Solution

Linear Algebra II - Midterm Spring 2026
Nagoya University, G30 Program Instructor: Henrik Bachmann

Total: 36 Points

1) (2+2+2+2=8 Points) Decide if the following statements are true or false. Justify your answers.

(i) The set $U = \left\{ f \in C^\infty(\mathbb{R},\mathbb{R}) \,\middle|\, \int_{-1}^{1} \big(f(x)+x\big)\,dx = f(0) \right\}$ is a subspace of $C^\infty(\mathbb{R},\mathbb{R})$.

(ii) The matrix $A_x = \begin{pmatrix} x & 3 \\ 1 & x-2 \end{pmatrix}$ is invertible for any $x \in \mathbb{R}$ with $x > 0$.

(iii) Every linear map $F : \mathbb{R}^3 \to \mathbb{R}^2$ has $\ker(F) \neq \{0\}$.

(iv) Let (b_1, b_2, b_3) be a basis of V. Then $(b_1 - b_2,\, b_2 - b_3,\, b_3 - b_1)$ is also a basis of V.

2) (5+3=8 Points) Consider the bases $B = (3x+1, x+1)$, $C = (x+1, 2x-1)$ of $\mathcal{P}_1$ and the linear map $F : \mathcal{P}_1 \to \mathcal{P}_1$ with

$$[F]_B^C = \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}.$$

(You do <u>not</u> need to show that B and C are bases.)

(i) Determine $F(x)$ and $F(1)$.

(ii) Find a basis for $\ker(F)$ and calculate $\det(F)$.

3) (3+3=6 Points) For $a \in \mathbb{R}$ define the following matrix

$$M_a = \begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix}.$$

(i) Determine all $a \in \mathbb{R}$ for which M_a is invertible.

(ii) Calculate $\ker(M_1)$ and $\ker(M_{-2})$.

4) (3+3+2=8 Points) We define the following elements in $\mathbb{R}^{2\times 2}$

$$m_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},\ m_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ m_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ m_4 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

and define the following linear map

$$\begin{aligned} T : \mathbb{R}^{2\times 2} &\longrightarrow \mathbb{R}^{2\times 2} \\ A &\longmapsto 3A - 2A^T. \end{aligned}$$

(You do <u>not</u> need to show that T is linear.)

(i) Show that $M = (m_1, m_2, m_3, m_4)$ is a basis of $\mathbb{R}^{2\times 2}$.

(ii) Determine $[T]_M$ and $\det(T)$.

(iii) Show that $U = \{A \in \mathbb{R}^{2\times 2} \mid T(A) = A\}$ is a subspace of $\mathbb{R}^{2\times 2}$ and determine $\dim(U)$.

5) (3+3=6 Points) Define the matrix $M = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$.

(i) Show by induction that for all $n \geq 1$ we have $M^n = \begin{pmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{pmatrix}$.

(ii) Let $F : \mathcal{P}_1 \to \mathcal{P}_1$ be the linear map with $[F]_B = M$, where $B = (x, 1)$. Determine $F^n(g)$ for all $n \geq 1$, where $g(x) = 2x + 1$.
(Here F^n means that we apply n times the map F.)

1) (2+2+2+2=8 Points) Decide if the following statements are true or false. Justify your answers.

(i) The set $U = \left\{ f \in C^\infty(\mathbb{R},\mathbb{R}) \,\middle|\, \int_{-1}^{1} \big(f(x)+x\big)\,dx = f(0) \right\}$ is a subspace of $C^\infty(\mathbb{R},\mathbb{R})$.

(ii) The matrix $A_x = \begin{pmatrix} x & 3 \\ 1 & x-2 \end{pmatrix}$ is invertible for any $x \in \mathbb{R}$ with $x > 0$.

(iii) Every linear map $F : \mathbb{R}^3 \to \mathbb{R}^2$ has $\ker(F) \neq \{0\}$.

(iv) Let (b_1, b_2, b_3) be a basis of V. Then $(b_1 - b_2,\, b_2 - b_3,\, b_3 - b_1)$ is also a basis of V.

(i) True: Since $\int_{-1}^{1} f(x) + x\,dx = \int_{-1}^{1} f(x)dx + \underbrace{\int_{-1}^{1} x\,dx}_{=0} = \int_{-1}^{1} f(x)\,dx$

we have $U = \{ f \in C^\infty(\mathbb{R},\mathbb{R}) \mid \int_{-1}^{1} f(x)dx = f(0)\}$.

