

Tutorial 2: Vector spaces

A (real) **vector space** is a tuple $(V, +, \cdot)$, where V is a set together with two functions

$$\begin{aligned} + : V \times V &\longrightarrow V & \cdot : \mathbb{R} \times V &\longrightarrow V \\ (u, v) &\longmapsto u + v & (\lambda, v) &\longmapsto \lambda v \end{aligned}$$

such that the following properties are satisfied:

- Properties of the addition:
 - (A.1) $\forall u, v, w \in V: (u + v) + w = u + (v + w)$. (Associativity)
 - (A.2) $\forall u, v \in V: u + v = v + u$. (Commutativity)
 - (A.3) $\exists n \in V, \forall u \in V: n + u = u$. (Identity/neutral element of addition)
 - (A.4) $\forall u \in V, \exists v \in V: u + v = n$. (Inverse elements of addition)
- Compatibility of addition and scalar multiplication:
 - (C.1) $\forall u, v \in V, \lambda \in \mathbb{R}: \lambda \cdot (u + v) = \lambda u + \lambda v$. (Distributivity I)
 - (C.2) $\forall u \in V, \lambda, \mu \in \mathbb{R}: (\lambda + \mu) \cdot u = \lambda u + \mu u$. (Distributivity II)
 - (C.3) $\forall u \in V, \lambda, \mu \in \mathbb{R}: \lambda \cdot (\mu u) = (\lambda \mu) \cdot u$.
 - (C.4) $\forall u \in V: 1 \cdot u = u$.

Exercise 1. Let $V = \{x \in \mathbb{R} \mid x > 0\}$ and define for $u, v \in V$ and $\lambda \in \mathbb{R}$:

$$\begin{aligned} u \oplus v &= uv, \\ \lambda \odot v &= v^\lambda. \end{aligned}$$

Show that (V, \oplus, \odot) is a vector space.

- A polynomial function is a function $f : \mathbb{R} \rightarrow \mathbb{R}$, such that there exist fixed $a_0, a_1, \dots, a_m \in \mathbb{R}$ with $f(x) = \sum_{j=0}^m a_j x^j$ for all $x \in \mathbb{R}$. The largest j with $a_j \neq 0$ is called the degree of f , denoted by $\deg(f)$.

- We denote the vector space of all polynomial functions by

$$\mathcal{P} = \{f : \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ is a polynomial function}\},$$

where the addition and scalar multiplication is the usual one given on functions $\mathbb{R} \rightarrow \mathbb{R}$.

- For $n \geq 0$ denote by $\mathcal{P}_n = \{f \in \mathcal{P} \mid \deg(f) \leq n\}$ the space of polynomial functions of degree $\leq n$.

For example, the function $f(x) = x^3 + 2x$ is an element in \mathcal{P}_m for all $m \geq 3$, but not in $\mathcal{P}_2, \mathcal{P}_1$ or \mathcal{P}_0 .

Exercise 2. Consider the following subset of \mathcal{P}_2

$$U = \{f \in \mathcal{P}_2 \mid f(1) = 0\}.$$

Find a basis of U .

Tut Exercise 1: Need to check these:

- *Properties of the addition:*

$$(A.1) \quad \forall u, v, w \in V: (u \oplus v) \oplus w = u \oplus (v \oplus w). \quad (\text{Associativity})$$

$$(A.2) \quad \forall u, v \in V: u \oplus v = v \oplus u. \quad (\text{Commutativity})$$

$$(A.3) \quad \exists n \in V, \forall u \in V: n \oplus u = u. \quad (\text{Identity/neutral element of addition})$$

$$(A.4) \quad \forall u \in V, \exists v \in V: u \oplus v = n. \quad (\text{Inverse elements of addition})$$

