# Linear Algebra II

## Lecture 1

(1 – 13 LA I)

## § 14 Vector spaces

Vector space = A space having "addition" and "scalar multiplication" such that all "usual computation" rules as in $\mathbb{R}^n$ are satisfied.

**Example 45** $\mathcal{F}(\mathbb{R},\mathbb{R})$: all functions $\mathbb{R} \to \mathbb{R}$

$e^x \in$, sin $\in$, cos $\in$, polynomials $\in$

If $f, g \in \mathcal{F}(\mathbb{R},\mathbb{R})$ we can define

$$(f+g)(x) := f(x) + g(x) \qquad \text{addition}$$

$$\lambda \in \mathbb{R} \qquad (\lambda \cdot f)(x) := \lambda \cdot f(x) \qquad \text{scalar multiplication}$$

Consider the subset $U = \{ f \in \mathcal{F}(\mathbb{R},\mathbb{R}) \mid f'' = f \}$

1) The function $n(x) = 0 \ \forall x \in \mathbb{R}$ is in $U$. "zero function"

2) If $f, g \in U$ then $(f+g)''(x) = f''(x) + g''(x)$

$$= f(x) + g(x) = (f+g)(x)$$

$\Rightarrow$ $f+g \in U$

3) If $f \in U$ and $\lambda \in \mathbb{R}$: $(\lambda f)''(x) = \lambda f''(x) = \lambda f(x)$

$\Rightarrow$ $\lambda f \in U$

$\rightsquigarrow$ "$U$ is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$"

Elements in $U$: $f(x) = e^{x}$, $g(x) = e^{-x}$

Also their linear combinations $\lambda_1 f + \lambda_2 g \in U$

$\lambda_1, \lambda_2 \in \mathbb{R}$

One can show that these are all, i.e. "$U = \operatorname{span}\{f, g\}$".

Moreover if $\underbrace{\lambda_1 f + \lambda_2 g = n}$ then $\lambda_1 = \lambda_2 = 0$

this means $\lambda_1 f(x) + \lambda_2 g(x) = n(x) = 0 \ \forall x \in \mathbb{R}$

$\Rightarrow$ $f, g$ are "linearly independent"

$\Rightarrow$ "$(f, g)$ is a basis of $U$"

$\Rightarrow$ "$\dim U = 2$" $\rightsquigarrow$ "$U \cong \mathbb{R}^2$"
"isomorphic"

To show this consider different values of $x$

e.g. $x=0$ $\lambda_1 f_1(0) + \lambda_2 g(0)$

$= \lambda_1 + \lambda_2 = 0$

$x = 1$ : $\lambda_1 e + \lambda_1 \frac{1}{e} = 0$

$\overset{LAI}{\Rightarrow}$ $\lambda_1 = \lambda_2 = 0$

Now let us make the notion precise:

**Definition 14.1** A (real) vector space is a tuple $(V,+,\cdot)$, where $V$ is a set and $+$ and $\cdot$ are two functions

addition $\qquad\qquad$ scalar multiplication

$$+ : V \times V \longrightarrow V \qquad\qquad \cdot : \mathbb{R} \times V \longrightarrow V$$
$$(u,v) \longmapsto u+v \quad , \qquad (\lambda, v) \longmapsto \lambda \cdot v = \lambda v$$

satisfying the following properties:

- Properties of addition

Example: $V = \mathbb{R}^n$

(A.1) $\forall u,v,w \in V: \quad (u+v)+w = u+(v+w)$ (Associativity) ✓

(A.2) $\forall u,v \in V: \quad u+v = v+u$ (Commutativity) ✓

$n = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

(A.3) $\exists n \in V, \forall u \in V: \quad n+u = u$ (Identity/neutral element of addition)

(A.4) $\forall u \in V, \exists v \in V: \quad u+v = n$ (Inverse element of addition)

$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \; v = \begin{pmatrix} -u_1 \\ \vdots \\ -u_n \end{pmatrix} = (-1)\cdot u = -u$

- Compatibility of $+$ and $\cdot$ :

