Linear Algebra II

Lecture 4

Last time: $F: V \to V$ linear map, $B = (b_1, \dots, b_n)$ basis of V:

$$[F]_B := [F]_B^B = \begin{pmatrix} | & & | \\ [F(b_1)]_B & \cdots & [F(b_n)] \\ | & & | \end{pmatrix} \in \mathbb{R}^{n \times n}$$

Matrix of F with respect to B.

Next few lectures: Determinants

§ 17 Determinants

Recall: Ex 55

$D: P_2 \to P_2$, $f \mapsto f'$, $B = (1, x, x^2)$

$C = (2, 2x+4, x^2)$

$$[D]_B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[D]_C = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$D(x^2) = 2x = -2 \cdot 2 + 2x + 4$

Motivation: $A \in \mathbb{R}^{n \times n}$

$$\det: \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}$$
$$A \longmapsto \det(A)$$

Nice properties: $\det(A) \neq 0 \iff A$ is invertible

$A, B \in \mathbb{R}^{n \times n}$: $\det(AB) = \det(A)\det(B)$

$F: V \to V$, B_1, B_2 bases of V $(\dim(V) = n)$

$$\det([F]_{B_1}) = \det([F]_{B_2}) =: \det(F)$$

17.1 Mathematical induction (Will be needed for some proofs)

P_n : Statement depending on a natural number n $(n \geq 1)$

Goal: Prove P_n is true for all $n \geq K$

1) Base step: Show P_1 (P_K) is true

2) Induction step: Assume P_m is true for all $K \leq m < n$. Then show that P_n is true.

1) & 2) shows that P_n is true for all $n \geq 1$.

$$\underset{\substack{P_K \\ \text{true because of 1)}}}{P_1} \xRightarrow{2)} \underset{P_{K+1}}{P_2} \xRightarrow{2)} \underset{P_{K+2}}{P_3} \Rightarrow \dots$$

Remark: If you want to show that P_n is just true for all $n \geq K$, then you prove P_K in 1) and assume P_m is true for $K \leq m < n$ in 2).

(Another example: P_n: $1+2+\ldots+n = \frac{n\cdot(n+1)}{2}$)

Example 56 (Sum of first n odd numbers)

$$\left(\begin{array}{ll} n=1: & 1 = 1^2 \\ n=2: & 1+3 = 4 = 2^2 \\ n=3: & 1+3+5 = 9 = 3^2 \\ & \quad \vdots \end{array}\right)$$

P_n: "We have $\sum_{i=1}^{n}(2i-1) = n^2$"

Prove by induction:

1) Base step: $n=1$ $\quad \sum_{i=1}^{1}(2i-1) = 2-1 = 1 = 1^2$

$\Rightarrow P_1$ is true.

2) Induction step: Fix n and assume P_m is true for $m<n$.

In particular, for $m=n-1$ we assume that

$$\sum_{i=1}^{n-1}(2i-1) = (n-1)^2.$$

Then we get

$$\sum_{i=1}^{n}(2i-1) = \sum_{i=1}^{n-1}(2i-1) + (2n-1) = (n-1)^2 + 2n-1$$

$$= n^2-2n+1+2n-1 = n^2.$$

$\Rightarrow P_n$ is true.

17.2 Determinants

Definition 17.1 A pattern in an $n \times n$-matrix is a choice of entries, such that precisely one entry from each row and each column is chosen.

Notation: $P = \{ (\underset{\text{row}}{i_1}, \underset{\text{column}}{j_1}), \dots, (i_n, j_n) \}$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$P = \{ (1,1), (2,3), (3,2) \}$$

Example 57 i) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$

Then there are 2 patterns:

$$P_1 = \{ (1,1), (2,2) \}$$

$$P_2 = \{ (1,2), (2,1) \}$$

In general there are $n! = 1 \cdot 2 \cdot \ldots \cdot n$ patterns.

ii) 3x3 case: $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ $\quad \det(A) = a\cdot e \cdot i - afh - bdi + bfg + cdh - ceg.$

$P_1 = \{(1,1),(2,2),(3,3)\}$ $\quad P_4 = \{(1,2),(2,3),(3,1)\}$

$P_2 = \{(1,1),(2,3),(3,2)\}$ $\quad P_5 = \{(1,3),(2,1),(3,2)\}$

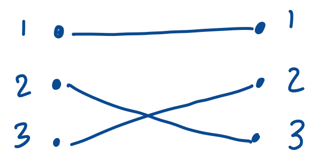
$P_3 = \{(1,2),(2,1),(3,3)\}$ $\quad P_6 = \{(1,3),(2,2),(3,1)\}$

Definition 17.2 i) A bijective map $\sigma: \{1,\dots,n\} \to \{1,\dots,n\}$ is called a <u>permutation</u> of $\{1,\dots,n\}$.

ii) S_n denotes the set of all permutations of $\{1,\dots,n\}$.

