

Linear Algebra II

Lecture 2

Last time: Def vector space $(V, +, \cdot)$

Example 46 i) $V = \mathbb{R}^n$

ii)

$$\mathcal{P}_n \subset \mathcal{P} \subset \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \subset \mathcal{C}^n(\mathbb{R}, \mathbb{R}) \subset \mathcal{C}^0(\mathbb{R}, \mathbb{R}) \subset \mathcal{F}(\mathbb{R}, \mathbb{R})$$

These are all vector spaces.

iii) $V = \mathcal{J} =$ set of all infinite sequences " \mathbb{R}^∞ "

$$\ni \underline{(a_n)_{n \geq 1}} = (a_1, a_2, a_3, \dots) \quad 0 = (0, 0, \dots)$$

$$a = (a_n), \quad b = (b_n), \quad a + b = (a_n + b_n)$$

$$\lambda a = (\lambda a_n)$$

iv) Matrices: $V = \mathbb{R}^{m \times n}$ $n=0 = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$

v) If $(V, +, \cdot)$ is a vector space and $f: V \rightarrow W$ is a bijective map, then you can define for

$$u, v \in W: \quad u \oplus v = f(f^{-1}(u) + f^{-1}(v))$$

$$\lambda \in \mathbb{R} \quad \lambda \odot u = f(\lambda \cdot f^{-1}(u)) \quad (\text{HW 1, Ex. 1})$$

and obtain a vector space (W, \oplus, \odot) .

Definition 14.3 Let $(V, +, \cdot)$ be a vector space. A subset $U \subset V$ is a subspace if

- i) $0 \in U$
- ii) $\forall u, v \in U: u + v \in U$
- iii) $\forall u \in U, \lambda \in \mathbb{R}: \lambda u \in U$

Example 47 i) The sets in Ex. 46 ii) are subspaces

$$\mathcal{P}_n \subset \mathcal{P} \subset \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \subset \mathcal{C}^n(\mathbb{R}, \mathbb{R}) \subset \mathcal{C}^0(\mathbb{R}, \mathbb{R}) \subset \mathcal{F}(\mathbb{R}, \mathbb{R})$$

ii) $\mathcal{J}^0 = \{ (a_n)_{n \geq 1} \in \mathcal{J} \mid \lim_{n \rightarrow \infty} a_n \text{ exists} \} \subset \mathcal{J}$
is a subspace.

Clearly $\lim_{n \rightarrow \infty} 0 = 0$

Calculus: If $\lim_{n \rightarrow \infty} a_n$ and $\lim_{n \rightarrow \infty} b_n$ exist then

$$\lim_{n \rightarrow \infty} a_n + b_n \text{ exists.}$$

iii) $GL_n(\mathbb{R}) = \{ A \in \mathbb{R}^{n \times n} \mid A^{-1} \text{ exists} \} \subset \mathbb{R}^{n \times n}$
is not a subspace! $0 \notin GL_n(\mathbb{R})$

Proposition 14.4 If $U \subset V$ is a subspace, then U is also a vector space with the operations inherited from V .

Proof: Clear.

Definition 14.5. Let $(V, +, \cdot)$ be a vector space and $v_1, \dots, v_n \in V$.

i) The span of v_1, \dots, v_n is given by

$$\text{span}\{v_1, \dots, v_n\} = \left\{ \underbrace{\sum_{j=1}^n \lambda_j \cdot v_j}_{\text{linear combination}} \in V \mid \lambda_1, \dots, \lambda_n \in \mathbb{R} \right\}$$

ii) The elements v_1, \dots, v_n span V if $V = \text{span}\{v_1, \dots, v_n\}$.

iii) NEW V is finitely generated if there exist v_1, \dots, v_n with $\text{span}\{v_1, \dots, v_n\} = V$.

(In LA I everything was fin. gen.!))

iv) The elements v_1, \dots, v_n are linearly independent if

$$\lambda_1 v_1 + \dots + \lambda_n v_n = 0 \Rightarrow \lambda_1 = \dots = \lambda_n = 0$$

v) If $V = \text{span}\{v_1, \dots, v_n\}$ and v_1, \dots, v_n are lin. indep., then $B = (v_1, \dots, v_n)$ is a basis of V .

