

Linear Algebra II

Tutorial 6

Definition 17.1 A **pattern** in an $n \times n$ -matrix is a choice of entries, such that precisely one entry from each row and each column is chosen.

Definition 17.2 i) A bijective map $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ is called a **permutation** of $\{1, \dots, n\}$.

ii) S_n denotes the set of all permutations of $\{1, \dots, n\}$.

Patterns in an $n \times n$ -matrix corresponds exactly to the permutations of $\{1, \dots, n\}$. For each $\sigma \in S_n$ we have the pattern

$$P = \{(1, \sigma(1)), (2, \sigma(2)), \dots, (n, \sigma(n))\},$$

where (i, j) denotes the choice of the i -th row and the j -th column.

Definition 17.3 i) The **number of inversion** of a permutation $\sigma \in S_n$, denoted by $\text{inv}(\sigma)$, is the number of pairs $(i, \sigma(i)), (j, \sigma(j))$ with $i < j$ and $\sigma(i) > \sigma(j)$.

ii) The **sign** of a permutation $\sigma \in S_n$ is defined by

$$\text{sign}(\sigma) = (-1)^{\text{inv}(\sigma)}.$$

Definition 17.4 The **determinant** of a $n \times n$ -matrix $A = (a_{i,j}) \in \mathbb{R}^{n \times n}$ is defined by

$$\det(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^n a_{i, \sigma(i)}.$$

Question: If $A = \begin{pmatrix} 1 & 0 & \dots & 0 \\ * & \boxed{B} & & \\ \vdots & & & \\ * & & & \end{pmatrix} \in \mathbb{R}^{n \times n}$ and

$B \in \mathbb{R}^{(n-1) \times (n-1)}$. What is the relation between $\det(A)$ and $\det(B)$?

Answer: The top left entry is the only non-zero entry in this row, therefore in $\det(A)$

just those σ give a contribution which have $\sigma(1) = 1$.

$$\det(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^n a_{i, \sigma(i)}$$

$$= \underbrace{a_{1,1}}_1 \cdot \sum_{\substack{\sigma \in S_n \\ \sigma(1)=1}} \text{sign}(\sigma) \prod_{i=2}^n a_{i, \sigma(i)}$$

$\sum_{\sigma' \in S_{n-1}} \text{sign}(\sigma') \prod_{i=1}^{n-1} b_{i, \sigma'(i)} = \det(B)$

If $\sigma(1) = 1$ is fixed fixed, we can

think of σ being an element σ' in S_{n-1} with $\sigma'(i) = \sigma(i+1) - 1$.

Note: $\text{sign}(\sigma') = \text{sign}(\sigma)$ and $b_{i,j} = a_{i+1, j+1}$.

(Useful for HW 3, Ex. 3).

Exercise: Calculate $\det \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 3 \\ 4 & 1 & 1 \end{pmatrix}$

$\begin{matrix} \text{---} \\ \text{---} \\ \text{---} \end{matrix} : b_1$
 $\begin{matrix} \text{---} \\ \text{---} \\ \text{---} \end{matrix} : b_2$

Sol: $\underbrace{\text{sign}(\sigma_1)}_{-1} \cdot 2 \cdot 3 \cdot 1 + \underbrace{\text{sign}(\sigma_2)}_{+1} \cdot 1 \cdot 3 \cdot 4 = -6 + 12 = 6$

(a lot of parts vanish.
Just patterns where we do not select "0" count)

Review lecture 1-5:

⑭ Vector spaces: $(V, +, \cdot)$

Examples:

$\mathcal{F}(\mathbb{R}, \mathbb{R})$
 \cup
 $\mathcal{C}^n(\mathbb{R}, \mathbb{R})$

\cup
 $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$

\cup
 \mathcal{P}

\cup
 \mathcal{P}_n

\mathbb{R}^n

$n=0,1,2,\dots$

\downarrow

$\mathbb{R}^{m \times n}$

usually not written down.
 Make sure that you know what they are!

• If V is **fin. gen.** then there exist basis $B=(b_1, \dots, b_n)$.

For $v \in V$ know how to calculate $[v]_B$!

• Subspaces

⑮ Linear maps, Isomorphisms

(Know the Theorems!)
 e.g. 15.3 & 15.5

⑯ Matrix of linear map

$[F]_{B_2}^{B_1} \leftarrow$ understand

Exercise : If $F: P_1 \rightarrow P_1$ is a lin. map.
Example

$B_1 = \begin{pmatrix} 1 & x \\ c_1 & c_2 \end{pmatrix}$, $B_2 = \begin{pmatrix} x+1 & x-1 \\ d_1 & d_2 \end{pmatrix}$ bases
of P_1 .

