.

ii) *If $n, \tilde{n} \in V$ both satisfy (A.3) in Definition 1.1, then $n = \tilde{n}$.*
(The identity element is unique)

iii) *If for a fixed $u \in V$ the elements $v, \tilde{v} \in V$ both satisfy (A.4), i.e. $u + v = u + \tilde{v} = n$, then $v = \tilde{v}$.*
(The inverse of an element u is unique)

iv) $u + (-1)u = n$.

The identity (also called neutral) element $n \in V$ of a vector space is usually (by abuse of notation) also denoted by 0. Be always aware in the following if 0 means the real number 0 or the identity element of a vector space. (These are two different things!)

Definition 1.3. *Let V be a vector space. A subset $U \subset V$ is a* **subspace** *if*

i) $0 \in U$.

ii) $\forall u, v \in U$: $u + v \in U$.

iii) $\forall u \in U$, $\lambda \in \mathbb{R}$: $\lambda u \in U$.

Proposition 1.4. *If $U \subset V$ is a subspace, then U is also a vector space with the operations inherited from V.*

Definition 1.5. *Let V be a vector space and $v_1, \dots, v_n \in V$.*

i) The **span** *of the elements $v_1, \dots, v_n$ is given by the set of all their* **linear combinations**, *i.e.*

$$\operatorname{span}\{v_1, \dots, v_n\} = \left\{ \sum_{i=1}^{n} \lambda_i v_i \in V \mid \lambda_1, \dots, \lambda_n \in \mathbb{R} \right\}.$$

ii) The elements $v_1, \dots, v_n$ **span (or generate) the space** *V if* $\operatorname{span}\{v_1, \dots, v_n\} = V$.

iii) V is **finitely generated** *if there exist $v_1, \dots, v_n \in V$ with* $\operatorname{span}\{v_1, \dots, v_n\} = V$.
(i.e. one just needs finitely many elements to generate the space)

iv) The elements $v_1, \dots, v_n$ are **linearly independent** *if*

$$\lambda_1 v_1 + \dots + \lambda_n v_n = 0 \implies \lambda_1 = \dots = \lambda_n = 0.$$

v) $B = (v_1, \dots, v_n)$ is a **basis** *of V if $v_1, \dots, v_n$ are linearly independent and* $\operatorname{span}\{v_1, \dots, v_n\} = V$.

Proposition 1.6. *Let V be a vector space and $v_1, \dots, v_n \in V$. The following statements are equivalent.*

i) $v_1, \dots, v_n$ are linearly dependent.

ii) There exist a $1 \leq j \leq n$ such that $v_j \in \operatorname{span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n\}$.

iii) There exist a $1 \leq j \leq n$ such that $\operatorname{span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n\} = \operatorname{span}\{v_1, \dots, v_n\}$.

Lemma 1.7. *If $v_1, \dots, v_l \in V$ are linearly independent and $V = \operatorname{span}\{w_1, \dots, w_m\}$, then $l \leq m$.*

Theorem 1.8. *Let V be a finitely generated vector space. Then we have the following*

i) V has a (finite) basis.

ii) All bases of V have the same number of elements.

iii) If $v_1, \dots, v_l \in V$ are linearly independent then there exist $v_{l+1}, \dots, v_n \in V$ such that $(v_1, \dots, v_n)$ is a basis of V.

iv) If $V = \operatorname{span}\{w_1, \dots, w_m\}$, then there exist a subset $\{u_1, \dots, u_l\} \subset \{w_1, \dots, w_m\}$, such that $(u_1, \dots, u_l)$ is a basis of V.

Definition 1.9. *Let V be a finitely generated vector space with basis $(v_1, \ldots, v_n)$. Then $\dim(V) = n$ is the* **dimension of V**.

