# Linear Algebra II

## Lecture 6

**Last time:** **Row operations:** There are 3 row operations for a matrix $A \in \mathbb{R}^{n\times n}$ $(1 \le i, j \le n,\ i \neq j,\ \lambda \in \mathbb{R})$.

(R1) Add $\lambda$ times row j-th to the i-th row

(R2) For $\lambda \neq 0$ multiply the ith row with $\lambda$.

(R3) Swap row j and i.

**Proposition 17.10** Let $A, B \in \mathbb{R}^{n\times n}$

i) $A \overset{R1}{\sim} B \implies \det(B) = \det(A)$

ii) $A \overset{R2}{\sim} B \implies \det(B) = \lambda \det(A) \quad (\lambda \neq 0)$

iii) $A \overset{R3}{\sim} B \implies \det(B) = -\det(A)$

i.e. $A \sim B \implies \det(B) = c \cdot \det(A)$ for $c \neq 0$

**Theorem 17.11** For any matrix $A \in \mathbb{R}^{n\times n}$ we have

$$A \text{ is invertible} \iff \det(A) \neq 0.$$

Idea: $A$ invertible $\iff A \sim \dots \sim I_n$

**Ex 62** $A = \overset{\textcircled{-1}}{\curvearrowleft}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 4 & 6 \end{pmatrix}$  Q: Find C with CA=B $\implies \det(I_n) = c \cdot \det(A)$

$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 1 & 4 & 6 \end{pmatrix} = B \sim \dots \sim I_3$  $\qquad \det(I_n) = 1 \qquad c \neq 0.$

$C = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

**Today:** Describe row operations by matrix multiplication to obtain the following:

**Theorem 17.12** i) For all $A, B \in \mathbb{R}^{n \times n}$ we have

$$\det(A \cdot B) = \det(A) \cdot \det(B)$$

ii) If $A$ is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

**Corollary 17.13** Let $V$ be fin. gen., $F : V \to V$ a linear map and $B_1, B_2$ bases of $V$. Then

$$\det([F]_{B_1}) = \det([F]_{B_2}).$$

**Proof:** Since $[F]_{B_1} = \underbrace{(S^{B_2}_{B_1})^{-1}}_{\begin{matrix}\shortparallel \\ S^{B_1}_{B_2}\end{matrix}} [F]_{B_2} S^{B_2}_{B_1}$ we get by Thm 17.12

$$\det([F]_{B_1}) = \underbrace{\det((S^{B_2}_{B_1})^{-1})}_{= \frac{1}{\det(S^{B_2}_{B_1})}} \det([F]_{B_2}) \det(S^{B_2}_{B_1}) = \det([F]_{B_2}).$$

**Definition 17.14** $V$ fin. gen., $F : V \to V$ linear map. We define the determinant of $F$ by

$$\det(F) = \det([F]_B),$$

where $B$ is any basis of $V$.

**Definition 17.15** For $\lambda \in \mathbb{R}$ with $\lambda \neq 0$, $1 \leq i,j \leq n$ $(i \neq j)$ we define the **elementary matrices**

$$R_i^{\lambda,j} = \begin{array}{cc} i & \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & \lambda & & \ddots & \\ & & & & 1 \end{pmatrix} \\ & \qquad j \end{array} \in \mathbb{R}^{n\times n}$$

"add $\lambda$-times row $j$ to $i$"

$$R_i^{\lambda} = \begin{array}{cc} i & \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \lambda & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \\ & \qquad i \end{array} \in \mathbb{R}^{n\times n}$$

$$R_{i,j} = \begin{array}{cc} & \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & \cdots & 1 & \\ & & \vdots & \ddots & \vdots & \\ & & 1 & \cdots & 0 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \\ & \qquad i \qquad j \end{array} \in \mathbb{R}^{n\times n}$$

with the entry $0$ at row $i$, column $i$ and the entry $1$ at row $j$, column $i$.

**Example 63** $n=3$

$$R_3^{\lambda,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix}, \quad R_2^{\lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_{1,2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The elementary matrices satisfy the following properties

1) $\det(R_i^{\lambda,j}) = 1$, $\det(R_i^{\lambda}) = \lambda$, $\det(R_{i,j}) = -1$

2) $\left(R_i^{\lambda,j}\right)^{-1} = R_i^{-\lambda,j}$, $(R_i^{\lambda})^{-1} = R_i^{\frac{1}{\lambda}}$, $(R_{i,j})^{-1} = R_{i,j}$.

In particular: Inverses of elementary matrices are again elementary matrices.

Recall:

**Row operations:** There are 3 row operations for a matrix $A \in \mathbb{R}^{n\times n}$ $(1 \leq i,j \leq n,\ i \neq j,\ \lambda \in \mathbb{R})$.

(R1) Add $\lambda$ times row j-th to the i-th row

(R2) For $\lambda \neq 0$ multiply the ith row with $\lambda$.

(R3) Swap row j and i.