Assume $[F]_{B_1}^{B_2} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

What is $F(2x+1)$?

Solution: We know that $[F(c_1)]_{B_2} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$
and $[F(c_2)]_{B_2} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

This means that

$$F(1) = 1 \cdot (x+1) + 3 \cdot (x-1) = 4x - 2$$

$$F(x) = 2 \cdot (x+1) + 4 \cdot (x-1) = 6x - 2$$

$$\text{Therefore } F(2x+1) \stackrel{\substack{\uparrow \\ F \text{ linear}}}{=} 2F(x) + F(1) = 2(6x-2) + (4x-2)$$

What is $[F]_B$? = $16x - 6$.

$$\begin{pmatrix} [F(1)]_B \\ [F(x)]_B \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 4 & 6 \end{pmatrix} \quad \det(F) = -2 \cdot 6 - (4) \cdot (-2) = -12 + 8 = -4$$

(17) Determinants

$$A \in \mathbb{R}^{n \times n} \rightsquigarrow \det(A) \in \mathbb{R}$$

$$A \text{ invertible} \Leftrightarrow \det(A) \neq 0$$

$$\det(A^T) = \det(A)$$

$$A \sim B \Rightarrow \det(B) = c \cdot \det(A) \quad c \neq 0$$

$$\text{Lecture 5: } \det(A^{-1}) = \frac{1}{\det(A)}$$

$$A, B \in \mathbb{R}^{n \times n} : \det(A) \det(B) = \det(AB)$$

$F: V \rightarrow V$ lin. map.

determinant of F : $\det(F) := \det([F]_B)$

for any basis B of V

WARNING: $\det([F]_B^C) \neq \det(F)$
if B and C are different

Homework 3: Induction & Determinants

Deadline: 28th May (23:55 JST), 2026

Exercise 1. (2+2+2 = 6 Points) Use mathematical induction to prove the following statements.

(i) Let $A, B \in \mathbb{R}^{m \times m}$ be matrices such that $AB = BA$. Show that for all $n \geq 1$, we have

$$(AB)^n = A^n B^n.$$

(ii) For all integers $n \geq 1$, prove that

$$\sum_{k=1}^n \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}.$$

(iii) Let $A \in \mathbb{R}^{m \times m}$ be a square matrix and I the $m \times m$ identity matrix. Show that for all $n \geq 1$,

$$(I - A) \sum_{k=0}^{n-1} A^k = I - A^n.$$

(Here we use the convention $A^0 = I$.)

Exercise 2. (2+4 = 6 Points) (Geometric interpretation of the determinant)

We define the vectors $v = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, u = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \in \mathbb{R}^2$.

(i) Draw a sketch in \mathbb{R}^2 by connecting the points $0, v, u,$ and $v + u$ to form a parallelogram.

(ii) Show that the area of this parallelogram is given by $\det \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}$, i.e. the determinant of the matrix which has v and u as columns.

(Remark: This works in general, i.e. if you write two vectors in \mathbb{R}^2 into the columns of a matrix $A \in \mathbb{R}^{2 \times 2}$ then $|\det(A)|$ gives the area of the parallelogram spanned by them.)

Exercise 3. (3+2 = 5 Points)

(i) Show (without using Proposition 17.7) that the determinant is linear in each row, i.e. for any $A = (a_{i,j}) \in \mathbb{R}^{n \times n}$ and $1 \leq l \leq n$ show that the map

$$F_{A,l} : \mathbb{R}^n \longrightarrow \mathbb{R} \\ x \longmapsto \det(A(l;x))$$

is linear. Here, $A(l;x)$ denotes the matrix A , where the l -th row is replaced by the vector x^T . (See page 126 in the lecture notes)

(ii) Assume that A is invertible. What is the kernel of $F_{A,1}$?

Exercise 4. (4 Points) For $a_1, a_2, \dots, a_n \in \mathbb{R}$ we define the matrix

$$A = \begin{pmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

Show that the determinant of A is given by

$$\det(A) = \prod_{1 \leq i < j \leq n} (a_j - a_i).$$

(Hint: Use that adding a multiple of rows/columns to other rows/columns does not change the determinant (Proposition 17.6 + 17.10). Try to prove the statement then by induction on n , i.e. try to use row/column operation to find a $n - 1 \times n - 1$ -version of such a matrix.)