Example 58 $n=3$, $\quad \sigma: \{1,2,3\} \to \{1,2,3\}$

$\sigma \in S_3$ $\quad \sigma(1)=1$, $\sigma(2)=3$, $\sigma(3)=2$

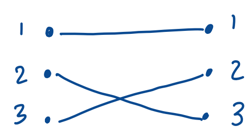


We have a 1:1 correspondence

Patterns of $n\times n$ matrices $\longleftrightarrow$ Permutations of $\{1,\dots,n\}$

$P = \{(1,\sigma(1)),(2,\sigma(2)),\dots,(n,\sigma(n))\} \longleftrightarrow \sigma$

$P = \{(1,1),(2,3),(3,2)\} \longleftrightarrow$

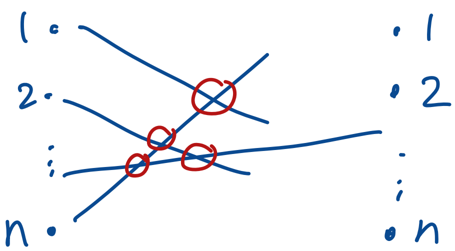


Definition 17.3 i) The number of inversions of a permutation $\sigma \in S_n$, denoted $\mathrm{inv}(\sigma)$, is the number of pairs $(i, \sigma(i))$, $(j, \sigma(j))$ with $i < j$ and $\sigma(i) > \sigma(j)$.

ii) The sign of a permutation $\sigma \in S_n$ is

$$\mathrm{sign}(\sigma) = (-1)^{\mathrm{inv}(\sigma)} = \begin{cases} 1 & \mathrm{inv}(\sigma) \text{ even} \\ -1 & \mathrm{inv}(\sigma) \text{ odd} \end{cases}$$

To calculate $\mathrm{inv}(\sigma)$ count the number of intersections in the picture



Definition 17.4 The determinant of $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is defined by

$$\det(A) = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} a_{i, \sigma(i)}.$$

Example 60:

i) In the case $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have

$S_2 = \{\sigma_1, \sigma_2\}$

σ_1: $1 \mapsto 1$, $2 \mapsto 2$ $\quad inv(\sigma_1) = 0$, $sign(\sigma_1) = 1$

σ_2: $1 \mapsto 2$, $2 \mapsto 1$ $\quad inv(\sigma_2) = 1$, $sign(\sigma_2) = -1$

$$\det(A) = sign(\sigma_1) \cdot a_{1,\sigma_1(1)} \cdot a_{2,\sigma_1(2)} + sign(\sigma_2) \cdot a_{1,\sigma_2(1)} \cdot a_{2,\sigma_2(2)}$$

$$= 1 \cdot a \cdot d + -1 \cdot b \cdot c = ad - bc$$

ii) Check that for $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ you get (by Example 57 ii)) "Sarrus rule"

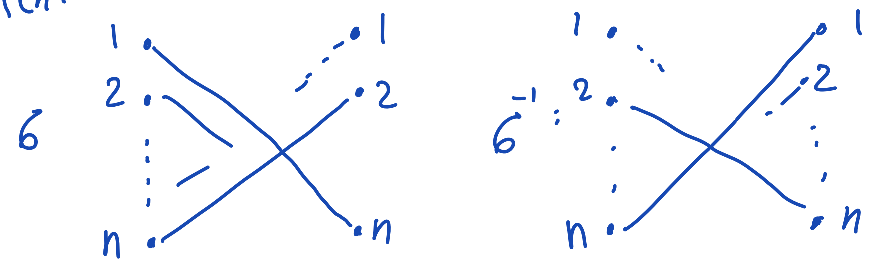
$$\det(A) = aei + bfg + cdh \quad \text{"}\backslash\text{"}$$
$$- gec - hfa - idb \quad \text{"}/\text{"}$$

17.3 Properties of determinants

Lemma 17.5 For all $\sigma \in S_n$ we have $\mathrm{inv}(\sigma) = \mathrm{inv}(\sigma^{-1})$

$\Rightarrow \mathrm{sign}(\sigma) = \mathrm{sign}(\sigma^{-1})$

Proof sketch:



$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det(A) = ad - bc, \quad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\det(A^T) = ad - cb.$$

Proposition 17.6: For any $A \in \mathbb{R}^{n \times n}$ we have $\det(A) = \det(A^T)$

Proof: $A = (a_{i,j})$, $A^T = (b_{i,j})$

we have $a_{i,j} = b_{j,i}$

$$\det(A) = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

$$\det(A^T) = \sum_{\tau \in S_n} \mathrm{sign}(\tau) \prod_{j=1}^{n} \underbrace{b_{j,\tau(j)}}_{a_{\underbrace{\tau(j)}_{k},\underbrace{j}_{\tau^{-1}(k)}}}$$

$k = \tau(j)$, $\tau^{-1}(k) = j$

$$= \sum_{\tau \in S_n} \mathrm{sign}(\tau) \prod_{k=1}^{n} a_{k,\tau^{-1}(k)}$$

$\tau^{-1} = \sigma$
$\sigma^{-1} = \tau$

$$= \sum_{\sigma \in S_n} \mathrm{sign}(\sigma^{-1}) \prod_{k=1}^{n} a_{k,\sigma(k)} \overset{\text{Lemma 4.5}}{=} \det(A) \square$$