- *Compatibility of addition and scalar multiplication:*

$$(C.1) \quad \forall u, v \in V, \lambda \in \mathbb{R}: \lambda \odot (u \oplus v) = \lambda u \oplus \lambda v. \quad (\text{Distributivity I})$$

$$(C.2) \quad \forall u \in V, \lambda, \mu \in \mathbb{R}: (\lambda + \mu) \odot u = \lambda u \oplus \mu u. \quad (\text{Distributivity II})$$

$$(C.3) \quad \forall u \in V, \lambda, \mu \in \mathbb{R}: \lambda \odot (\mu u) = (\lambda \mu) \odot u.$$

$$(C.4) \quad \forall u \in V: 1 \cdot u = u.$$

$$\begin{aligned} (A.1) \quad (u \oplus v) \oplus w & \qquad u \oplus v := uv \\ & \qquad \lambda \odot v := v^\lambda \\ & = (uv) \oplus w = (uv)w \\ & = u(vw) = u \oplus (vw) = u \oplus (v \oplus w). \end{aligned}$$

$$(A.2) \quad \text{clear}$$

$$(A.3) \quad \text{choose } n=1, \text{ then } \forall u \in V: n \oplus u = 1 \cdot u = u$$

$$(A.4) \quad \text{if } u \in V \text{ we can choose } v = \frac{1}{u} \text{ (since } u > 0\text{).}$$
$$\text{then } u \oplus v = u \cdot v = u \cdot \frac{1}{u} = 1 = n.$$

$$(C.1): \quad \lambda \odot (u \oplus v) = \lambda \odot (uv) = (uv)^\lambda = u^\lambda v^\lambda = u^\lambda \oplus v^\lambda \\ = (\lambda \odot u) \oplus (\lambda \odot v)$$

$$(c.2): (\lambda + \mu) \odot u = u^{\lambda + \mu} = u^\lambda u^\mu = u^\lambda \oplus u^\mu \\ = \lambda \odot u \oplus \mu \odot u$$

$$(c.3): \lambda \odot (\mu \odot u) = \lambda \odot u^\mu = (u^\mu)^\lambda = u^{\mu\lambda} \\ = (\mu\lambda) \odot u = (\lambda\mu) \odot u$$

$$(c.4) \quad 1 \odot u = u^1 = u. \quad ((-1) \odot u = \bar{u} = \frac{1}{u} = "-u")$$

Tut Exercise 2:

$$U = \{ f \in \mathcal{P}_2 \mid f(1) = 0 \} \subset \mathcal{P}_2$$

Let $f \in \mathcal{P}_2$, then $f(x) = ax^2 + bx + c$ for some $a, b, c \in \mathbb{R}$.

If $f \in U$, then $f(1) = 0$, i.e. $a + b + c = 0$.

We have two free variables, ↑
linear system!

Set $b = t_1$, and $c = t_2$ with $t_1, t_2 \in \mathbb{R}$. Then $a = -t_1 - t_2$

i.e. we have $f(x) = (-t_1 - t_2)x^2 + t_1x + t_2$

$$= (-x^2 + x)t_1 + (-x^2 + 1)t_2.$$

$$\Rightarrow U = \text{span} \left\{ \underbrace{-x^2 + x}_{f_1(x)}, \underbrace{-x^2 + 1}_{f_2(x)} \right\}$$

Claim: f_1 and f_2 are lin. indep. and form a basis of U .

Assume $\lambda_1 f_1 + \lambda_2 f_2 = 0$, i.e.

$$\forall x \in \mathbb{R} \quad \lambda_1 f_1(x) + \lambda_2 f_2(x) = 0$$

Choose some values for x to show that $\lambda_1 = \lambda_2 = 0$:

$$\left. \begin{array}{l} x=0 \quad \lambda_1 f_1(0) + \lambda_2 f_2(0) = \lambda_2 = 0 \\ x=-1 \quad \lambda_1 f_1(-1) + \lambda_2 f_2(-1) = -2\lambda_1 = 0 \end{array} \right\} \lambda_1 = \lambda_2 = 0$$

$\Rightarrow (f_1, f_2)$ is a basis of U .

Claim: We can set $f_3(x) = 1$ and get a basis (f_1, f_2, f_3) for \mathcal{P}_2

Homework 1: Vector spaces

Deadline: 24th April (23:55 JST), 2025

Exercise 0. (2 Points)

- (i) Try to solve the exercises below and write the solutions down by hand (paper, tablet) or by computer (Latex only). Create **one pdf-file** which contains your name on the first page and submit it before the deadline ends in TACT at the Assignment "Homework 1". Use precisely the following format as a filename: "**Familiyname_Givenname_LA2_HW1.pdf**". Repeat this for future Homework by replacing HW1 with HW2, HW3, etc.. Points will be removed in future homeworks if this is not the case.
- (ii) Read Chapter 14 of the lecture notes and compare the results and definitions with the corresponding results in Linear Algebra I (Chapters 1-13).