Corollary 1.10. *Let V be a vector space with $\dim(V) = n$ and $v_1, \ldots, v_n \in V$. Then the following statements are equivalent.*

i) $v_1, \ldots, v_n$ *are linearly independent.*

ii) $V = \operatorname{span}\{v_1, \ldots, v_n\}$.

iii) $(v_1, \ldots, v_n)$ *is a basis of* V.

Proposition 1.11. *Let V be finitely generated and $U \subset V$ a subspace. Then U is also finitely generated.*

Proposition 1.12. *Let $B = (v_1, \ldots, v_n)$ be a basis of V. Then for all $u \in V$ there exist unique $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, such that*

$$u = \sum_{i=1}^{n} \lambda_i v_i \,.$$

Definition 1.13. *Let $B = (v_1, \ldots, v_n)$ be a basis of V.*

i) The $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ in Proposition 1.12 are called the **coordinates** *of $u \in V$ in the basis B.*

ii) The vector $[u]_B \in \mathbb{R}^n$ given by

$$[u]_B = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

is called the **coordinate vector** *of u with respect to the basis B.*

2 Linear maps

Definition 2.1. *Let V, W be vector spaces. A* **linear map** *is a function $F : V \to W$ satisfying*

i) $F(u + v) = F(u) + F(v)$ *for all* $u, v \in V$.

ii) $F(\lambda \cdot u) = \lambda \cdot F(u)$ *for all* $u \in V, \lambda \in \mathbb{R}$.

Definition 2.2. *Let $F : V \to W$ be a linear map.*

i) The **kernel of** F *is given by*

$$\ker(F) = \{u \in V \mid F(u) = 0\} \subset V \,.$$

ii) The **image of** F *is given by*

$$\operatorname{im}(F) = \{w \in W \mid \exists u \in V : w = F(u)\} \subset W \,.$$

With the same arguments as in the $\mathbb{R}^n$-case we see that $\ker(F)$ *is a subspace of* V *and* $\operatorname{im}(F)$ *is a subspace of* W*. If* $\operatorname{im}(F)$ *is finitely generated, we define the* **rank of** F *by* $\operatorname{rk}(F) = \dim(\operatorname{im}(F))$*.*

Theorem 2.3 (kernel-image theorem)**.** *Let* V *be finitely generated and let* $F : V \to W$ *be a linear map to an arbitrary vector space* W*. Then*

$$\dim V = \dim(\ker(F)) + \dim(\operatorname{im}(F)) \,.$$

Definition 2.4. *i) (Recall) A function* $f : X \to Y$ *is* **invertible** *if there exist a function* $g : Y \to X$ *such that* $f \circ g = \mathrm{id}_Y$ *and* $g \circ f = \mathrm{id}_X$*.* f *is invertible iff* f *is bijective, i.e. injective and surjective.*

ii) An invertible linear map $F : V \to W$ *is called an* **isomorphism***.*

iii) Two vector spaces V *and* W *are called* **isomorphic** *(Notation:* $V \cong W$*) if there exists an isomorphism* $F : V \to W$*.*

Theorem 2.5. *i) A linear map* $F : V \to W$ *is an isomorphism iff* $\ker(F) = \{0\}$ *(*F *is injective) and* $\operatorname{im}(F) = W$ *(*F *is surjective).*

ii) Let $F : V \to W$ *be an isomorphism and* $(b_1, \ldots, b_n)$ *a basis of* V*. Then* $(F(b_1), \ldots, F(b_n))$ *is a basis of* W*.*

iii) Let V, W *be finitely generated and* $V \cong W$ *then* $\dim(V) = \dim(W)$*.*

iv) Let V, W *be finitely generated and* $\dim(V) = \dim(W)$*. Then for a linear map* $F : V \to W$ *the following three statements are equivalent*

(a) F *is an isomorphism.*

(b) $\ker(F) = \{0\}$.

