Linear Algebra II

Tutorial 5, 14th May 2026

Last lecture:

Definition 17.1 *A* **pattern** *in an $n \times n$-matrix is a choice of entries, such that precisely one entry from each row and each column is chosen.*

Definition 17.2 *i) A bijective map $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is called a* **permutation** *of $\{1, \ldots, n\}$.*

ii) S_n denotes the set of all permutations of $\{1, \ldots, n\}$.

Patterns in an $n \times n$-matrix corresponds exactly to the permutations of $\{1, \ldots, n\}$. For each $\sigma \in S_n$ we have the pattern

$$P = \big\{(1, \sigma(1)), (2, \sigma(2)), \ldots, (n, \sigma(n))\big\},$$

where (i, j) denotes the choice of the i-th row and the j-th column.

Definition 17.3 *i) The* **number of inversion** *of a permutation $\sigma \in S_n$, denoted by $\mathrm{inv}(\sigma)$, is the number of pairs $(i, \sigma(i)), (j, \sigma(j))$ with $i < j$ and $\sigma(i) > \sigma(j)$.*

ii) The sign of a permutation $\sigma \in S_n$ is defined by

$$\mathrm{sign}(\sigma) = (-1)^{\mathrm{inv}(\sigma)}.$$

Definition 17.4 *The* **determinant** *of a $n \times n$-matrix $A = (a_{i,j}) \in \mathbb{R}^{n \times n}$ is defined by*

$$\det(A) = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

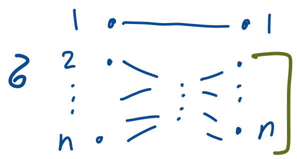
Question: If $A = \begin{pmatrix} 1 & 0 \cdots 0 \\ * & \\ \vdots & \boxed{B} \\ * & \end{pmatrix} \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n-1 \times n-1}$. What is the relation between $\det(A)$ and $\det(B)$?

Answer: The top left entry is the only non-zero entry in this row, therefore in $\det(A)$

just those σ give a contribution which have $\sigma(1)=1$.

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

$$= \underbrace{a_{1,1}}_{1} \cdot \sum_{\substack{\sigma \in S_n \\ \sigma(1)=1}} \operatorname{sign}(\sigma) \prod_{i=2}^{n} a_{i,\sigma(i)}$$



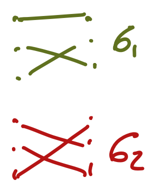
$$= \sum_{\sigma' \in S_{n-1}} \operatorname{sign}(\sigma') \prod_{i=1}^{n-1} b_{i,\sigma'(i)} = \det(B).$$

If $\sigma(1)=1$ is fixed fixed, we can think of σ being an element σ' in S_{n-1} with $\sigma'(i)=\sigma(i-1)-1$.

Note: $\operatorname{sign}(\sigma') = \operatorname{sign}(\sigma)$ and $b_{i,j} = a_{i+1,j+1}$.

(Useful for HW3, Ex. 3).

Exercise: Calculate $\det\begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 3 \\ 4 & 1 & 1 \end{pmatrix}$



Sol: $\underbrace{\operatorname{sign}(\sigma_1)}_{-1} \cdot 2 \cdot 3 \cdot 1 + \underbrace{\operatorname{sign}(\sigma_2)}_{+1}\, 1 \cdot 3 \cdot 4 = -6 + 12 = 6$

(a lot of parts vanish. Just patterns where we do not select "0" count)

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Homework 2: Linear maps and their matrices

Deadline: 15th May (23:55 JST), 2026

Exercise 1. (2+2+2+2+2 = 10 Points) For $n \geq 0$ we define the map $E_n : \mathcal{P}_n \to \mathbb{R}^3$ for $p \in \mathcal{P}_n$ by

$$E_n(p) = \begin{pmatrix} p(-1) \\ p(0) \\ p(1) \end{pmatrix}.$$

(i) Show that E_n is a linear map for any $n \geq 0$.

