Linear Algebra II

Lecture 5

Spring 2025
19th May

Last lecture: For $A=(a_{ij})\in\mathbb{R}^{n\times n}$

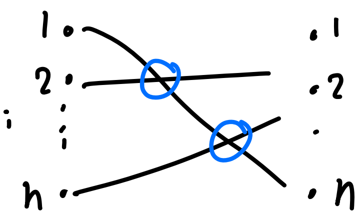
$\operatorname{sign}(\sigma)=(-1)^{\operatorname{inv}(\sigma)}$

$$\det(A)=\sum_{\sigma\in S_n}\operatorname{sign}(\sigma)\prod_{i=1}^{n} a_{i,\sigma(i)}.$$

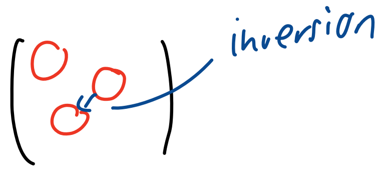
Permutation

$\operatorname{inv}(\sigma)$ = number of inversion

σ:



I_n pattern



$$\det\begin{pmatrix} a & b \\ c & d\end{pmatrix} = ad-bc$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i\end{pmatrix} = \underbrace{aei+bfg+cdh}_{+\text{ diagonals}}\ \underbrace{-gec-hfa-idb}_{-\text{ anti diagonals}}$$

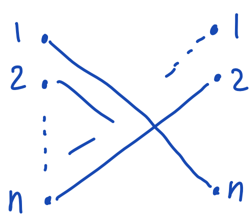
17.3 Properties of determinants

Lemma 17.5 For all $\sigma\in S_n$ we have $\operatorname{inv}(\sigma)=\operatorname{inv}(\sigma^{-1})$.

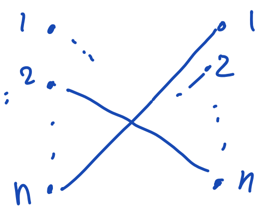
$\Rightarrow \operatorname{sign}(\sigma)=\operatorname{sign}(\sigma^{-1})$

Proof sketch:

σ



σ^{-1}:



$A=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$, $\det(A)=ad-bc$, $A^T=\begin{pmatrix} a & c \\ b & d\end{pmatrix}$ $\det(A^T)=ad-cb$.

Proposition 17.6 For all $A\in\mathbb{R}^{n\times n}$: $\det(A)=\det(A^T)$

Today: Show that $\det(A) \neq 0 \iff A$ invertible.

For a matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, $1 \le \ell \le n$ and $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ define

$$A(\ell; x) = \begin{pmatrix} a_{11} & \cdots & a_{1,n} \\ a_{\ell-1,1} & & a_{\ell-1,n} \\ x_1 & \cdots & x_n \\ a_{\ell+1,1} & \cdots & a_{\ell+1,n} \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{array}{l} \\ \\ \leftarrow \ell\text{-th row} \\ \\ \\ \end{array},$$

i.e. we replace the ℓ-th row of A by x^T.

<u>Proposition 17.7</u> For any $A \in \mathbb{R}^n$ and $1 \le \ell \le n$ the map

$$F_{A,\ell}: \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$x \longmapsto \det(A(\ell; x))$$

is linear.

<u>Proof</u>: This is Homework 3, Exercise 3

check $\det(A(\ell; x+y)) = \det(A(\ell;x)) + \det(A(\ell;y))$

$$\overset{\text{use def}}{=} \sum_{\sigma \in S_n} \cdots \quad = \sum_{\sigma} \cdots \quad + \sum_{\sigma} \cdots$$

similar for $\det(A(\ell; \lambda x))$

Example 61: For $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ and $\ell = 2$ the map $F_{A,2}$ is given by

$$F_{A,2} : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \det \begin{pmatrix} 1 & 2 & 3 \\ x_1 & x_2 & x_3 \\ 7 & 8 & 9 \end{pmatrix}$$

Use the formula for 3x3 det

$$6x_1 - 12x_2 + 6x_3 = \begin{matrix} 9x_2 + 14x_3 + 24x_1 \\ -21x_2 - 8x_3 - 18x_1 \end{matrix}$$

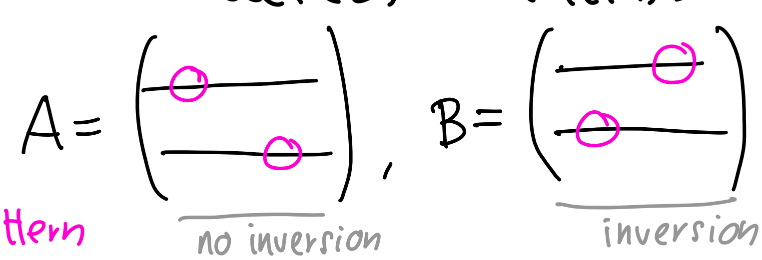
In this case, it is easy to see that this map is linear, since

$$F_{A,2}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 6 & -12 & 6 \end{pmatrix}}_{[F_{A,2}]} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Proposition 17.8 For $A \in \mathbb{R}^{n \times n}$ let $B \in \mathbb{R}^{n \times n}$ be a matrix obtained from A by swapping two rows. Then

$$\det(B) = -\det(A).$$

Proof idea



○ : choices of a Pattern

Corollary 17.9 If a matrix A contains two equal rows or columns, then $\det(A) = 0$.

Prop 17.6

Recall (LAI):

Row operations: There are 3 row operations for a matrix $A \in \mathbb{R}^{n \times n}$ ($1 \le i, j \le n$, $i \ne j$, $\lambda \in \mathbb{R}$).