Example 48 i) Set $f_j(x) = x^j$. Then

f_0, f_1, \dots, f_n are linearly independent.

We will show this later by using determinants.

For $n=3$ (HW 1, Ex 3) please check it directly. For this we use different values for x (e.g., $-1, 0, 1, 2$) and we LAI.

In particular, $\mathcal{P}_n = \text{span}\{f_0, \dots, f_n\}$
is fin. gen.

ii) \mathcal{P} is not fin. gen., because if

$g_1, \dots, g_n \in \mathcal{P}$ with $\mathcal{P} = \text{span}\{g_1, \dots, g_n\}$

then set $d = \max_{i=1, \dots, n} \{\deg(g_i)\}$
 \uparrow
degree of g_i

Then f_{d+1} is not in $\text{span}\{g_1, \dots, g_n\}$ by i).

Check yourself:

Prop. 14.6, Lemma 14.7, Thm 14.8 (existence of
basis)

Def. 14.9 (dim), Cor 14.10,

Def 14.13: coordinates & coordinate vector.

§ 15 Linear maps

In the following we consider vector spaces V, W and write $+_V, \cdot_V$ for addition/scalar multiplication and 0_V for the neutral elements. Later we will usually just write $+$, \cdot and 0 .

Definition 15.1 Let $(V, +_V, \cdot_V)$, $(W, +_W, \cdot_W)$ be vector spaces.

A linear map is a function $F: V \rightarrow W$ satisfying

$$i) F(u +_V v) = F(u) +_W F(v)$$

$$ii) F(\lambda \cdot_V u) = \lambda \cdot_W F(u)$$

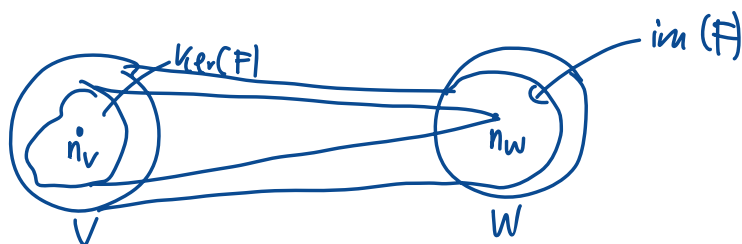
Definition 15.2 Let $F: V \rightarrow W$ be a linear map.

i) The kernel of F is

$$\ker(F) = \{u \in V \mid F(u) = 0_W\} \subset V$$

ii) The image of F is

$$\text{im}(F) = \{w \in W \mid \exists u \in V: w = F(u)\} \subset W$$



Example 50 i) The map

$$(n \geq 1): \quad D: \mathcal{C}^1(\mathbb{R}, \mathbb{R}) \longrightarrow \mathcal{F}(\mathbb{R}, \mathbb{R})$$
$$f \longmapsto f'$$

is linear, since

$$D(f+g) = (f+g)' = f' + g' = D(f) + D(g) \quad \forall f, g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$$
$$D(\lambda f) = \lambda D(f) \quad \lambda \in \mathbb{R}$$

$$\text{Ker}(D) = \{ \text{constant functions} \} = \mathcal{P}_0$$

$$\text{im}(D) = \mathcal{C}^{n-1}(\mathbb{R}, \mathbb{R})$$

ii) The map

$$e_v_a: \mathcal{F}(\mathbb{R}, \mathbb{R}) \longmapsto \mathbb{R}$$
$$f \longmapsto f(a)$$

is linear for any $a \in \mathbb{R}$.

$$e_v_a(f+g) = (f+g)(a) = f(a) + g(a)$$

$$\text{Ker}(e_v_a) = \{ f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \mid f(a) = 0 \} = e_v_a(f) + e_v_a(g).$$

$$\text{im}(e_v_a) = \mathbb{R}$$

iii) $F: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^2$ is linear
 $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ d \end{pmatrix}$

$$\text{Ker}(F) = \text{span} \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}, \quad \dim(\text{Ker}(F)) = 2$$

$$\text{im}(F) = \mathbb{R}^2 \quad \dim(\text{im}(F)) = 2$$

$$\dim \mathbb{R}^{2 \times 2} = 4 = 2 + 2$$

Theorem 15.3 Let V be fin. gen. and let
 $F: V \rightarrow W$ be a linear map. Then

$$\dim V = \underbrace{\dim(\text{Ker}(F))}_k + \underbrace{\dim(\text{im}(F))}_n$$

Proof sketch: • $\text{Ker}(F) \subset V$ subspace, i.e. $\text{Ker}(F)$ is fin. gen.