(c) $\operatorname{im}(F) = W$*.*

Proposition 2.6. *Let* V *be finitely generated with basis* $B = (b_1, \ldots, b_n)$*, i.e.* $\dim(V) = n$*.*
Then the **coordinate map**

$$c_B : \mathbb{R}^n \longrightarrow V \,,$$
$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \longmapsto \sum_{i=1}^{n} \lambda_i b_i$$

is an isomorphism. The inverse is given by $c_B^{-1}(u) = [u]_B$ *for* $u \in V$*.*

Corollary 2.7. *Let* V, W *be finitely generated. Then the following two statements are equivalent*

i) $V \cong W$*.*

ii) $\dim(V) = \dim(W)$*.*

3 The matrix of a linear map

In the following V and W are finitely generated vector spaces.

Definition 3.1. *Let $B_V = (v_1, \dots, v_n)$ be a basis of V, $B_W = (w_1, \dots, w_m)$ be a basis of W and let $F : V \to W$ be a linear map. The* **matrix of F with respect to B_V and B_W** *is defined by*

$$[F]_{B_V}^{B_W} = \left[c_{B_W}^{-1} \circ F \circ c_{B_V}\right].$$

Here $c_{B_W}^{-1} \circ F \circ c_{B_V}$ is the linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ for which the corresponding matrix was defined before. We have the following diagram

$$\begin{array}{ccc} V & \xrightarrow{\quad F \quad} & W \\ {\scriptstyle c_{B_V}} \big\uparrow & & \big\downarrow {\scriptstyle c_{B_W}^{-1}} \\ \mathbb{R}^n & \xrightarrow[c_{B_W}^{-1} \circ F \circ c_{B_V}]{} & \mathbb{R}^m \end{array}$$

We have

$$[F]_{B_V}^{B_W} = \begin{pmatrix} | & \cdots & | \\ [F(v_1)]_{B_W} & \cdots & [F(v_n)]_{B_W} \\ | & \cdots & | \end{pmatrix}.$$

In other words: The j-th column of $[F]_{B_V}^{B_W}$ is given by the vector $\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}$, where $F(v_j) = \sum_{i=1}^m \lambda_i w_i$.

Definition 3.2. *Let $B_1 = (v_1, \dots, v_n)$ and $B_2 = (u_1, \dots, u_n)$ be bases of V. The* **change-of-basis matrix from B_1 to B_2** *is the matrix*

$$S_{B_1}^{B_2} = [\mathrm{id}_V]_{B_1}^{B_2} = [c_{B_2}^{-1} \circ c_{B_1}] = \begin{pmatrix} | & \cdots & | \\ [v_1]_{B_2} & \cdots & [v_n]_{B_2} \\ | & \cdots & | \end{pmatrix}.$$

4 Determinants

Definition 4.1. *A* **pattern** *in an $n \times n$-matrix is a choice of entries, such that precisely one entry from each row and each column is chosen.*

Definition 4.2. *i) A bijective map $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is called a* **permutation** *of $\{1, \ldots, n\}$.*

ii) S_n denotes the set of all permutations of $\{1, \ldots, n\}$.

Patterns in an $n \times n$-matrix corresponds exactly to the permutations of $\{1, \ldots, n\}$. For each $\sigma \in S_n$ we have the pattern

$$P = \big\{(1, \sigma(1)), (2, \sigma(2)), \ldots, (n, \sigma(n))\big\}\,,$$

where (i, j) denotes the choice of the i-th row and the j-th column.

Definition 4.3. *i) The* **number of inversion** *of a permutation $\sigma \in S_n$, denoted by $\operatorname{inv}(\sigma)$, is the number of pairs $(i, \sigma(i)), (j, \sigma(j))$ with $i < j$ and $\sigma(i) > \sigma(j)$.*

ii) The sign of a permutation $\sigma \in S_n$ is defined by

$$\operatorname{sign}(\sigma) = (-1)^{\operatorname{inv}(\sigma)}\,.$$

Definition 4.4. *The* **determinant** *of a $n \times n$-matrix $A = (a_{i,j}) \in \mathbb{R}^{n \times n}$ is defined by*