(You don't need to write down anything for Exercise 0)

Exercise 1. (3+2+2+1 = 8 Points) Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be an injective function. Define on $V := \text{im}(\varphi)$ the addition \oplus and the scalar multiplication \odot for $u, v \in V$ and $\lambda \in \mathbb{R}$ by

$$\begin{aligned}u \oplus v &= \varphi(\varphi^{-1}(u) + \varphi^{-1}(v)), \\ \lambda \odot v &= \varphi(\lambda \cdot \varphi^{-1}(v)).\end{aligned}$$

Here $+$ and \cdot denote the usual addition and multiplication in \mathbb{R} .

- (i) Show that (V, \oplus, \odot) is a vector space. What is the neutral element of (V, \oplus, \odot) ? (i.e. check that the operations \oplus and \odot satisfy the properties (A.1) – (A.4) and (C.1) – (C.4).)
- (ii) Determine all subspaces of (V, \oplus, \odot) .
- (iii) Find an isomorphism

$$F : (\mathbb{R}, +, \cdot) \longrightarrow (V, \oplus, \odot).$$

Here $(\mathbb{R}, +, \cdot)$ denotes the vector space \mathbb{R}^1 with the usual addition and multiplication of real numbers.

- (iv) Do (ii) and (iii) explicitly for the case $\varphi(x) = e^x$.

Exercise 2. (2+2+2+2 = 8 Points) Let \mathcal{P} denote the set of all polynomial functions from \mathbb{R} to \mathbb{R} . Define the following subsets

$$\begin{aligned}\mathcal{P}_3 &= \{f \in \mathcal{P} \mid \deg(f) \leq 3\}, \\ U &= \{f \in \mathcal{P}_3 \mid f(-1) = f(0) = 0\} \subset \mathcal{P}_3.\end{aligned}$$

- (i) Show that U is a subspace of \mathcal{P}_3 .
- (ii) Determine a basis $B = (b_1, \dots, b_n)$ of U .
- (iii) Determine the coordinate vector $[f]_B$ for the function $f \in U$ given by $f(x) = x(x+1)^2$.
- (iv) Extend the basis B to a basis \tilde{B} of \mathcal{P}_3 . (i.e. find a basis of \mathcal{P}_3 , which contains all the basis elements of your basis B of U)

Exercise 3. (2+2+2 = 6 Points) Define for $M \in \mathbb{R}^{2 \times 2}$ the following set

$$C(M) = \{A \in \mathbb{R}^{2 \times 2} \mid AM = MA\}.$$

- (i) Show that for a given fixed $M \in \mathbb{R}^{2 \times 2}$ the set $C(M)$ is a subspace of $\mathbb{R}^{2 \times 2}$.
- (ii) For $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ determine a basis of $C(T)$.
- (iii) Show that for all $M \in \mathbb{R}^{2 \times 2}$ we have

$$2 \leq \dim(C(M)) \leq 4.$$

(i.e. show that there exists no matrix M , such that $C(M)$ has dimension 0 or 1.)

Hints for HW1:

Ex 1 similar to tut Ex 1.

Ex 2 ——— " ——— Ex 2

Ex 3: i) Check $0 \in C(M)$ (clear)

$$\text{If } A, B \in C(M) : (A+B)M = \overset{\text{check}}{\dots} = M(A+B)$$
$$(\lambda A)M = \dots = M(\lambda A)$$

ii) Assume $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. What does

$$TA = AT \text{ imply for } a, b, c, d?$$

iii) $\mathbb{R}^{2 \times 2}$ has basis $\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$

$$\Rightarrow \dim \mathbb{R}^{2 \times 2} = 4$$

$$C(I_2) = \left\{ A \in \mathbb{R}^{2 \times 2} \mid A I_2 \overset{\text{always true}}{=} I_2 A \right\}$$
$$= \mathbb{R}^{2 \times 2}$$

$$\dim C(I_2) = 4$$

To show $\dim(C(M)) \geq 2$: Find matrices which are always in $C(M)$.