(C.1) $\forall u,v \in V, \lambda \in \mathbb{R}: \quad \lambda \cdot (u+v) = \lambda \cdot u + \lambda \cdot v$ (Distributivity I)

(C.2) $\forall u \in V, \lambda, \mu \in \mathbb{R}: \quad (\lambda \underset{\text{in } \mathbb{R}}{+} \mu) \cdot u = \lambda \cdot u + \mu \cdot u$ (Distributivity II)

(C.3) $\forall u \in V, \lambda, \mu \in \mathbb{R}: \quad \lambda \cdot (\mu \cdot u) = (\lambda \mu) \cdot u$

(C.4) $\forall u \in V: \quad 1 \cdot u = u$

- If $+$ and $\cdot$ are clear from context we just write $V$ instead of $(V,+,\cdot)$.
- "$+$" and "$\cdot$" are the usual symbols used, but any other symbol is possible. If we consider different vector spaces at the same time we might use other symbols, e.g. "$\oplus$", "$\odot$" or $(V,+_V,\cdot_V)$, $(W,+_W,\cdot_W)$

**Proposition 14.2** Let $(V,+,\cdot)$ be a vector space and $u\in V$.

i) $u+n=u$

ii) If $n,\tilde{n}\in V$ both satisfy (A.3) then $n=\tilde{n}$

(The neutral element is unige. Notation $n=0$)

If you see this symbol always make sure what it means!!

iii) If for a fixed $u\in V$ the elements $v,\tilde{v}\in V$ both satisfy (A.4), i.e. $u+v=u+\tilde{v}=n$, then $v=\tilde{v}$.

(The inverse is unique. Notation $v=-u$)

iv) $-u=(-1)\cdot u$ (i.e. $u+(-1)u=n$)

Proof: Do yourself (good Exercise!) or see lecture notes.

**Example 46** Example of vector spaces:

i) $V=\mathbb{R}^n$, $n=0=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$, $-u=\begin{pmatrix}-u_1\\ \vdots\\ -u_n\end{pmatrix}$

ii) $V=\mathcal{F}(\mathbb{R},\mathbb{R})=\{f:\mathbb{R}\to\mathbb{R}\}$ with addition & scalar mult. from Example 1.

$0=n \leftarrow$ zero function ($n(x)=0$ $\forall x\in\mathbb{R}$)

$$\mathcal{C}^0(\mathbb{R},\mathbb{R}) = \{ f \in \mathcal{F}(\mathbb{R},\mathbb{R}) \mid f \text{ is continuous} \}$$
$$\cup$$
$$\mathcal{C}^n(\mathbb{R},\mathbb{R}) = \{ f \in \mathcal{F}(\mathbb{R},\mathbb{R}) \mid f^{(n)} \text{ exists and is continuous} \}$$
$$\cup$$
Smooth functions
$$\mathcal{C}^\infty(\mathbb{R},\mathbb{R}) = \{ f \in \mathcal{F}(\mathbb{R},\mathbb{R}) \mid f^{(n)} \text{ exists for all } n \geq 0 \}$$
$$\cup$$
$$\mathcal{P} = \{ f \in \mathcal{F}(\mathbb{R},\mathbb{R}) \mid f \text{ is a polynomial fct.} \}$$
$$\cup$$
$$\mathcal{P}_n = \{ f \in \mathcal{F}(\mathbb{R},\mathbb{R}) \mid f(x) = \sum_{j=0}^{n} a_j x^j \text{ for some } a_1, \dots, a_n \in \mathbb{R} \}$$

These are all vector spaces.

iii) $V = \mathcal{J}$ = set of all infinite sequences "$\mathbb{R}^\infty$"

$\ni (a_n)_{\geq 1} = (a_1, a_2, a_3, \dots)$ $\quad 0 = (0, 0, \dots)$

$a = (a_n)$, $b = (b_n)$, $\quad a+b = (a_n + b_n)$

$\lambda a = (\lambda a_n)$

iv) Matrices: $V = \mathbb{R}^{m \times n}$ $\quad n = 0 = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix}$

v) If $(V, +, \cdot)$ is a vector space and $f: V \to W$ is a bijective map, then you can define for $u, v \in W$:

$$u \oplus v = f(f^{-1}(u) + f^{-1}(v))$$

$\lambda \in \mathbb{R}$

$$\lambda \odot u = f(\lambda \cdot f^{-1}(u)) \quad \text{(HW 1, Ex. 1)}$$

and obtain a vector space $(W, \oplus, \odot)$.