Hints HW3:

Ex 4: Induction on n (Base case $n=1$ $A = (1)$)

$$A = \begin{pmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{pmatrix}$$

$\prod_{1 \leq i < j \leq n} (a_j - a_i) := 1$
↑ convention empty prod.

$$A^T \xrightarrow{\text{row } 1 \rightarrow \text{row } 1 - (-a_1)} \begin{pmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & a_2 - a_1 & a_3 - a_1 \\ 0 & a_2(a_2 - a_1) & a_3(a_3 - a_1) \end{pmatrix} = B$$

$$B^T \xrightarrow{\text{col } 2 \rightarrow \text{col } 2 - (-a_1)} \begin{pmatrix} 1 & 0 & 0 \\ 1 & a_2 - a_1 & a_2(a_2 - a_1) \\ 1 & a_3 - a_1 & a_3(a_3 - a_1) \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} = C$$

$$\det(C) = (a_2 - a_1) \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & a_2 \\ 0 & a_3 - a_1 & a_3(a_3 - a_1) \end{vmatrix} = (a_2 - a_1)(a_3 - a_1)$$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a_2 \\ 0 & 1 & a_3 \end{pmatrix}$ - $n=2$ case

Total: 36 Points

- 1) (2+2+2+2=8 Points) Decide if the following statements are true or false. Justify your answers.
- i) The set $U = \{f \in C^\infty(\mathbb{R}, \mathbb{R}) \mid f(1)^2 \geq f(1)\}$ is a subspace of $C^\infty(\mathbb{R}, \mathbb{R})$.
 - ii) If V, W are two vector spaces with $\dim(V) = 3$ and $\dim(W) = 2$ then there exists a surjective linear map $F : V \rightarrow W$.
 - iii) For any $A \in \mathbb{R}^{n \times n}$ we have $\det(A) = \det(\text{rref}(A))$.
 - iv) If (A, B, C, D) is a basis of $\mathbb{R}^{2 \times 2}$, then (AA, AB, AC, AD) is also a basis of $\mathbb{R}^{2 \times 2}$.

- 2) (4+4=8 Points) Consider the bases $B = (2x + 1, x + 1)$ and $C = (x, x - 1)$ of \mathcal{P}_1 and the linear map $F : \mathcal{P}_1 \rightarrow \mathcal{P}_1$ with

$$[F]_C^B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and $g \in \mathcal{P}_1$ with $g(x) = x$.

- i) Determine $F(g)$.
 - ii) Calculate the determinant of F .
- (You do not need to show that B and C are bases)

- 3) (6 Points) For $n \geq 1$ consider the matrix $A_n = (a_{ij}) \in \mathbb{R}^{n \times n}$ with $a_{ij} = i + j$, i.e.

$$A_n = \begin{pmatrix} 2 & 3 & \dots & n+1 \\ 3 & 4 & \dots & n+2 \\ \vdots & \dots & \ddots & \vdots \\ n+1 & n+2 & \dots & n+n \end{pmatrix}.$$

Determine $\det(A_n)$ for all $n \geq 1$.

- 4) (3+3+3=9 Points) We define the following elements in $\mathbb{R}^{2 \times 2}$

$$m_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, m_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, m_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, m_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

and define the following linear map

$$G : \mathbb{R}^{2 \times 2} \longrightarrow \mathbb{R}^{2 \times 2}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} b & a \\ d & c \end{pmatrix}.$$

(You do not need to show that G is linear)

- i) Show that $M = (m_1, m_2, m_3, m_4)$ is a basis of $\mathbb{R}^{2 \times 2}$.
- ii) Show that $N = (G(m_1), G(m_2), G(m_3), G(m_4))$ is a basis of $\mathbb{R}^{2 \times 2}$.
- iii) Determine $[G]_M$ and $[G]_M^N$.

- 5) (5 Points) The Fibonacci numbers F_n are defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Show by induction that for all $n \geq 0$ we have

$$\sum_{j=0}^n F_j^2 = F_n F_{n+1}.$$

Total: 36 Points

1) (2+2+2+2=8 Points) Decide if the following statements are true or false. Justify your answers.