(ii) Show that E_2 is an isomorphism and compute its inverse.

(iii) Determine $[E_4]_B^C$, where $B = (1, x+1, x^2-1, x^3+1, x^4-1)$ is a basis of $\mathcal{P}_4$ and C is the standard basis of $\mathbb{R}^3$.

(iv) Determine whether E_1 and E_3 are injective and/or surjective.

(v) Determine a basis of $\mathrm{im}(E_1)$ and a basis of $\ker(E_3)$.

Exercise 2. (1+1+2+2 = 6 Points) The Fibonacci numbers F_n are defined by $F_0 = 0$, $F_1 = 1$ and

$$F_n = F_{n-1} + F_{n-2}. \qquad (n \geq 2)$$

In this exercise, we want to prove the following explicit formula:

$$F_n = \frac{1}{2^n \sqrt{5}} \left((1+\sqrt{5})^n - (1-\sqrt{5})^n \right). \tag{😺}$$

To do so, follow these steps:

(i) Find a linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}$ for all $n \geq 1$, where $F^n = \underbrace{F \circ \cdots \circ F}_{n}$.

(ii) Define the following two bases of $\mathbb{R}^2$ (you do not need to check that these are bases):

$$B_1 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right), \quad B_2 = \left(\begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix}, \begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix} \right).$$

Determine the change-of-basis matrices $S_{B_1}^{B_2}$ and $\left(S_{B_1}^{B_2}\right)^{-1}$.

(iii) Compute $[F]_{B_1}$ and $[F]_{B_2}$.

(iv) Compute $[F]_{B_1}^n$ by using

$$[F]_{B_1} = \left(S_{B_1}^{B_2}\right)^{-1} [F]_{B_2} S_{B_1}^{B_2}$$

and prove (😺) using (i).

Exercise 3. (2+2+2 = 6 Points) The Tribonacci numbers T_n are defined by $T_0 = 0$, $T_1 = 0$, $T_2 = 1$ and

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}. \qquad (n \geq 3)$$

We define the space of Tribonacci-like sequences by

$$\mathcal{T} = \{(a_n)_{n \geq 0} \in \mathcal{J} \mid a_n = a_{n-1} + a_{n-2} + a_{n-3} \text{ for all } n \geq 3\},$$

where $\mathcal{J}$ denotes the vector space of all infinite sequences (see Lecture 1).

(i) Show that $\mathcal{T}$ is a subspace of $\mathcal{J}$.

(ii) Show that $\mathcal{T}$ is finitely generated and find a basis B of $\mathcal{T}$.

(iii) Define the map $G : \mathcal{T} \to \mathcal{T}$ on a sequence $a = (a_n)_{n \geq 0}$ by $G(a) = b$, where $b = (b_n)_{n \geq 0}$ is given by $b_n = a_{n+3}$. Show that G is an isomorphism and determine $[G]_B$, where B is the basis from (ii).

For:

Ex 2 Let $F: \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 2x_1 \\ -x_1+3x_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$F\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$F^2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = F\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2-3 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}^2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$F^3\begin{pmatrix} 1 \\ 0 \end{pmatrix} = F\begin{pmatrix} 4 \\ -5 \end{pmatrix} = \begin{pmatrix} 8 \\ -4-3\cdot 5 \end{pmatrix} = \begin{pmatrix} 8 \\ -19 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}^3\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Goal: Calculate $F^n\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $n \geq 1$

$$\parallel$$

$$\begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}^n\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Notice: $\begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix} = [F] = [F]_B \quad B = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$

Idea: Find basis C of $\mathbb{R}^2$ such that $[F]_C^n$ is easy to calculate.

