Tutorial 4

HW1 ex1

(ii) $\boxed{\text{prove that } \{\varphi(0)\},\ V \text{ are subspaces of } V \text{ but does not prove they are the only subspace.}}$

To prove:

$V := \mathrm{im}(\varphi)$. $\forall u \in V$ s.t $u \neq \varphi(0)$

$\lambda \odot u = \varphi(\underbrace{\lambda \varphi^{-1}(u)}_{\text{can be anything } \in \mathbb{R}})$ can be anything $\in \mathbb{R}$

Thus $\lambda \odot u \in V$ can be anything $\in V$.

OR if $\forall u \in W$, W is subspace of V and $u \neq \varphi(0)$,

$\varphi^{-1}(u) \neq 0$ (injectivity).

$\forall v \in V$, let $\lambda = \dfrac{\varphi^{-1}(v)}{\varphi^{-1}(u)}$, thus

$$\lambda \odot u = \varphi(\lambda \varphi^{-1}(u)) = \varphi\left(\frac{\varphi^{-1}(v)}{\varphi^{-1}(u)} \cdot \varphi^{-1}(u)\right) = v$$

Thus W and V are the same subspaces.

(iii) $\boxed{\text{Speifying } \varphi(u) = u}$

We need to find a isomorphism that suits all φ, not a speific φ

HW1 ex2

(iii) $\boxed{\text{Forget to talk about the case when } M \text{ and } I \text{ are linearly dependent}}$

To consider: $M = \lambda I \to C(M) = \mathbb{R}^{2\times 2}$

which has $\dim = 4$

Hints for HW2:

HW2 ex1.

For understanding the context:

if $n=0$ $E_0 : P_0 \to \mathbb{R}^3$

what is inside E_0 ?

if $f \in P_0$ then $f(x) = a$ for some $a \in \mathbb{R}$

$$E_0(f) = \begin{pmatrix} a \\ a \\ a \end{pmatrix}$$

$$\operatorname{im}(E_0) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\},$$

$$\ker(E_0) = \{ 0 \}$$

(iii) Recall: $F: V \to M$

B_V is basis of V, B_W is basis of W

$\{v_1, \dots, v_n\}$ $\qquad\qquad$ $\{w_1, \dots, w_m\}$

$$\begin{tikzcd}
V \arrow[r, "F"] & W \arrow[d, "C_{B_W}^{-1}"] \\
\mathbb{R}^n \arrow[u, "C_{B_V}"] \arrow[r, "{[F]_{B_V}^{B_W}}"] & \mathbb{R}^m
\end{tikzcd}$$

"The matrix of F with respect to B_V and B_W"

$$[F]_{B_V}^{B_W} = \begin{pmatrix} | & & | \\ [F(v_1)]_{B_W} & \cdots & [F(v_n)]_{B_W} \\ | & & | \end{pmatrix}$$

Interpretation: if $u \in V$ $(F(u) \in W)$.

we know that $[u]_{B_V} \in \mathbb{R}^n$, then

$[F]_{B_V}^{B_W}$ tells us how to get $[F(u)]_{B_W} \in \mathbb{R}^m$

$$[F]_{B_V}^{B_W} [u]_{B_V} = [F(u)]_{B_W}$$

Notation: if $V=W$ and $B_W = B_V$

$$[F]_{B_V}^{B_V} = [F]_{B_V}$$

if $V=W$, $F = id_V$

$$[F]_{B_V}^{B_W} = S_{B_V}^{B_W}$$

Change of basis matrix from B_V to B_W

An explicit example related to this exercise:

let Basis $D = (\overset{d_1}{X+2}, \overset{d_2}{2X-1})$ a basis of P_1

Basis $C = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$ a basis of $\mathbb{R}^3$.

Calculate $[E_1]_D^C$

what is $[E_1]_D^C$?

$$[E_1]_D^C = \begin{pmatrix} | & | \\ [E_1(d_1)]_C & [E_1(d_2)]_C \\ | & | \end{pmatrix}$$

$$E_1(d_1) = \begin{pmatrix} -1+2 \\ 0+2 \\ 1+2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2\cdot\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

thus $[E_1(d_1)]_C = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ → C is the standard basis

$$E_1(d_2) = \begin{pmatrix} 2(-1)-1 \\ 2(0)-1 \\ 2(1)-1 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$[E_1(d_2)]_C = \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$$

thus $[E_1]_D^C = \begin{bmatrix} 1 & -3 \\ 2 & -1 \\ 3 & 1 \end{bmatrix}$

Therefore if we have $f(x) = 3x+1 \in P_1$

$f(x) = 1\cdot(x+2) + 1\cdot(2x-1) \rightarrow [f(x)]_D = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$E_1(f) = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$; $[E_1]_D^C [f(x)]_D = \begin{bmatrix} 1 & -3 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$

same result!

(iv) To prove not injective/surjective, provide one counter example.

HW2 ex2.

(i). It's better to use f to replace F to representing the linear map, which avoids confusion.

HW2. ex3

Recall $\mathcal{T}$ — the vector space of all infinite series.

$$\mathcal{T} = \{ (a_n)_{n\geq 0} \mid a_n \in \mathbb{R} \}. \quad (a_n)_{n\geq 0} = (a_0, a_1, a_2, \dots)$$

$$(0)_{n\geq 0} = (0, 0, \dots)$$

$$a = (a_n)_{n\geq 0},\ b = (b_n)_{n\geq 0},\quad a+b = (a_n + b_n)_{n\geq 0}$$
$$\lambda a = (\lambda a_n)_{n\geq 0}$$