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}\,.$$

4.1 Properties of determinants

Lemma 4.5. *For all $\sigma \in S_n$ we have $\operatorname{inv}(\sigma) = \operatorname{inv}(\sigma^{-1})$.*

Proposition 4.6. *For any $A \in \mathbb{R}^{n \times n}$ we have $\det(A) = \det(A^T)$.*

For $A = (a_{i,j}) \in \mathbb{R}^{n \times n}$ define for a vector $x \in \mathbb{R}^n$ and $1 \le l \le n$ the matrix $A(l; x)$ as the matrix where the l-th row of A gets replaced by x, i.e.

$$A(l;x) = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & & & \vdots \\ a_{l-1,1} & a_{l-1,2} & \cdots & a_{l-1,n} \\ x_1 & x_2 & \cdots & x_n \\ a_{l+1,1} & a_{l+1,2} & \cdots & a_{l+1,n} \\ \vdots & & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \in \mathbb{R}^{n \times n}, \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n\,.$$

Proposition 4.7. *For any $A = (a_{i,j}) \in \mathbb{R}^{n\times n}$ and $1 \le l \le n$ the map*

$$\begin{aligned} F_{A,l} : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ x &\longmapsto \det(A(l;x)) \end{aligned}$$

is a linear map, i.e. the determinant is linear in each row,

Proposition 4.8. *For $A \in \mathbb{R}^{n\times n}$ let $B \in \mathbb{R}^{n\times n}$ be a matrix obtained from the matrix A by swapping two rows. Then we have*

$$\det(A) = -\det(B)\,.$$

Corollary 4.9. *If a matrix $A \in \mathbb{R}^{n\times n}$ contains two equal rows or columns, then $\det(A) = 0$.*

Recall from Linear Algebra I that there are three types of **row operations** for a matrix $A \in \mathbb{R}^{n\times n}$. $(1 \le i, j \le n\,, i \ne j\,, \lambda \in \mathbb{R})$.

(R1) Add λ-times the j-th row to the i-th row.

(R2) For $\lambda \ne 0$ multiply the i-th row with λ.

(R3) Swap the j-th row with the i-th row.

Two matrices $A, B \in \mathbb{R}^{n\times n}$ are called **row equivalent**, if one can obtain B from A by using the row operations (R1), (R2) and (R3). Notation: $A \sim B$.

Proposition 4.10. *Let $A, B \in \mathbb{R}^{n\times n}$.*

i) If B is obtained from A by using (R1), then $\det(B) = \det(A)$.

ii) If B is obtained from A by using (R2), then $\det(B) = \lambda \det(A)$.

iii) If B is obtained from A by using (R3), then $\det(B) = -\det(A)$.

Theorem 4.11. *A matrix $A \in \mathbb{R}^{n\times n}$ is invertible if and only if $\det(A) \ne 0$.*

Theorem 4.12. *i) For all $A, B \in \mathbb{R}^{n\times n}$ we have $\det(AB) = \det(A)\det(B)$.*

ii) If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

Corollary 4.13. *Let V be a finitely generated vector space, $F : V \to V$ a linear map and B_1, B_2 two bases of V. Then*

$$\det([F]_{B_1}) = \det([F]_{B_2})\;,$$

where $[F]_B = [F]_B^B$ denotes the matrix of F with respect to the basis B (Definition 3.1).

Definition 4.14. *Let V be a finitely generated vector space, $F : V \to V$ a linear map and B any basis of V. We define the* **determinant of the linear map** *F by*

$$\det(F) = \det([F]_B)\;.$$

Definition 4.15. *For $\lambda \in \mathbb{R}$ with $\lambda \neq 0$ and $1 \leq i,j \leq n$ we define the* **elementary matrices** $R_i^{\lambda,j}, R_i^{\lambda}, R_{i,j} \in \mathbb{R}^{n\times n}$ *by*

$$R_i^{\lambda,j} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & \lambda & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}, \quad R_i^{\lambda} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \lambda & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}, \quad R_{i,j} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & & 1 & \\ & & & \ddots & & \\ & & 1 & & 0 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}.$$

Here the λ in $R_i^{\lambda,j}$ is in the i-th row and j-th column, in R_i^{λ} it is in the i-th row, and in $R_{i,j}$ the 0 are on the diagonal in the i-th row and j-th column.

