Linear Algebra II   Spring 2025

Tutorial 3

Matrix of a linear map

$$D: \mathcal{P}_2 \longrightarrow \mathcal{P}_2$$
$$f \longmapsto f'$$

$$f(x) = ax^2 + bx + c \qquad (Df)(x) = 2ax + b$$

Let $B_1 = (x^2, x, 1)$. Claim: B_1 is a basis of $\mathcal{P}_2$

linear independence: Want to show that $\underline{\text{IF}}$ for all $x \in \mathbb{R}$

$$\lambda_1 x^2 + \lambda_2 x + \lambda_3 = 0,$$

then $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Choose $x = 0, -1, 1$:

$x=0$: $\lambda_1 0 + \lambda_2 0 + \lambda_3 = 0$

$x=-1$: $\lambda_1 - \lambda_2 + \lambda_3 = 0$

$x=1$: $\lambda_1 + \lambda_2 + \lambda_3 = 0$

$$\Longleftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \overset{\text{LAI}}{\Longrightarrow}$$ Only solution is $\lambda_1 = \lambda_2 = \lambda_3 = 0$

Clearly $\mathcal{P}_2 = \operatorname{span}\{x^2, x, 1\} \Rightarrow B_1$ is basis

We get an isomorphism

$$C_{B_1} : \mathbb{R}^3 \longrightarrow P_2$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \longmapsto \lambda_1 x^2 + \lambda_2 x + \lambda_3$$

We define (Lecture 3) the matrix of $F: V \to V$ with respect to a basis B by: $(\dim V = n)$

$$D: \begin{tikzcd} V \arrow[r, "F"] & V \\ \mathbb{R}^n \arrow[u, "C_B"] \arrow[r, "{[F]_B}"'] & \mathbb{R}^n \arrow[u, "C_B"'] \end{tikzcd}$$

i.e. $[F]_B$ is the matrix of the linear map

$$C_B^{-1} \circ F \circ C_B : \mathbb{R}^n \to \mathbb{R}^n$$

In our case:

$$\left(C_{B_1}^{-1} \circ D \circ C_{B_1}\right)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = C_B^{-1}\left(D\left(C_{B_1}\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right)\right)$$

$$= C_{B_1}^{-1}\left(D\left(ax^2+bx+c\right)\right) = C_B^{-1}\left(2ax+b\right)$$

$$= \begin{pmatrix} 0 \\ 2a \\ b \end{pmatrix}$$ and therefore

$$C_B^{-1} \circ D \circ C_{B_1} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 2a \\ a \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{[D]_{B_1}} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Notice: This matrix will look completely different for a different basis.

Homework 1: Vector spaces

Deadline: 24th April (23:55 JST), 2025

Exercise 0. (2 Points)

(i) Try to solve the exercises below and write the solutions down by hand (paper, tablet) or by computer (Latex only). Create **one pdf-file** which contains your name on the first page and submit it before the deadline ends in TACT at the Assignment "Homework 1". Use precisely the following format as a filename: "**Familyname_Givenname_LA2_HW1.pdf**". Repeat this for future Homework by replacing HW1 with HW2, HW3, etc.. Points will be removed in future homeworks if this is not the case.

(ii) Read Chapter 14 of the lecture notes and compare the results and definitions with the corresponding results in Linear Algebra I (Chapters 1-13).

(You don't need to write down anything for Exercise 0)

Exercise 1. (3+2+2+1 = 8 Points) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an injective function. Define on $V := \mathrm{im}(\varphi)$ the addition $\oplus$ and the scalar multiplication $\odot$ for $u, v \in V$ and $\lambda \in \mathbb{R}$ by

$$\begin{aligned} u \oplus v &= \varphi(\varphi^{-1}(u) + \varphi^{-1}(v))\,, \\ \lambda \odot v &= \varphi(\lambda \cdot \varphi^{-1}(v)). \end{aligned}$$

Here $+$ and $\cdot$ denote the usual addition and multiplication in $\mathbb{R}$.

(i) Show that $(V, \oplus, \odot)$ is a vector space. What is the neutral element of $(V, \oplus, \odot)$?
(i.e. check that the operations $\oplus$ and $\odot$ satisfy the properties $(A.1) - (A.4)$ and $(C.1) - (C.4)$.)

(ii) Determine all subspaces of $(V, \oplus, \odot)$.

(iii) Find an isomorphism

$$F : (\mathbb{R}, +, \cdot) \longrightarrow (V, \oplus, \odot)\,.$$

Here $(\mathbb{R}, +, \cdot)$ denotes the vector space $\mathbb{R}^1$ with the usual addition and multiplication of real numbers.

(iv) Do (ii) and (iii) explicitly for the case $\varphi(x) = e^x$.

Exercise 2. (2+2+2+2 = 8 Points) Let $\mathcal{P}$ denote the set of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$. Define the following subsets

$$\begin{aligned} \mathcal{P}_3 &= \{f \in \mathcal{P} \mid \deg(f) \leq 3\}\,, \\ U &= \{f \in \mathcal{P}_3 \mid f(-1) = f(0) = 0\} \subset \mathcal{P}_3\,. \end{aligned}$$

(i) Show that U is a subspace of $\mathcal{P}_3$.

(ii) Determine a basis $B = (b_1, \ldots, b_n)$ of U.

(iii) Determine the coordinate vector $[f]_B$ for the function $f \in U$ given by $f(x) = x(x+1)^2$.

(iv) Extend the basis B to a basis $\tilde{B}$ of $\mathcal{P}_3$.
(i.e. find a basis of $\mathcal{P}_3$, which contains all the basis elements of your basis B of U)

Exercise 3. (2+2+2 = 6 Points) Define for $M \in \mathbb{R}^{2\times 2}$ the following set

$$C(M) = \left\{A \in \mathbb{R}^{2\times 2} \mid AM = MA\right\}.$$

(i) Show that for a given fixed $M \in \mathbb{R}^{2\times 2}$ the set $C(M)$ is a subspace of $\mathbb{R}^{2\times 2}$.

(ii) For $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ determine a basis of $C(T)$.

(iii) Show that for all $M \in \mathbb{R}^{2\times 2}$ we have

$$2 \leq \dim(C(M)) \leq 4\,.$$

(i.e. show that there exists no matrix M, such that $C(M)$ has dimension 0 or 1.)

Hints for HW1:

Ex 1 similar to tut Ex 1.
Ex 2 —— " —— Ex 2

Ex 3: i) Check $0 \in C(M)$ (clear)

If $A, B \in C(M)$: $(A+B)M = \underset{\text{check}}{\dots} = M(A+B)$

$(\lambda A)M = \dots = M(\lambda A)$

ii) Assume $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. What does

$TA = AT$ imply for a, b, c, d?

iii) $\mathbb{R}^{2\times 2}$ has basis $\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$

$\Rightarrow \dim \mathbb{R}^{2\times 2} = 4$

$$C(I_2) = \{ A \in \mathbb{R}^{2\times 2} \mid \overset{\text{always true}}{A I_2 = I_2 A} \}$$

$$= \mathbb{R}^{2\times 2}$$

$$\dim C(I_2) = 4$$

To show $\dim(C(M)) \geq 2$: Find matrices which are always in $C(M)$