

Total: 36 Points

1) (2+2+2+2=8 Points) Decide if the following statements are true or false. Justify your answers.

- (i) The set $U = \{f \in C^\infty(\mathbb{R}, \mathbb{R}) \mid f'(x) = xf(x) \text{ for all } x \in \mathbb{R}\}$ is a subspace of $C^\infty(\mathbb{R}, \mathbb{R})$.
- (ii) There exists an invertible linear map $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ that satisfies $F(F(x)) = 0$ for all $x \in \mathbb{R}^3$.
- (iii) The map $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $F(x) = \det(xx^T)$ for $x \in \mathbb{R}^2$ is a linear map.
- (iv) Let (b_1, b_2) be a basis of a vector space V then $(b_1 + 2b_2, b_1 - b_2)$ is also a basis of V .

2) (5+3=8 Points) Consider the bases $B = (2x - 1, x + 1)$, $C = (-3, x - 2)$ of \mathcal{P}_1 and the linear map $F : \mathcal{P}_1 \rightarrow \mathcal{P}_1$ with

$$[F]_B^C = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}.$$

(You do not need to show that B and C are bases.)

- (i) Determine $F(x)$ and $F(1)$.
- (ii) Find a basis for $\ker(F)$ and calculate $\det(F)$.

3) (3+3=6 Points) For $n \geq 1$ consider the matrix $A_n = (a_{ij}) \in \mathbb{R}^{n \times n}$ with $a_{ij} = i - j$, i.e.

$$A_n = \begin{pmatrix} 0 & -1 & -2 & \dots & 1 - n \\ 1 & 0 & -1 & \dots & 2 - n \\ 2 & 1 & 0 & \dots & 3 - n \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ n - 1 & n - 2 & n - 3 & \dots & 0 \end{pmatrix}.$$

- (i) Decide for which n the matrix A_n is invertible.
- (ii) Calculate $\dim(\ker(A_n))$ for all $n \geq 1$.

4) (3+3+2=8 Points) We define the following elements in $\mathbb{R}^{2 \times 2}$

$$m_1 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, m_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, m_3 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, m_4 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

and define the following linear map

$$H : \mathbb{R}^{2 \times 2} \longrightarrow \mathbb{R}^{2 \times 2} \\ A \longmapsto 2A - A^T.$$

(You do not need to show that H is linear.)

- (i) Show that $M = (m_1, m_2, m_3, m_4)$ is a basis of $\mathbb{R}^{2 \times 2}$.
- (ii) Determine $[H]_M$ and $\det(H)$.
- (iii) Show that $U = \{A \in \mathbb{R}^{2 \times 2} \mid H(A) = 3A\}$ is a subspace of $\mathbb{R}^{2 \times 2}$ and determine $\dim(U)$.

5) (2+4=6 Points) We define for $n \geq 1$ the following number

$$C(n) = \sum_{m=1}^n (m \cdot m!),$$

where $m! = 1 \cdot 2 \cdots m$ is the factorial (e.g. $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24, \dots$).

- (i) Evaluate $C(1), C(2)$ and $C(3)$. Make a guess for a simple explicit formula for $C(n)$.
- (ii) Prove your guess by using induction on n .