Linear Algebra II   Lecture 3

$$F: \mathbb{R}^{2\times 2} \to \mathbb{R}^2 \quad \text{is linear}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ d \end{pmatrix}$$

$\mathrm{Ker}(F) = \mathrm{span}\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$, $\quad \dim(\mathrm{Ker}(F)) = 2$

$\mathrm{im}(F) = \mathbb{R}^2 \qquad \dim(\mathrm{im}(F)) = 2$

$$\dim \mathbb{R}^{2\times 2} = 4 = 2+2$$

Theorem 15.3 Let V be fin. gen. and let $F: V \to W$ be a linear map. Then

$$\dim V = \underbrace{\dim(\mathrm{Ker}(F))}_{= \ell} + \underbrace{\dim(\mathrm{im}(F))}_{= n}.$$

Proof sketch: • $\mathrm{Ker}(F) \subset V$ subspace, i.e $\mathrm{Ker}(F)$ is fin. gen.

Let $(v_1, \dots, v_\ell)$ be a basis of $\mathrm{Ker}(F)$.

• We can extent this to a basis $(v_1, \dots, v_\ell, u_1, \dots, u_n)$ of V. (Thm. 14.8)

• Show: $(F(u_1), \dots, F(u_n))$ is a basis of $\mathrm{im}(F)$.
(Think about that!)

Example 51 Consider the linear map

$$D : P_2 \to P_2$$
$$f \mapsto f'$$

$f(x) = ax^2 + bx + c$, $(D f)(x) = 2ax + b$

$\ker(D) = P_0$, $\operatorname{im}(D) = P_1$

$\dim P_n = n+1$, $B = (1, x, \dots, x^n)$

will later show that this is a basis for arbitrary n (Vandermonde determinant)

Notice: P_n behaves like $\mathbb{R}^{n+1}$ " P_n and $\mathbb{R}^{n+1}$ are isomorphic"

$$ax^2 + bx + c \quad \rightsquigarrow \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Above linear map can be described as a lin. map $\mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 2a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

We will make this precise in the following:

Definition 15.4 i) (Recall) A fct. $f: X \to Y$ is <u>invertible</u> if there exists a fct. $g: Y \to X$ with $f \circ g = id_Y$ and $g \circ f = id_X$. f is invertible iff f is bijective.

(ii) An invertible lin. map $F: V \to W$ is called an <u>isomorphism</u>. (isos = equal, morphe = form / shape)

(iii) Two vector spaces V and W are called isomorphic (Notation $V \cong W$) if there exists an isomorphism $F: V \to W$.

(In example: $P_2 \cong \mathbb{R}^3$, $ax^2+bx+c \mapsto \begin{pmatrix} a \\ b \\ c \end{pmatrix}$)

Theorem 15.5 i) A lin. map $F: V \to W$ is an isomorphism iff $\ker(F) = \{0_V\}$ and $\operatorname{im}(F) = W$.

ii) Let $F: V \to W$ be an isomorphism an $(b_1, \dots, b_n)$ a basis of V. Then $(F(b_1), \dots, F(b_n))$ is a basis of W.

iii) Let V, W be fin. gen. and $V \cong W$ then $\dim(V) = \dim(W)$.

iv) Let V, W be fin. gen. and $\dim(V) = \dim(W)$. Then for a lin. map $F: V \to W$ the following statements are equivalent:

(a) F is an isomorphism
(b) $\ker(F) = \{0_V\}$
(c) $\operatorname{im}(F) = W$

Proposition 15.6 Let V be fin. gen. with basis $B=(b_1,\dots,b_n)$, i.e. $\dim(V)=n$. Then the <u>coordinate map</u>

$$C_B : \mathbb{R}^n \longrightarrow V$$
$$\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_n\end{pmatrix} \longmapsto \sum_{i=1}^{n} \lambda_i b_i$$

is an isomorphism. The inverse is given by $C_B^{-1}(x) = [u]_B$ for $u \in V$.

Example 53 $V = \mathbb{P}_2$, $B = (\underset{b_1}{x+1}, \underset{b_2}{x^2-1}, \underset{b_3}{x+3})$ basis of V.

$$C_B : \mathbb{R}^3 \longrightarrow \mathbb{P}_2$$
$$\begin{pmatrix}a\\ b\\ c\end{pmatrix} \longmapsto a\overset{b_1}{(x+1)} + b\overset{b_2}{(x^2-1)} + c\overset{b_3}{(x+3)}$$
$$= (b)x^2 + (a+c)x + (a+3c)$$

$$C_B\left(\begin{pmatrix}1\\-1\\2\end{pmatrix}\right) = -x^2+3x+8$$

$\Rightarrow$ If $f(x) = -x^2+3x+8$ then $[f]_B = \begin{pmatrix}1\\-1\\2\end{pmatrix}$.