Proposition 4.16. *Let $A \in \mathbb{R}^{n\times n}$.*

i) $R_i^{\lambda,j}A$ is the matrix obtained from A by row operation (R1). *(Add λ-times the j-th row to the i-th row)*

ii) $R_i^{\lambda}A$ is the matrix obtained from A by row operation (R2). *(Multiply the i-th row with λ)*

iii) $R_{i,j}A$ is the matrix obtained from A by row operation (R3). *(Swap the j-th row with the i-th row)*

Corollary 4.17. *Let $A \in \mathbb{R}^{n\times n}$.*

i) The matrix A is invertible if and only if it is a product of elementary matrices.

ii) If C is an elementary matrix then $\det(CA) = \det(C)\det(A)$.

For a matrix $A \in \mathbb{R}^{n\times n}$ and $1 \leq i,j \leq n$ we denote by $A_{i,j} \in \mathbb{R}^{(n-1)\times(n-1)}$ the matrix which is obtained from A by removing the i-th row and the j-th column.

Theorem 4.18 (Laplace expansion)**.** *For a matrix $A = (a_{i,j}) \in \mathbb{R}^{n\times n}$ and $1 \leq i,j \leq n$ we have*

$$\begin{aligned} \det(A) &= \sum_{l=1}^{n} (-1)^{i+l} a_{i,l} \det(A_{i,l}) \\ &= \sum_{l=1}^{n} (-1)^{j+l} a_{l,j} \det(A_{l,j}) \,. \end{aligned}$$

5 Eigenvalues and eigenvectors

In this section V always denotes a vector space.

Definition 5.1. *Let $F : V \to V$ be a linear map.*

i) A $\lambda \in \mathbb{R}$ is called an **eigenvalue** *of F, if there exist a vector $v \in V$ with $v \neq 0$, such that*

$$F(v) = \lambda v \,. \tag{5.1}$$

ii) A vector $v \in V$ with $v \neq 0$, satisfying (5.1)*, is called an* **eigenvector** *of F with eigenvalue λ.*

Notice that $v = 0$ always satisfies (5.1) for any $\lambda \in \mathbb{R}$, since F is a linear map. This is one of many reasons why $v = 0$ is not called an eigenvector of F.

In the following, we always assume that V is a finitely generated vector space.

Definition 5.2. *Let $F : V \to V$ be a linear map let $\mathrm{id}_V : V \to V$ be the identity map on V.*

i) The polynomial $f_F(\lambda) = \det(F - \lambda\,\mathrm{id}_V)$ is called the **characteristic polynomial** *of F.*

ii) Let $\lambda \in \mathbb{R}$ be an eigenvalue of F. Then the space

$$\begin{aligned} E_\lambda(F) &= \ker(F - \lambda\,\mathrm{id}_V) \\ &= \{v \in V \mid F(v) = \lambda v\} \end{aligned}$$

is called the **eigenspace** *of F with respect to the eigenvalue λ.*

The eigenspace $E_\lambda(F)$ contains therefore all eigenvectors of F with eigenvalue λ and the zero vector.

Definition 5.3. *i) Let $\dim V = n$. A linear map $F : V \to V$ is called* **diagonalizable** *if there exist a basis B of V, such that*

$$[F]_B = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

for some $d_1, \ldots, d_n \in \mathbb{R}$.

ii) A matrix $A \in \mathbb{R}^{n\times n}$ is called **diagonalizable** *if there exists an invertible matrix $S \in \mathbb{R}^{n\times n}$ with*

$$S^{-1}AS = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

for some $d_1, \ldots, d_n \in \mathbb{R}$.