$[F]_C = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \rightsquigarrow [F]^n_C = \begin{pmatrix} d_1^n & 0 \\ 0 & d_2^n \end{pmatrix}$

Motivation: $[F]_B = S_C^B [F]_C S_B^C$

Then $[F]_B^2 = S_C^B [F]_C \underbrace{S_B^C S_C^B}_{I_2} [F]_C S_B^C$

$$= S_C^B [F]_C^2 S_B^C$$

In general: $\underset{\text{want}}{[F]_B^n} = \underbrace{S_C^B [F]_C^n S_B^C}_{\text{easy}}$.

In example: $C = \left(\underset{c_1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}, \underset{c_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \right)$

(Later we learn how to find such C)

Then $[F]_C = \left([F(c_1)]_C \quad [F(c_2)]_C \right)$

$F\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$F\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$$S_C^B = \begin{pmatrix} | & | \\ [c_1]_B & [c_2]_B \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

$$S_B^C = \left(S_C^B\right)^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\begin{matrix} -1 \\ \downarrow \end{matrix}\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$$\Rightarrow \quad [F]_B^n = S_C^B \, [F]_C^n \, S_B^C.$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ -3^n & 3^n \end{pmatrix} = \begin{pmatrix} 2^n & 0 \\ 2^n - 3^n & 3^n \end{pmatrix}$$

$$\Rightarrow \quad [F]_B^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2^n \\ 2^n - 3^n \end{pmatrix}}}.$$

Homework 3: Induction & Determinants

Deadline: 28th May (23:55 JST), 2026

Exercise 1. (2+2+2 = 6 Points) Use mathematical induction to prove the following statements.

(i) Let $A, B \in \mathbb{R}^{m\times m}$ be matrices such that $AB = BA$. Show that for all $n \geq 1$, we have

$$(AB)^n = A^n B^n.$$

(ii) For all integers $n \geq 1$, prove that

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}\,.$$

(iii) Let $A \in \mathbb{R}^{m\times m}$ be a square matrix and I the $m \times m$ identity matrix. Show that for all $n \geq 1$,

$$(I - A)\sum_{k=0}^{n-1} A^k = I - A^n\,.$$

(Here we use the convention $A^0 = I$.)

Exercise 2. (2+4 = 6 Points) (Geometric interpretation of the determinant)
We define the vectors $v = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, u = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \in \mathbb{R}^2$.

(i) Draw a sketch in $\mathbb{R}^2$ by connecting the points 0, v, u, and $v + u$ to form a parallelogram.

(ii) Show that the area of this parallelogram is given by $\det \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}$, i.e. the determinant of the matrix which has v and u as columns.

(Remark: This works in general, i.e. if you write two vectors in $\mathbb{R}^2$ into the columns of a matrix $A \in \mathbb{R}^{2\times 2}$ then $|\det(A)|$ gives the area of the parallelogram spanned by them.)

Exercise 3. (3+2 = 5 Points)

(i) Show (without using Proposition 17.7) that the determinant is linear in each row, i.e. for any $A = (a_{i,j}) \in \mathbb{R}^{n\times n}$ and $1 \leq l \leq n$ show that the map

$$\begin{aligned} F_{A,l} : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ x &\longmapsto \det(A(l;x)) \end{aligned}$$

is linear. Here, $A(l;x)$ denotes the matrix A, where the l-th row is replaced by the vector x^T. (See page 126 in the lecture notes)

(ii) Assume that A is invertible. What is the kernel of $F_{A,1}$?

Exercise 4. (4 Points) For $a_1, a_2, \ldots, a_n \in \mathbb{R}$ we define the matrix

$$A = \begin{pmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \ldots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \ldots & a_n^{n-1} \end{pmatrix} \in \mathbb{R}^{n\times n}\,.$$

Show that the determinant of A is given by

$$\det(A) = \prod_{1\leq i<j\leq n} (a_j - a_i)\,.$$

(Hint: Use that adding a multiple of rows/columns to other rows/columns does not change the determinant (Proposition 17.6 + 17.10). Try to prove the statement then by induction on n, i.e. try to use row/column operation to find a $n-1 \times n-1$-version of such a matrix.)