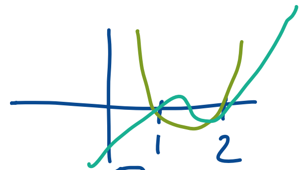
Corollary 15.7 Let V, W be fin. gen.. Then the following two statements are equivalent

i) $V \cong W$

ii) $\dim(V) = \dim(W)$.

(In particular: $V \cong \mathbb{R}^n$ for some $n \geq 1$)

Example 54 (Similar to HW 1)



$$U = \{ f \in P_3 \mid f(1) = f(2) = 0 \} \subset P_3$$

A possible basis of U is $B = (b_1, b_2)$

$$b_1(x) = (x-1)(x-2) = x^2 - 3x + 2 \quad , \quad b_2(x) = (x-1)(x-2)(x+3) = x^3 - 7x + 6$$

$$\Rightarrow \quad \dim U = 2 \quad \Rightarrow \quad U \cong \mathbb{R}^2$$

Isomorphism is given by

$C_B : \mathbb{R}^2 \to U$

$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \mapsto \lambda_1 b_1 + \lambda_2 b_2$

or

$C_B^{-1} : U \to \mathbb{R}^2$

$f \mapsto [f]_B$

Let $f(x) = (x-1)^2(x-2)$. What is $[f]_B$?

$$f(x) = \lambda_1 b_1(x) + \lambda_2 b_2(x) \quad \forall x \in \mathbb{R}.$$

$$(x-1)^2(x-2) = \lambda_1 (x-1)(x-2) + \lambda_2 (x-1)(x-2)(x+3)$$

$$x - 1 = \lambda_1 + \lambda_2 (x+3)$$

$$0 = \underbrace{(\lambda_2 - 1)}_{=0} x + \underbrace{\lambda_1 + 1 + 3\lambda_2}_{=0}$$

Here we use x and 1 are lin. indep.

$$\Rightarrow \quad \lambda_2 = 1 \qquad \lambda_1 = -4$$

$$[f]_B = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

§ 16 The matrix of a linear map

Goal: For fin. gen. vector spaces V, W and a linear map $F: \underset{\mathbb{R}^n}{\overset{\shortparallel}{V}} \to \underset{\mathbb{R}^m}{\overset{\shortparallel}{W}}$ define "the" matrix of F.

There are a lot of choices depending on bases of V and W.

Definition 16.1 Let V, W be fin. gen. vector spaces with bases $B_V = (v_1, \dots, v_n)$ and $B_W = (w_1, \dots, w_m)$. For a lin. map $F: V \to W$ we define the matrix of F with respect to B_V and B_W by

$$[F]^{B_W}_{B_V} := [C_{B_W}^{-1} \circ F \circ C_{B_V}].$$

$$\begin{tikzcd} V \arrow[r, "F"] & W \\ \mathbb{R}^n \arrow[u, "C_{B_V}"] \arrow[r, "G"'] & \mathbb{R}^m \arrow[u, "C_{B_W}"'] \end{tikzcd}$$

Here $\underbrace{C_{B_W}^{-1} \circ F \circ C_{B_V}}_{=G}$ is the linear map $\mathbb{R}^n \to \mathbb{R}^m$ for which we defined the matrix $[\,..\,]$ in LA I. We have:

$$[G] = \begin{pmatrix} | & & | \\ G(e_1) & \dots & G(e_n) \\ | & & | \end{pmatrix}$$

And by $G(e_j) = C_{B_W}^{-1}\Big(\underbrace{F\big(\underbrace{C_{B_V}(e_j)}_{v_j}\big)}_{F(v_j)}\Big)$ we get

$$\underbrace{\phantom{C_{B_W}^{-1}(F(C_{B_V}(e_j)))}}_{[F(v_j)]_{B_W}}$$

$$[F]_{B_V}^{B_W} = \begin{pmatrix} | & & | \\ [F(v_1)]_{B_W} & \cdots & [F(v_n)]_{B_W} \\ | & & | \end{pmatrix}$$

Special case: $V = W$ and $F =$ identity (id_V):

<u>Definition 16.2</u> Let $B_1 = (v_1, \dots, v_n)$ and $B_2 = (u_1, \dots, u_n)$ be bases of V. The <u>change-of-basis matrix from B_1 to B_2</u> is the matrix

$$S_{B_1}^{B_2} = [id_V]_{B_1}^{B_2} = [C_{B_2}^{-1} \circ C_{B_1}] = \begin{pmatrix} | & & | \\ [v_1]_{B_2} & \cdots & [v_n]_{B_2} \\ | & & | \end{pmatrix}.$$