Lemma 5.4. *Let B be a basis of V and let $F : V \to V$ be a linear map. Then the following two statements are equivalent*

i) The linear map F is diagonalizable.

ii) The matrix $[F]_B$ is diagonalizable.

Lemma 5.5. *Let $F : V \to V$ be a linear map and $B = (b_1, \ldots, b_n)$ be a basis of V, such that all b_i are eigenvectors of F, i.e. $F(b_i) = d_i b_i$ for some $d_i \in \mathbb{R}$ and $i = 1, \ldots, n$. Then F is diagonalizable and*

$$[F]_B = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

Conversely, if F is diagonalizable then there exists a basis of eigenvectors.

Theorem 5.6. *Let $v_1, \dots, v_m \in V$ be eigenvectors of a linear map $F : V \to V$ with different eigenvalues $\lambda_1, \dots, \lambda_m \in \mathbb{R}$. Then $v_1, \dots, v_m$ are linearly independent.*

Corollary 5.7. *Let $F : V \to V$ be a linear map with eigenvalues $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ and $\dim V = n$.*

i) If F has n distinct eigenvalues, i.e. $m = n$, then F is diagonalizable.

ii) If $B_1, \dots, B_m$ are bases of $E_{\lambda_1}(F), \dots, E_{\lambda_m}(F)$, then $B_1 \cup \dots \cup B_m$ are linearly independent.

iii) The map F is diagonalizable if and only if

$$\sum_{j=1}^{m} \dim E_{\lambda_j}(F) = n \,.$$

Definition 5.8. *Let $F : V \to V$ be a linear map and let $\lambda \in \mathbb{R}$ be an eigenvalue of F.*

i) The **algebraic multiplicity** *of λ, denoted by $\operatorname{algmu}_F(\lambda)$, is the multiplicity of λ in the characteristic polynomial f_F.*

ii) The **geometric multiplicity** *of λ is given by $\operatorname{geomu}_F(\lambda) = \dim E_\lambda(F)$.*

Theorem 5.9. *Let $F : V \to V$ be a linear map and $\lambda \in \mathbb{R}$ be an eigenvalue of F. Then*

$$\operatorname{geomu}_F(\lambda) \leq \operatorname{algmu}_F(\lambda) \,.$$

Corollary 5.10. *If F is diagonalizable then $\operatorname{geomu}_F(\lambda) = \operatorname{algmu}_F(\lambda)$ for all eigenvalues λ of F.*

5.1 The spectral theorem

In this section we will just consider the vector space $V = \mathbb{R}^n$. Recall that the **norm** of a vector $x \in \mathbb{R}^n$ is defined by

$$\|x\| = \sqrt{x_1^2 + \dots + x_n^2} \,, \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \,.$$

Definition 5.11. *An* **orthogonal map** *is a linear map $F : \mathbb{R}^n \to \mathbb{R}^n$, such that*

$$\|F(x)\| = \|x\| \,, \qquad \forall x \in \mathbb{R}^n \,,$$

i.e. the map F does not change the norm of a vector. We call a matrix $A \in \mathbb{R}^{n\times n}$ orthogonal if $\|Ax\| = \|x\|$ for all $x \in \mathbb{R}^n$.

Recall that the **dot product** $\bullet$ for two vectors $x, y \in \mathbb{R}^n$ is defined by

$$x \bullet y = x^T y = x_1 y_1 + \dots + x_n y_n \in \mathbb{R} \,.$$

With this the norm of a vector can also be written as $\|x\| = \sqrt{x \bullet x}$.

Lemma 5.12. *For all $x, y \in \mathbb{R}^n$ we have*

$$x \bullet y = \frac{1}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right) .$$

Proposition 5.13. *A linear map $F : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if and only if*

$$F(x) \bullet F(y) = x \bullet y$$

for all $x, y \in \mathbb{R}^n$.

