Homework 3: Induction & Determinants

Deadline: 28th May (23:55 JST), 2026

Exercise 1. (2+2+2 = 6 Points) Use mathematical induction to prove the following statements.

(i) Let $A, B \in \mathbb{R}^{m\times m}$ be matrices such that $AB = BA$. Show that for all $n \geq 1$, we have

$$(AB)^n = A^n B^n.$$

(ii) For all integers $n \geq 1$, prove that

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}\,.$$

(iii) Let $A \in \mathbb{R}^{m\times m}$ be a square matrix and I the $m \times m$ identity matrix. Show that for all $n \geq 1$,

$$(I - A)\sum_{k=0}^{n-1} A^k = I - A^n\,.$$

(Here we use the convention $A^0 = I$.)

Exercise 2. (2+4 = 6 Points) (Geometric interpretation of the determinant)
We define the vectors $v = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, u = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \in \mathbb{R}^2$.

(i) Draw a sketch in $\mathbb{R}^2$ by connecting the points 0, v, u, and $v + u$ to form a parallelogram.

(ii) Show that the area of this parallelogram is given by $\det \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}$, i.e. the determinant of the matrix which has v and u as columns.

(Remark: This works in general, i.e. if you write two vectors in $\mathbb{R}^2$ into the columns of a matrix $A \in \mathbb{R}^{2\times 2}$ then $|\det(A)|$ gives the area of the parallelogram spanned by them.)

Exercise 3. (3+2 = 5 Points)

(i) Show (without using Proposition 17.7) that the determinant is linear in each row, i.e. for any $A = (a_{i,j}) \in \mathbb{R}^{n\times n}$ and $1 \leq l \leq n$ show that the map

$$\begin{aligned} F_{A,l} : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ x &\longmapsto \det(A(l;x)) \end{aligned}$$

is linear. Here, $A(l;x)$ denotes the matrix A, where the l-th row is replaced by the vector x^T. (See page 126 in the lecture notes)

(ii) Assume that A is invertible. What is the kernel of $F_{A,1}$?

Exercise 4. (4 Points) For $a_1, a_2, \ldots, a_n \in \mathbb{R}$ we define the matrix

$$A = \begin{pmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \ldots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \ldots & a_n^{n-1} \end{pmatrix} \in \mathbb{R}^{n\times n}\,.$$

Show that the determinant of A is given by

$$\det(A) = \prod_{1\leq i<j\leq n} (a_j - a_i)\,.$$

(Hint: Use that adding a multiple of rows/columns to other rows/columns does not change the determinant (Proposition 17.6 + 17.10). Try to prove the statement then by induction on n, i.e. try to use row/column operation to find a $n - 1 \times n - 1$-version of such a matrix.)