**Proposition 17.16** Let $A \in \mathbb{R}^{n\times n}$

i) $R_i^{\lambda,j}A$ is the matrix obtained from A by the row operation (R1). $\left(A \overset{(R1)}{\sim} R_i^{\lambda,j}A\right)$

ii) $R_j^{\lambda}A$ is the matrix obtained from A by (R2) $\left(A \overset{(R2)}{\sim} R_j^{\lambda}A\right)$

iii) $R_{i,j} A$ is the matrix obtained from A by (R3)

$(A \overset{(R3)}{\sim} R_{i,j} A)$

Proof:

i) We have $R_i^{\lambda,j} = \begin{pmatrix} 1 & & \\ & \ddots & \\ \lambda & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & & \\ & \ddots & \\ 1 & & 0 \end{pmatrix}$

$$= I_n + \lambda E_{i,j} \text{,}$$

where $E_{i,j} = \begin{array}{cc} i & \begin{pmatrix} 0 & & \\ 1 & \ddots & \\ & & 0 \end{pmatrix} \\ & j \end{array} = (e_{kl})_{k,l}$

$$e_{kl} = \begin{cases} 1 & k=i, l=j \\ 0 & \text{else.} \end{cases}$$

Therefore $R_i^{\lambda,j} A = (I_n + \lambda E_{i,j}) A$

$$= \underline{A + \lambda E_{i,j} A} \text{.}$$

We have with $A = (a_{ij})$

$$E_{i,j} A = \begin{array}{cc} i & \begin{pmatrix} 0 & & \\ 1 & \ddots & \\ & & 0 \end{pmatrix} \\ & j \end{array} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = \begin{array}{cc} \begin{pmatrix} 0 & \dots & & 0 \\ a_{j1} & a_{j2} & \dots & a_{jn} \\ 0 & \dots & & 0 \end{pmatrix} & \begin{array}{c} \\ - i \\ \\ \end{array} \end{array} \text{,}$$

i.e. this gives the matrix which just contains the j-th row of A in the i-th row.

Therefore $A + \lambda E_{ij} A$ is the matrix obtained by adding $\lambda$-times the j-th row to the ith row of A. (R1)

ii) Similar to i) we see that

$$R_i^{\lambda} A = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & \lambda & \\ 0 & & & \ddots 1 \end{pmatrix} A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \lambda a_{i1} & \cdots & \lambda a_{in} \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

iii) Try by yourself.

**Corollary 17.17** Let $A \in \mathbb{R}^{n \times n}$

i) A is invertible $\Longleftrightarrow$ A is a product of elementary matrices

ii) If C is an elementary matrix then $\det(CA) = \det(C)\det(A)$.

Proof: i) "$\Rightarrow$": If A is invertible we can use row operations R1, R2 and R3 to obtain $\mathrm{rref}(A) = I_n$.
By Proposition 17.16 these correspond to multiplication of elementary matrices.

$$A \sim \ldots \sim I_n = \mathrm{rref}(A) \Rightarrow A = C_1 \cdot \ldots \cdot C_m \cdot I_n$$

with elementary matrices $C_1, \ldots, C_m$.

"$\Leftarrow$": If $A = C_1 \cdot \ldots \cdot C_m$, then

$$A^{-1} = (C_1 \cdot \ldots \cdot C_m)^{-1}$$
$$= C_m^{-1} \cdot \ldots \cdot C_1^{-1},$$

since all elementary matrices are invertible.

ii) This is just a reformulation of Proposition 17.16 together with Proposition 17.10 and the properties

$$\det(R_i^{\lambda,j}) = 1, \quad \det(R_i^{\lambda}) = \lambda, \quad \det(R_{i,j}) = -1$$

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Now we can prove Theorem 17.12 i), i.e

$$\det(AB) = \det(A)\det(B).$$

for any $A, B \in \mathbb{R}^{n \times n}$.

**Case 1:** $A$ is invertible, i.e $A = C_1 \cdot \ldots \cdot C_m$ for elementary matricer $C_1, \ldots, C_m$.

We want to show $\det(A \cdot B) = \det(A)\det(B)$ by induction on $m$.

**Base step $m=1$:** In this case $A = C_1$ and by Corollary 17.17 ii) we have

$$\det(A \cdot B) = \det(C_1 \cdot B) = \underbrace{\det(C_1)}_{=\det(A)} \cdot \det(B).$$

**Induction step:** Assume that

$$\det(\underbrace{C_2 \cdot \ldots \cdot C_m}_{\text{Product of } m-1 \text{ elem. matricer}} \cdot B) = \det(C_2 \cdot \ldots \cdot C_m)\det(B).$$

Induction hypothesis

Corollary 17.17 ii)

Then we have for $A = C_1 \cdot \ldots \cdot C_m$

$$\det(AB) = \det(C_1 C_2 \cdots C_m B) = \det(C_1)\det(C_2 \cdot \ldots \cdot C_m B)$$

$$= \det(C_1) \cdot \det(C_2 \cdot \ldots \cdot C_m)\det(B) = \det(C_1 \cdot C_2 \cdot \ldots \cdot C_m)\det(B)$$

$$= \det(A)\det(B).$$

Case 2: If $A$ is not invertible $\Rightarrow \det(A)=0$

(Recall from LA1: a $n\times n$-matrix $A \in \mathbb{R}^{n\times n}$ is invertible iff $\operatorname{im}(A)=\mathbb{R}^n$.)

But since $\operatorname{im}(AB) \subset \operatorname{im}(A) \neq \mathbb{R}^n$

$\Rightarrow$ $AB$ is not invertible $\Rightarrow \det(AB)=0$.

$$\Rightarrow \quad \det(AB) = 0 = \underbrace{\det(A)}_{=0}\cdot\det(B).$$