

Homework 2: Linear maps and their matrices

Deadline: 15th May (23:55 JST), 2026

Exercise 1. (2+2+2+2+2 = 10 Points) For $n \geq 0$ we define the map $E_n : \mathcal{P}_n \rightarrow \mathbb{R}^3$ for $p \in \mathcal{P}_n$ by

$$E_n(p) = \begin{pmatrix} p(-1) \\ p(0) \\ p(1) \end{pmatrix}.$$

- (i) Show that E_n is a linear map for any $n \geq 0$.
- (ii) Show that E_2 is an isomorphism and compute its inverse.
- (iii) Determine $[E_4]_B^C$, where $B = (1, x+1, x^2-1, x^3+1, x^4-1)$ is a basis of \mathcal{P}_4 and C is the standard basis of \mathbb{R}^3 .
- (iv) Determine whether E_1 and E_3 are injective and/or surjective.
- (v) Determine a basis of $\text{im}(E_1)$ and a basis of $\text{ker}(E_3)$.

Exercise 2. (1+1+2+2 = 6 Points) The Fibonacci numbers F_n are defined by $F_0 = 0, F_1 = 1$ and

$$F_n = F_{n-1} + F_{n-2}. \quad (n \geq 2)$$

In this exercise, we want to prove the following explicit formula:

$$F_n = \frac{1}{2^n \sqrt{5}} \left((1 + \sqrt{5})^n - (1 - \sqrt{5})^n \right). \quad (\boxtimes)$$

To do so, follow these steps:

- (i) Find a linear map $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $F^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}$ for all $n \geq 1$, where $F^n = \underbrace{F \circ \dots \circ F}_n$.
- (ii) Define the following two bases of \mathbb{R}^2 (you do not need to check that these are bases):

$$B_1 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right), \quad B_2 = \left(\begin{pmatrix} 2 \\ 1 + \sqrt{5} \end{pmatrix}, \begin{pmatrix} 2 \\ 1 - \sqrt{5} \end{pmatrix} \right).$$

Determine the change-of-basis matrices $S_{B_1}^{B_2}$ and $(S_{B_1}^{B_2})^{-1}$.

- (iii) Compute $[F]_{B_1}$ and $[F]_{B_2}$.
- (iv) Compute $[F]_{B_1}^n$ by using

$$[F]_{B_1} = (S_{B_1}^{B_2})^{-1} [F]_{B_2} S_{B_1}^{B_2}$$

and prove (\boxtimes) using (i).

Exercise 3. (2+2+2 = 6 Points) The Tribonacci numbers T_n are defined by $T_0 = 0, T_1 = 0, T_2 = 1$ and

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}. \quad (n \geq 3)$$

We define the space of Tribonacci-like sequences by

$$\mathcal{T} = \{(a_n)_{n \geq 0} \in \mathcal{J} \mid a_n = a_{n-1} + a_{n-2} + a_{n-3} \text{ for all } n \geq 3\},$$

where \mathcal{J} denotes the vector space of all infinite sequences (see Lecture 1).

- (i) Show that \mathcal{T} is a subspace of \mathcal{J} .
- (ii) Show that \mathcal{T} is finitely generated and find a basis B of \mathcal{T} .
- (iii) Define the map $G : \mathcal{T} \rightarrow \mathcal{T}$ on a sequence $a = (a_n)_{n \geq 0}$ by $G(a) = b$, where $b = (b_n)_{n \geq 0}$ is given by $b_n = a_{n+3}$. Show that G is an isomorphism and determine $[G]_B$, where B is the basis from (ii).