Recall: We say that x and y are **orthogonal** if $x \bullet y = 0$. A basis $B = (b_1, \ldots, b_n)$ of $\mathbb{R}^n$ is called an **orthonormal basis** if b_i and b_j for $i \neq j$ are orthogonal and $\|b_i\| = 1$ for all i, i.e.

$$b_i \bullet b_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} .$$

Theorem 5.14. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ a linear map and $A = [F]_B$ the matrix of F for $B = (e_1, \ldots, e_n)$. The following statements are equivalent.*

- *i) F is orthogonal.*
- *ii) A is orthogonal.*
- *iii) For all $x, y \in \mathbb{R}^n$ we have $F(x) \bullet F(y) = x \bullet y$.*
- *iv) A is invertible and $A^{-1} = A^T$.*
- *v) $(F(e_1), \ldots, F(e_n))$ (the columns of A) is an orthonormal basis of $\mathbb{R}^n$.*
- *vi) If $(b_1, \ldots, b_n)$ is an orthonormal basis of $\mathbb{R}^n$ then $(F(b_1), \ldots, F(b_n))$ is also an orthonormal basis.*

Corollary 5.15.

- *i) $A \in \mathbb{R}^{n\times n}$ is orthogonal if and only if A^T is orthogonal.*
- *ii) If $A, B \in \mathbb{R}^{n\times n}$ are orthogonal then AB is orthogonal.*
- *iii) If B_1 and B_2 are two orthonormal bases, then the change of basis matrix $S_{B_1}^{B_2}$ is orthogonal.*

Definition 5.16.

- *i) An* **eigenbasis** *of a linear map $F : \mathbb{R}^n \to \mathbb{R}^n$ is a basis consisting of eigenvectors of F.*
- *ii) Let $U \subset \mathbb{R}^n$ be a subspace. A linear map $F : U \to U$ is called* **symmetric** *if we have for all $x, y \in U$*

$$x \bullet F(y) = F(x) \bullet y .$$

Theorem 5.17. *(Spectral theorem) Let $U \subset \mathbb{R}^n$ be a subspace and $F : U \to U$ a linear map. Then F is symmetric if and only if there exists an orthonormal eigenbasis of F.*

Corollary 5.18. *A matrix $A \in \mathbb{R}^{n\times n}$ is symmetric if and only if there exists an orthogonal matrix $S \in \mathbb{R}^{n\times n}$, such that*

$$S^T A S = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix},$$

for some $d_1, \ldots, d_n \in \mathbb{R}$.

Lemma 5.19. *Every symmetric linear map $F : U \to U$ has an eigenvalue.*

6 Linear differential equations

Let $x : \mathbb{R} \to \mathbb{R}^n$ be a function written as

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix},$$

where the entries $x_1, ..., x_n$ are differentiable functions in $C^{(1)}(\mathbb{R}, \mathbb{R})$. By $x'(t) = \frac{d}{dt}x(t)$ we denote

$$x'(t) = \begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}.$$

For a matrix $A \in \mathbb{R}^{n\times n}$ the equation

$$x'(t) = Ax(t)$$

is called a **continuous (linear) dynamical system**.

One dimensional ($n = 1$) continuous dynamical systems have the following solutions:

Proposition 6.1. *Let $a \in \mathbb{R}$. The only solutions to*

$$x'(t) = a\,x(t)$$

in $C^{(1)}(\mathbb{R}, \mathbb{R})$ are given by $x(t) = c\,e^{at}$ for $c \in \mathbb{R}$.

Recall that the space $C^\infty(\mathbb{R}, \mathbb{R})$, the space of **smooth functions**, denotes the space of all functions $f : \mathbb{R} \to \mathbb{R}$ for which derivatives of all orders exist. This means that for any $n \geq 0$ and $f \in C^\infty(\mathbb{R}, \mathbb{R})$, the n-th derivative $f^{(n)} \in C^\infty(\mathbb{R}, \mathbb{R})$ exists. The space $C^\infty(\mathbb{R}, \mathbb{R})$ is a vector space.