- (i) The matrix $A_x = \begin{pmatrix} x & -1 \\ 2 & x-3 \end{pmatrix}$ is invertible for any $x \in \mathbb{R}$ with $x < 2$.
- (ii) For an orthogonal matrix $A \in \mathbb{R}^{n \times n}$ we have $\det(A) \in \{1, -1\}$.
(Reminder: A is orthogonal iff $A^T = A^{-1}$)
- (iii) Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear map which is not injective and $\text{im}(F) \neq \{0\}$. Then $\ker(F) \cong \text{im}(F)$.
- (iv) There exists a basis (A, B, C, D) of $\mathbb{R}^{2 \times 2}$ with $A \sim B$, $A \sim C$ and $A \sim D$.

2) (4+3=7 Points) Consider the bases $B = (x+1, x-1)$ and $C = (3x+1, x-1)$ of \mathcal{P}_1 , the linear map $F : \mathcal{P}_1 \rightarrow \mathcal{P}_1$ with

$$[F]_B^C = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

and $g \in \mathcal{P}_1$ defined by $g(x) = 3x + 1$.

(You do not need to show that B and C are bases.)

- (i) Determine $F(g)$.
- (ii) Determine $[F]_C^B$.

3) (4+3=7 Points) For $a \in \mathbb{R}$ define the following matrix

$$M_a = \begin{pmatrix} a & 2 & 4 \\ 0 & -1 & a \\ a & 0 & 2 \end{pmatrix}.$$

- (i) Determine all $a \in \mathbb{R}$ for which M_a is invertible.
- (ii) Calculate $\ker(M_1)$ and $\ker(M_{-1})$.

4) (3+3+2=8 Points) We define the following elements in $\mathbb{R}^{2 \times 2}$

$$m_1 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, m_2 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, m_3 = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}, m_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and define the following linear map

$$T : \mathbb{R}^{2 \times 2} \longrightarrow \mathbb{R}^{2 \times 2} \\ A \longmapsto (m_1 - m_3)A.$$

(You do not need to show that T is linear.)

- (i) Show that $M = (m_1, m_2, m_3, m_4)$ is a basis of $\mathbb{R}^{2 \times 2}$.
- (ii) Determine $[T]_M$ and $\det(T)$.
- (iii) Calculate $\dim(\ker(T))$ and $\dim(\text{im}(T))$.

5) (3+3=6 Points) Define the matrix $M = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$.

- (i) Show by induction that for all $n \geq 1$ we have $M^n = \begin{pmatrix} 2^n & 0 \\ 1 - 2^n & 1 \end{pmatrix}$.
- (ii) Let $F : \mathcal{P}_1 \rightarrow \mathcal{P}_1$ be the linear map with $[F]_B = M$, where $B = (x, 1)$. Determine $F^n(g)$ for all $n \geq 1$, where $g(x) = x - 1$.
(Here F^n means that we apply n times the map F .)

Total: 36 Points

1) (2+2+2+2=8 Points) Decide if the following statements are true or false. Justify your answers.

- (i) The matrix $A_x = \begin{pmatrix} x & -2x \\ 3 & 3x \end{pmatrix}$ is invertible for any $x \in \mathbb{R}$ with $x \neq 0$.
- (ii) If V, W are two vector spaces with $\dim(V) = 3$ and $\dim(W) = 2$ then there exists an injective linear map $F : V \rightarrow W$.
- (iii) If $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a surjective linear map then $\det(F) \neq 0$.
- (iv) If (f_1, f_2, f_3) is a basis of \mathcal{P}_2 then $(f_1 + f_2', f_2 + f_3', f_3 + f_1')$ is a basis of \mathcal{P}_2 .

2) (5+3=8 Points) Consider the bases $B = (2x + 1, x + 1)$ of \mathcal{P}_1 and $D = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right)$ of \mathbb{R}^2 , the linear map $F : \mathcal{P}_1 \rightarrow \mathbb{R}^2$ with

$$[F]_B^D = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and $g \in \mathcal{P}_1$ defined by $g(x) = x - 1$.

(You do not need to show that B and D are bases)

- (i) Determine $F(g)$.
- (ii) Let $c_B : \mathbb{R}^2 \rightarrow \mathcal{P}_1$ be the coordinate map with respect to the basis B . Consider the linear map $G = F \circ c_B : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and determine its matrix $[G]$.

3) (3+4=7 Points) For $\mu \in \mathbb{R}$ let $U_\mu = \{f \in \mathcal{P}_3 \mid f(\mu x) = \mu f(x) \text{ for all } x \in \mathbb{R}\}$.