**Definition 14.3** Let $(V,+,\cdot)$ be a vector space. A subset $U \subset V$ is a <u>subspace</u> if

i) $0 \in U$

ii) $\forall u,v \in U: \ u+v \in U$

iii) $\forall u \in U, \lambda \in \mathbb{R}: \ \lambda u \in U$

**Example 47** i) The sets in Ex. 46 ii) are subspaces

$$\mathcal{P}_n \subset \mathcal{P} \subset \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \subset \mathcal{C}^n(\mathbb{R},\mathbb{R}) \subset \mathcal{C}^0(\mathbb{R},\mathbb{R}) \subset \mathcal{F}(\mathbb{R},\mathbb{R})$$

ii) $\mathcal{J}^0 = \{ (a_n)_{n\geq 1} \in \mathcal{J} \mid \lim\limits_{n\to\infty} a_n \text{ exists} \} \subset \mathcal{J}$

is a subspace.

Clearly $\lim\limits_{n\to\infty} 0 = 0$

Calculus: If $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ exist then

$\lim\limits_{n\to\infty} a_n + b_n$ exists.

iii) $GL_n(\mathbb{R}) = \{ A \in \mathbb{R}^{n\times n} \mid A^{-1} \text{ exists} \} \subset \mathbb{R}^{n\times n}$

is not a subspace! $\quad 0 \notin GL_n(\mathbb{R})$

**Proposition 14.4** If $U \subset V$ is a subspace, then $U$ is also a vector space with the operations inherited from $V$.

Proof: Clear.

**Definition 14.5.** Let $(V, +, \cdot)$ be a vector space and $v_1, \dots, v_n \in V$.

i) The span of $v_1, \dots, v_n$ is given by

$$\operatorname{span}\{v_1, \dots, v_n\} = \Big\{ \underbrace{\sum_{j=1}^{n} \lambda_i \cdot v_i}_{\text{linear combination.}} \in V \,\Big|\, \lambda_1, \dots, \lambda_n \in \mathbb{R} \Big\}$$

ii) The elements $v_1, \dots, v_n$ span $V$ if $V = \operatorname{span}\{v_1, \dots, v_n\}$.

iii) $V$ is finitely generated if there exist $v_1, \dots, v_n$ with $\operatorname{span}\{v_1, \dots, v_n\} = V$.

NEW

(In LA I everything was fin. gen.!)

iv) The elements $v_1, \dots, v_n$ are linearly independent if

$$\lambda_1 v_1 + \dots + \lambda_n v_n = 0 \Rightarrow \lambda_1 = \dots = \lambda_n = 0$$

v) If $V = \operatorname{span}\{v_1, \dots, v_n\}$ and $v_1, \dots, v_n$ are lin. indep., then $B = (v_1, \dots, v_n)$ is a basis of $V$.

**Example 48** i) Set $f_j(x) = x^j$. Then $f_0, f_1, \dots, f_n$ are linearly independent.

We will show this later by using determinants.

For $n=3$ (HW 1, Ex 2) please check it directly. For this use different values for $x$ (e.g. $-1, 0, 1, 2$) and use LAI.

In particular, $\mathcal{P}_n = \text{span}\{f_0, \dots, f_n\}$ is fin. gen.

ii) $\mathcal{P}$ is <u>not</u> fin. gen., because if $g_1, \dots, g_n \in \mathcal{P}$ with $\mathcal{P} = \text{span}\{g_1, \dots, g_n\}$ then set $d = \max\limits_{i=1,\dots,n} \{\deg(g_i)\}$

degree of $g_i$

Then $f_{d+1}$ is not in $\text{span}\{g_1, \dots, g_n\}$ by i).

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Homework 1, Exercise 0: Read the rest of chapter 14.