For $v \in V$ we have

$$S_{B_1}^{B_2} [v]_{B_1} = [v]_{B_2}.$$

Example 55 Consider the linear map

$$D : \mathcal{P}_2 \longrightarrow \mathcal{P}_2$$
$$f \longmapsto f'$$

$$B = (\underset{b_1}{1}, \underset{b_2}{x}, \underset{b_3}{x^2}) \quad , \quad C = (\underset{c_1}{2}, \underset{c_2}{2x+4}, \underset{c_3}{x^2})$$

We could calculate $[D]^B_B$, $[D]^C_B$, $[D]^C_C$, $[D]^B_C$

$$f(x) = ax^2 + bx + c \overset{D}{\longmapsto} \overset{D(f)(x)}{\overset{/\!/}{f'(x)}} = 2ax + b$$

$$\begin{tikzcd}
{[f]_B = \begin{pmatrix} c \\ b \\ a \end{pmatrix}} \arrow[r, "{[D]^B_B}"] \arrow[dr, "{[D]^C_B}"] \arrow[d, "{S^C_B}"'] & {[D(f)]_B = \begin{pmatrix} b \\ 2a \\ 0 \end{pmatrix}} \\
{[f]_C = \begin{pmatrix} \frac{1}{2}c - b \\ \frac{1}{2}b \\ a \end{pmatrix}} \arrow[ur, "{[D]^B_C}"] \arrow[r, "{[D]^C_C}"'] & {[D(f)]_C = \begin{pmatrix} \frac{1}{2}b - 2a \\ a \\ 0 \end{pmatrix}}
\end{tikzcd}$$

Calculate the matrices:

$$[D]^B_B = \begin{pmatrix} | & | & | \\ [D(b_1)]_B & [D(b_2)]_B & [D(b_3)]_B \\ | & | & | \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}}$$

$$[D]_B^C = \begin{pmatrix} | & | & | \\ [D(b_1)]_C & [D(b_2)]_C & [D(b_3)]_C \\ | & | & | \end{pmatrix} = \boxed{\begin{pmatrix} 0 & \frac{1}{2} & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}$$

$B = (\underset{b_1}{1}, \underset{b_2}{x}, \underset{b_3}{x^2})$, $C = (\underset{c_1}{2}, \underset{c_2}{2x+4}, \underset{c_3}{x^2})$

$D(b_1) = 0$, $[D(b_1)]_B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $[D(b_1)]_C = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$D(b_2) = 1$, $[D(b_2)]_B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $[D(b_2)]_C = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$

$D(b_3) = 2x$, $[D(b_3)]_B = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$, $[D(b_3)]_C = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$$S_B^C = \begin{pmatrix} | & | & | \\ [b_1]_C & [b_2]_C & [b_3]_C \\ | & | & | \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$c_1 = \frac{1}{2} b_1$, $c_2 = \frac{1}{2} b_2 - b_1$, $c_3 = b_3$

$$S_C^B = \begin{pmatrix} | & | & | \\ [c_1]_B & [c_2]_B & [c_3]_B \\ | & | & | \end{pmatrix} = \begin{pmatrix} 2 & 4 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} \frac{1}{2} \\ \frac{1}{2} \\ \end{matrix} \underset{S_C^B}{\left(\begin{array}{ccc|ccc} 2 & 4 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)} \overset{-2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -1 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$S_C^B = (S_B^C)^{-1}$

Remarks: i) We have $\left(S^{B_2}_{B_1}\right)^{-1} = S^{B_1}_{B_2}$

ii) If we have base B_1, B_2, B_3, B_4 of V and a linear map $F: V \to V$, then

$$[F]^{B_4}_{B_1} = S^{B_4}_{B_3} [F]^{B_3}_{B_2} S^{B_2}_{B_1}$$

Often: $B_1 = B_4$, $B_2 = B_3$

$$[F]^{B_1}_{B_1} \qquad [F]_{B_2}$$

Notation → !!

$$[F]_{B_1} = S^{B_1}_{B_2} \overbrace{[F]^{B_2}_{B_2}} S^{B_2}_{B_1}$$

$$= \left(S^{B_2}_{B_1}\right)^{-1} [F]_{B_2} S^{B_2}_{B_1}.$$

iii) $$[F]^2_{B_1} = \left(S^{B_2}_{B_1}\right)^{-1} [F]_{B_2} \cancel{S^{B_2}_{B_1} \left(S^{B_2}_{B_1}\right)^{-1}} [F]_{B_2} S^{B_2}_{B_1}$$

$$= \left(S^{B_2}_{B_1}\right)^{-1} [F]^2_{B_2} S^{B_2}_{B_1}$$

Motivation: In applications we want to calculate $[F]_{B_1}^n$. For this we try to find B_2 such that $[F]_{B_2}^n$ is easy to calculate.

e.g. diagonal matrix