- (i) Show that U_μ is a subspace of \mathcal{P}_3 for any $\mu \in \mathbb{R}$.
- (ii) Determine a basis of U_μ for any $\mu \in \mathbb{R}$.

4) (3+3+2=8 Points) We define the following elements in $\mathbb{R}^{2 \times 2}$

$$m_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, m_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, m_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, m_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and define the following linear map

$$T : \mathbb{R}^{2 \times 2} \longrightarrow \mathbb{R}^{2 \times 2} \\ A \longmapsto A + A^T.$$

(You do not need to show that T is linear)

- (i) Show that $M = (m_1, m_2, m_3, m_4)$ is a basis of $\mathbb{R}^{2 \times 2}$.
- (ii) Determine $[T]_M$ and $\det(T)$.
- (iii) Calculate $\dim(\ker(T))$ and $\dim(\text{im}(T))$.

5) (2+3=5 Points) We define for $n \geq 1$ the following number

$$C(n) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} = \sum_{i=1}^n \frac{1}{i(i+1)}.$$

- (i) Evaluate $C(1), C(2)$ and $C(3)$. Make a guess for an explicit formula for $C(n)$.
- (ii) Prove your guess by using induction.

Total: 36 Points

1) (2+2+2+2=8 Points) Decide if the following statements are true or false. Justify your answers.

- (i) The set $U = \{f \in C^\infty(\mathbb{R}, \mathbb{R}) \mid f'(x) = xf(x) \text{ for all } x \in \mathbb{R}\}$ is a subspace of $C^\infty(\mathbb{R}, \mathbb{R})$.
- (ii) There exists an invertible linear map $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ that satisfies $F(F(x)) = 0$ for all $x \in \mathbb{R}^3$.
- (iii) The map $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $F(x) = \det(xx^T)$ for $x \in \mathbb{R}^2$ is a linear map.
- (iv) Let (b_1, b_2) be a basis of a vector space V then $(b_1 + 2b_2, b_1 - b_2)$ is also a basis of V .

2) (5+3=8 Points) Consider the bases $B = (2x - 1, x + 1)$, $C = (-3, x - 2)$ of \mathcal{P}_1 and the linear map $F : \mathcal{P}_1 \rightarrow \mathcal{P}_1$ with

$$[F]_B^C = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}.$$

(You do not need to show that B and C are bases.)

- (i) Determine $F(x)$ and $F(1)$.
- (ii) Find a basis for $\ker(F)$ and calculate $\det(F)$.

3) (3+3=6 Points) For $n \geq 1$ consider the matrix $A_n = (a_{ij}) \in \mathbb{R}^{n \times n}$ with $a_{ij} = i - j$, i.e.

$$A_n = \begin{pmatrix} 0 & -1 & -2 & \dots & 1 - n \\ 1 & 0 & -1 & \dots & 2 - n \\ 2 & 1 & 0 & \dots & 3 - n \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ n - 1 & n - 2 & n - 3 & \dots & 0 \end{pmatrix}.$$

- (i) Decide for which n the matrix A_n is invertible.
- (ii) Calculate $\dim(\ker(A_n))$ for all $n \geq 1$.

4) (3+3+2=8 Points) We define the following elements in $\mathbb{R}^{2 \times 2}$

$$m_1 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, m_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, m_3 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, m_4 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

and define the following linear map

$$H : \mathbb{R}^{2 \times 2} \longrightarrow \mathbb{R}^{2 \times 2} \\ A \longmapsto 2A - A^T.$$

(You do not need to show that H is linear.)

- (i) Show that $M = (m_1, m_2, m_3, m_4)$ is a basis of $\mathbb{R}^{2 \times 2}$.
- (ii) Determine $[H]_M$ and $\det(H)$.
- (iii) Show that $U = \{A \in \mathbb{R}^{2 \times 2} \mid H(A) = 3A\}$ is a subspace of $\mathbb{R}^{2 \times 2}$ and determine $\dim(U)$.

5) (2+4=6 Points) We define for $n \geq 1$ the following number

$$C(n) = \sum_{m=1}^n (m \cdot m!),$$

where $m! = 1 \cdot 2 \cdot \dots \cdot m$ is the factorial (e.g. $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, \dots).

- (i) Evaluate $C(1)$, $C(2)$ and $C(3)$. Make a guess for a simple explicit formula for $C(n)$.
- (ii) Prove your guess by using induction on n .