- The function $n(x) = 0 \ \forall x \in \mathbb{R}$ is in U, since $\int_{-1}^{1} n(x)dx = 0 = n(0)$.
- If $f, g \in U$ then

$$\int_{-1}^{1} (f+g)(x)dx = \int_{-1}^{1} f(x) + g(x)dx = \int_{-1}^{1} f(x)dx + \int_{-1}^{1} g(x)dx = f(0) + g(0) = (f+g)(0)$$

$\Rightarrow f + g \in U$

- If $f \in U$ and $\lambda \in \mathbb{R}$ $\int_{-1}^{1} \lambda f(x)dx = \lambda \int_{-1}^{1} f(x)\,dx = \lambda f(0)$

$\Rightarrow \lambda f \in U$

Therefore U is a subspace of $C^\infty(\mathbb{R},\mathbb{R})$.

(ii) False: We have $\det(A_x) = \det\begin{pmatrix} x & 3 \\ 1 & x-2 \end{pmatrix} = x(x-2) - 3$

$= x^2 - 2x - 3$

$= (x+1)(x-3)$

Therefore $\det(A_3) = 0 \Rightarrow A_3$ not invertible

(iii) True: Since $3 = \dim \mathbb{R}^3 = \dim \operatorname{Ker}(F) + \dim \operatorname{im}(F)$ and $\operatorname{im}(F) \subset \mathbb{R}^2 \Rightarrow \dim \operatorname{im}(F) \leq 2$, we get $\dim \operatorname{Ker}(F) \geq 1 \Rightarrow \operatorname{Ker}(F) \neq \{0\}$.

(iv) False: We have $(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_1) = 0$

$\Rightarrow ((b_1 - b_2), (b_2 - b_3), (b_3 - b_1))$ are not linearly independent.

2) (5+3=8 Points) Consider the bases $B = (\underset{b_1}{3x+1}, \underset{b_2}{x+1})$, $C = (\underset{c_1}{x+1}, \underset{c_2}{2x-1})$ of $\mathcal{P}_1$ and the linear map $F : \mathcal{P}_1 \to \mathcal{P}_1$ with

$$[F]^C_B = \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}.$$

(You do <u>not</u> need to show that B and C are bases.)

(i) Determine $F(x)$ and $F(1)$.

(ii) Find a basis for $\ker(F)$ and calculate $\det(F)$.

(i)

From the matrix $[F]^C_B$ we can read off

$$F(3x+1) = F(b_1) = 2c_1 - 1c_2 = 2(x+1) - (2x-1) = 3$$

$$F(x+1) = F(b_2) = F(b_1) = 3$$

Therefore $3F(x) + F(1) = F(3x+1) = 3$

$$F(x) + F(1) = F(x+1) = 3$$

Subtracting the second eq. from the first gives $2F(x) = 0$ and therefore

$$F(x) = 0, \quad F(1) = 3.$$

(ii) With $D = (x, 1)$ we get by (i), $[F]_D = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$

and therefore $\det(F) = 0$.

If $F(ax+b) = 0$, then by (i)

$$F(ax+b) = aF(x) + bF(1) = 3b = 0$$

$\Rightarrow$ $b = 0$. Therefore $\ker(F) = \{ax \mid a \in \mathbb{R}\}$.

$\Rightarrow$ $\ker(F) = \operatorname{span}\{x\}$ $\Rightarrow$ $\dim \ker(F) = 1$.

3) (3+3=6 Points) For $a \in \mathbb{R}$ define the following matrix

$$M_a = \begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix}.$$

(i) Determine all $a \in \mathbb{R}$ for which M_a is invertible.

(ii) Calculate $\ker(M_1)$ and $\ker(M_{-2})$.

(i) We have by Sarrus rule

$$\det(M_a) = a^3 + 1 + 1 - a - a - a = a^3 - 3a + 2$$

$$= (a-1)(a^2 + a - 2) = (a-1)^2 (a+2)$$

See that 1 is a root and then factor out a-1

$$\Rightarrow \quad \det(M_a) = 0 \quad \text{for } a \in \{1, -2\}$$

$$\Rightarrow \quad M_a \text{ is invertible for all } a \in \mathbb{R} \text{ with } a \neq 1 \text{ and } a \neq -2$$

(ii)

$$M_1 = \begin{array}{c} \scriptstyle (-1)(-1) \\ \\ \\ \end{array} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \operatorname{Ker} M_1 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

$$M_{-2} = \begin{array}{c} \\ \scriptstyle (1) \\ \scriptstyle (1) \end{array} \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \sim \begin{array}{c} \\ \scriptstyle (-1) \\ \\ \end{array} \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \sim \begin{array}{c} \\ \\ \scriptstyle (\frac{1}{3}) \end{array} \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & 3 & -3 \end{pmatrix}$$

$$\sim \begin{array}{c} \\ \scriptstyle (2) \\ \\ \end{array} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \operatorname{Ker} M_{-2} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

4) (3+3+2=8 Points) We define the following elements in $\mathbb{R}^{2\times 2}$

$$m_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad m_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad m_4 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

and define the following linear map

$$T : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{2\times 2}$$
$$A \longmapsto 3A - 2A^T .$$

(You do <u>not</u> need to show that T is linear.)