Definition 6.2. *i) A* **differential operator of order** *n is a map $T : C^\infty(\mathbb{R}, \mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R})$ of the form*

$$T(f) = a_0 f + a_1 f' + a_2 f^{(2)} + \cdots + a_n f^{(n)}$$

for some $a_0, a_1, \ldots, a_n \in \mathbb{R}$ with $a_n \neq 0$.

(More precisely this is a "linear differential operator of order n with constant coefficients".)

ii) A **linear differential equation** *is an equation of the form $T(f) = g$, where T is a differential operator and $g \in \mathbb{C}^\infty(\mathbb{R}, \mathbb{R})$.*

iii) A linear differential equation is called **homogeneous** *if $g = 0$, i.e. if $T(f) = 0$.*

Lemma 6.3. *Let $F : V \to W$ be a linear map between two vector spaces V and W. Assume that $F(v) = w$ for a fixed $v \in V$ and $w \in W$. Then the following two statements are equivalent:*

i) $F(x) = w$.

ii) $x = v + u$ for some $u \in \ker(F)$.

Theorem 6.4. *Let $T : C^\infty(\mathbb{R}, \mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R})$ be a differential operator of order n. Then we have*

$$\dim(\ker(T)) = n\,.$$

Definition 6.5. *Let $T(f) = a_0 f + a_1 f' + \cdots + a_n f^{(n)}$ be a differential operator of order n. The* **characteristic polynomial** *of T is defined by*

$$p_T(x) = \sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + \cdots + a_n x^n\,.$$

In the following, T always denotes a differential operator.

Proposition 6.6. *i) The function $e^{\lambda t}$ is an eigenvector of T with eigenvalue $p_T(\lambda)$.*

ii) We have $e^{\lambda t} \in \ker(T)$ if and only if $p_T(\lambda) = 0$.

Corollary 6.7. *Let T be a differential operator of order n.*

i) If $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ are distinct, then $e^{\lambda_1 t}, \dots, e^{\lambda_n t}$ are linearly independent.

ii) If p_T has n distinct zeroes $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ then $(e^{\lambda_1 t}, \dots, e^{\lambda_n t})$ is a basis of $\ker(T)$.

Lemma 6.8. *For two differential operators T_1 and T_2 we have $T_1 \circ T_2 = T_2 \circ T_1$.*

Theorem 6.9. *Let T be a differential operator with characteristic polynomial*

$$p_T(x) = (x - \lambda_1)^{m_1} \cdots (x - \lambda_r)^{m_r}$$

where $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ with $\lambda_i \neq \lambda_j$ for $i \neq j$.
Then $B = B_1 \cup \cdots \cup B_r$ is a basis of $\ker(T)$, where we have for $1 \leq j \leq r$

$$B_j = (e^{\lambda_j t}, te^{\lambda_j t}, \dots, t^{m_j - 1} e^{\lambda_j t})\,.$$

Theorem 6.10. *Let T be a differential operator. If $p_T(x)$ contains a factor $((x-a)^2 + b^2)^m$, then*

$$\{e^{at}\cos(bt), e^{at}\sin(bt), te^{at}\cos(bt), te^{at}\sin(bt), \dots, t^{m-1}e^{at}\cos(bt), t^{m-1}e^{at}\sin(bt)\}$$

are $2m$ linearly independent elements in $\ker(T)$.

Lemma 6.11. *Let $F : U \to V$ and $G : V \to W$ be surjective linear maps between vector spaces U, V, W, such that $\ker(F)$ and $\ker(G)$ are finitely generated. Then we have*

$$\dim(\ker(G \circ F)) = \dim(\ker(F)) + \dim(\ker(G))\,.$$