(R1) Add λ times row j-th to the i-th row

(R2) For $\lambda \ne 0$ multiply the ith row with λ.

(R3) Swap row j and i.

A and B are row equivalent ($A \sim B$) if one can obtain B from A by using (R1), (R2) and (R3).

(In LAI: $A \sim \dots \sim \operatorname{rref}(A)$)

(Write $A \overset{R1}{\sim} B$ if B is obtained from A by using R1. Similar for R2 and R3.)

Proposition 17.10 Let $A, B \in \mathbb{R}^{n \times n}$

(i) $A \overset{R1}{\sim} B \Rightarrow \det(B) = \det(A)$

(ii) $A \overset{R2}{\sim} B \Rightarrow \det(B) = \lambda \det(A)$

(iii) $A \overset{R3}{\sim} B \Rightarrow \det(B) = -\det(A)$

Proof:
- ii) follows from Prop. 17.7 since the det is linear in the rows. $x =$ ℓ-th row of A; $A = A(\ell; x)$, $B = A(\ell; \lambda x)$; $\det(A(\ell;\lambda x)) = \lambda \det(A(\ell; x))$
- iii) is Prop. 17.8
- i): Using Prop. 17.7

r_j: j-th row of A

$$\underbrace{\begin{pmatrix} — r_1 — \\ \vdots \\ — r_n — \end{pmatrix}}_{A} \overset{R1}{\sim} \underbrace{\begin{array}{c} \\ j \\ i \\ \\ \end{array}\begin{pmatrix} — r_1 — \\ — r_j — \\ — r_i + \lambda r_j — \\ — r_n — \end{pmatrix}}_{B} = \underbrace{\begin{pmatrix} — r_1 — \\ — r_j — \\ — r_i — \\ — r_n — \end{pmatrix}}_{A} + \lambda \cdot \begin{pmatrix} — r_1 — \\ — r_j — \\ — r_j — \\ — r_n — \end{pmatrix}$$

Since det is linear in the rows we get

$$\det(B) = \det(A) + \lambda \cdot \underbrace{\det\begin{pmatrix} - r_1 - \\ - r_j - \\ - r_j - \\ - r_n - \end{pmatrix}}_{=0 \text{ by Cor } 4.9}$$
$$= \det(A)$$

Example 62 Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 4 & 6 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 4 & 6 \end{pmatrix} \overset{R1}{\sim} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 2 & 3 \end{pmatrix} \overset{R3}{\sim} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix} \overset{R1}{\sim} \frac{1}{2}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underset{\lambda = \frac{1}{2}}{\overset{R2}{\sim}} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{R1}{\sim} \underset{I_3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} = \operatorname{rref}(A) \qquad \det(I_3) = 1$$

Prop. 17.10:

$$1 = \det(I) = \underbrace{1 \cdot \frac{1}{2} \cdot 1 \cdot (-1) \cdot 1}_{-\frac{1}{2}} \cdot \det(A)$$

$$\Longrightarrow \det(A) = -2.$$

$$\overbrace{\det(A) = c \cdot \det(B)}^{c \neq 0}$$

We see that if $A \sim B$ then $\begin{cases} \det(A) = 0 \iff \det(B) = 0 \\ \det(A) \neq 0 \iff \det(B) \neq 0 \end{cases}$

(In particular $\det(A) = 0 \iff \det(\operatorname{rref}(A)) = 0$)

Theorem 17.11 For any matrix $A \in \mathbb{R}^{n \times n}$ we have

$$A \text{ is invertible} \iff \det(A) \neq 0.$$

Proof: "$\Rightarrow$": Assume that A is invertible. Then $A \sim \mathrm{rref}(A) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ and $\det\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = 1$.

$\Rightarrow \det(A) \neq 0$.

"$\Leftarrow$": We assume A is not invertible and will show $\det(A) = 0$.

If A is not invertible, then $rk(A) < n$, i.e. $\mathrm{rref}(A) = \begin{pmatrix} & * & \\ 0 & \cdots & 0 \end{pmatrix}$. Therefore

$\det(\mathrm{rref}(A)) = 0 \Rightarrow \det(A) = 0$.

<u>Theorem 17.12</u> i) For all $A, B \in \mathbb{R}^{n \times n}$ we have

$$\det(A \cdot B) = \det(A) \cdot \det(B)$$

ii) If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

Proof: i) will be proven in the next lecture

ii) Since $AA^{-1} = I_n$ and $\det(I_n) = 1$ we get by i)

$$1 = \det(AA^{-1}) = \det(A)\det(A^{-1})$$

$$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}.$$

<u>Corollary 17.13</u> Let V be fin. gen., $F: V \to V$ a linear map and B_1, B_2 bases of V. Then

$$\det([F]_{B_1}) = \det([F]_{B_2}).$$

Proof: Since $[F]_{B_1} = \underbrace{\left(S_{B_1}^{B_2}\right)^{-1}}_{= S_{B_2}^{B_1}} [F]_{B_2} S_{B_1}^{B_2}$ we get by Thm 4.12

$$\det([F]_{B_1}) = \underbrace{\det\left(\left(S_{B_1}^{B_2}\right)^{-1}\right)}_{= \frac{1}{\det(S_{B_1}^{B_2})}} \det([F]_{B_2}) \det\left(S_{B_1}^{B_2}\right) = \det([F]_{B_2}).$$

Thanks to this Corollary the following definition makes sense

Definition 17.14 V fin. gen., $F: V \to V$ linear map.
We define the determinant of F by

$$\det(F) = \det([F]_B),$$

where B is any basis of V.