(i) Show that $M = (m_1, m_2, m_3, m_4)$ is a basis of $\mathbb{R}^{2\times 2}$.

(ii) Determine $[T]_M$ and $\det(T)$.

(iii) Show that $U = \{A \in \mathbb{R}^{2\times 2} \mid T(A) = A\}$ is a subspace of $\mathbb{R}^{2\times 2}$ and determine $\dim(U)$.

(i) Since $\dim \mathbb{R}^{2\times 2} = 4$ it suffices to show $\mathbb{R}^{2\times 2} = \operatorname{span}\{m_1, m_2, m_3, m_4\}$

For $a, b, c, d \in \mathbb{R}$ we want to solve

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = a \\ \lambda_1 + \lambda_2 + \lambda_4 = b \\ \lambda_1 + \lambda_3 + \lambda_4 = c \\ \lambda_2 + \lambda_3 + \lambda_4 = d \end{cases}$$

(Sol. 2: Do this for $a = \ldots = d = 0$ to show lin. indep.)

$$\Leftrightarrow \begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = a \\ -\lambda_3 + \lambda_4 = b - a \\ -\lambda_2 + \lambda_4 = c - a \\ \lambda_2 + \lambda_3 + \lambda_4 = d \end{cases}$$

$$\Leftrightarrow \begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = a \\ -\lambda_2 + \lambda_4 = c - a \\ -\lambda_3 + \lambda_4 = b - a \\ \lambda_3 + 2\lambda_4 = -a + c + d \end{cases} \Leftrightarrow \begin{cases} \lambda_1 + 2\lambda_4 = -a + b - c \\ -\lambda_2 + \lambda_4 = c - a \\ -\lambda_3 + \lambda_4 = b - a \\ 3\lambda_4 = -2a + b + c + d \end{cases}$$

$$\Rightarrow \begin{aligned} \lambda_1 &= \tfrac{1}{3}(a+b+c-2d) \\ \lambda_2 &= \tfrac{1}{3}(a+b-2c+d) \\ \lambda_3 &= \tfrac{1}{3}(a-2b+c+d) \\ \lambda_4 &= \tfrac{1}{3}(-2a+b+c+d) \end{aligned}$$

Therefore $\mathbb{R}^{2\times 2} = \operatorname{span}\{m_1, m_2, m_3, m_4\}$

(ii) We calculate first $T(m_j)$ for $j=1,\dots,4$

$$T(m_1) = 3m_1 - 2\overbrace{m_1^T}^{m_1} = 3m_1 - 2m_1 = m_1$$

$$T(m_2) = 3m_2 - 2\overbrace{m_2^T}^{m_3} = 3m_2 - 2m_3$$

$$T(m_3) = 3m_3 - 2\overbrace{m_3^T}^{m_2} = 3m_3 - 2m_2$$

$$T(m_4) = 3m_4 - 2\overbrace{m_4^T}^{m_4} = 3m_4 - 2m_4 = m_4$$

We get

$$[T]_M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\det(T) = \det([T]_M) = \det\begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} = \underline{\underline{5}}$

(iii) $T(A) = A$ means $3A - 2A^T = A$

$$\iff 2(A - A^T) = 0$$

$$\iff A = A^T$$

Therefore

$$U = \{ A \in \mathbb{R}^{2\times 2} \mid A = A^T \}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2} \mid c = b \right\}$$

$$= \operatorname{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a subspace.

$\dim U = 3.$

5) (3+3=6 Points) Define the matrix $M = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$.

(i) Show by induction that for all $n \geq 1$ we have $M^n = \begin{pmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{pmatrix}$.

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(Here F^n means that we apply n times the map F.)

(i) Base step $n=1$: $M^1 = M = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2^1 - 1 \\ 0 & 2^1 \end{pmatrix}$ ✓

Assume $M^m = \begin{pmatrix} 1 & 2^m - 1 \\ 0 & 2^m \end{pmatrix}$ for $m < n$

then $M^n = M \cdot M^{n-1} \overset{m=n-1}{=} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2^{n-1} - 1 \\ 0 & 2^{n-1} \end{pmatrix}$

$= \begin{pmatrix} 1 & 2^{n-1} - 1 + 2^{n-1} \\ 0 & 2^n \end{pmatrix} = \begin{pmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{pmatrix}$

which proves the statement.

(ii) We have

$[F^n(g)]_B = [F]_B^n [g]_B$

$= M^n \begin{pmatrix} 2 \\ 1 \end{pmatrix} \overset{(i)}{=} \begin{pmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2^n + 1 \\ 2^n \end{pmatrix}$

Therefore $F^n(g) = (2^n + 1)x + 2^n$.