Let (v_1, \dots, v_k) be a basis of $\text{Ker}(F)$.

- We can extend this to a basis $(v_1, \dots, v_k, u_1, \dots, u_n)$ of V . (Thm. 1.8)
- Show: $(F(u_1), \dots, F(u_n))$ is a basis of $\text{im}(F)$.
(Think about that!)

Example 51 Consider the linear map

$$D: P_2 \rightarrow P_2$$

$$f \mapsto f'$$

$$f(x) = ax^2 + bx + c, \quad Df(x) = 2ax + b$$

$$\text{Ker}(D) = P_0, \quad \text{im}(D) = P_1$$

$$\dim P_n = n+1, \quad B = (1, x, \dots, x^n)$$

will later show that this is a basis
for arbitrary n (Vandermonde
determinant)

Notice: P_n behaves like \mathbb{R}^{n+1} " P_n and
 \mathbb{R}^{n+1} are isomorphic"
 $ax^2 + bx + c \mapsto \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Above linear map can be described as a lin.
map $\mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 2a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

We will make this precise in the following:

Definition 15.4 i) (Recall) A fct. $f: X \rightarrow Y$ is invertible if there exists a fct. $g: Y \rightarrow X$ with $f \circ g = \text{id}_Y$ and $g \circ f = \text{id}_X$. f is invertible iff f is bijective.

ii) An invertible lin. map $F: V \rightarrow W$ is called an isomorphism. (isos = equal, morphe = form / shape)

iii) Two vector spaces V and W are called isomorphic (Notation $V \cong W$) if there exists an isomorphism $F: V \rightarrow W$.

(In example: $\mathcal{P}_2 \cong \mathbb{R}^3$, $ax^2+bx+c \mapsto \begin{pmatrix} a \\ b \\ c \end{pmatrix}$)

Theorem 15.5 i) A lin. map $F: V \rightarrow W$ is an isomorphism iff $\text{Ker}(F) = \{0_V\}$ and $\text{im}(F) = W$.

ii) Let $F: V \rightarrow W$ be an isomorphism and (b_1, \dots, b_n) a basis of V . Then $(F(b_1), \dots, F(b_n))$ is a basis of W .

iii) Let V, W be fin. gen. and $V \cong W$ then $\dim(V) = \dim(W)$.

iv) Let V, W be fin. gen. and $\dim(V) = \dim(W)$. Then for a lin. map $F: V \rightarrow W$ the following statements are equivalent:

- (a) F is an isomorphism
- (b) $\text{Ker}(F) = \{0_V\}$
- (c) $\text{im}(F) = W$

Proposition 15.6 Let V be fin. gen. with basis $B = (b_1, \dots, b_n)$, i.e. $\dim(V) = n$. Then the coordinate map

$$C_B: \mathbb{R}^n \rightarrow V$$

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \mapsto \sum_{i=1}^n \lambda_i b_i$$

is an isomorphism. The inverse is given by $C_B^{-1}(u) = [u]_B$ for $u \in V$.

Example 53 $V = \mathcal{P}_2$, $B = \left(\begin{matrix} x+1 & x^2-1 & x+3 \\ b_1 & b_2 & b_3 \end{matrix} \right)$

$C_B: \mathbb{R}^3 \rightarrow \mathcal{P}_2$ basis of V .

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto a \overset{b_1}{(x+1)} + b \overset{b_2}{(x^2-1)} + c \overset{b_3}{(x+3)}$$

$$= (b)x^2 + (a+c)x + (a+3c)$$

$$C_B\left(\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}\right) = -x^2 + 3x + 8$$

\Rightarrow If $f(x) = -x^2 + 3x + 8$ then $[f]_B = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$.

Corollary 15.7 Let V, W be fin. gen.. Then the following two statements are equivalent

- i) $V \cong W$
 - ii) $\dim(V) = \dim(W)$.
- (In particular: $V \cong \mathbb{R}^n$ for some $n \geq 1$)