

Linear Algebra II

Lecture 4

Spring 2025
12th May

Last time: $F: V \rightarrow V$ linear map, $B = (b_1, \dots, b_n)$

basis of V :

$$[F]_B := [F]_B^B = \begin{pmatrix} | & | \\ [F(b_1)]_B & \dots & [F(b_n)]_B \\ | & | \end{pmatrix} \in \mathbb{R}^{n \times n}$$

Next few lectures: Determinants

Matrix of F
with respect to B .

§ 17 Determinants

Recall: Ex 55

Motivation: $A \in \mathbb{R}^{n \times n}$

$$\det: \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}$$
$$A \mapsto \det(A)$$

$$D: P_2 \rightarrow P_1 \quad B = (1, x, x^2)$$
$$f \mapsto f' \quad C = (4x^2 + 4, x^3)$$

$$[D]_B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[D]_C = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$D(x^2) = 2x = -2 \cdot 2 + 2x + 4$$

Nice properties: $\det(A) \neq 0 \Leftrightarrow A$ is invertible

$$A, B \in \mathbb{R}^{n \times n}: \det(AB) = \det(A)\det(B)$$

$F: V \rightarrow V$, B_1, B_2 bases of V ($\dim(V) = n$)

$$\det([F]_B) = \det([F]_{B_2}) =: \det(F)$$

17.1 Mathematical induction (will be needed for some proofs)

P_n : Statement depending on a natural number n
($n \geq 1$)

Goal: Prove P_n is true for all $n \geq k$

1) Base step: Show P_1 is true

2) Induction step: Assume P_m is true for all
 $k \leq m < n$. Then show that
 P_n is true.

1) & 2) shows that P_n is true for all $n \geq 1$.

$P_1 \xrightarrow[true \text{ because of 1)}]{2)} P_2 \xrightarrow[2)]{P_{k+1}} P_3 \xrightarrow[2)]{P_{k+2}} \dots$

Remark: If you want to show that P_n is just true for all $n \geq k$, then you prove P_k in 1) and assume P_m is true for $k \leq m < n$ in 2).

(Another example: $P_n: 1+2+\dots+n = \frac{n \cdot (n+1)}{2}$)

Example 56 (Sum of first n odd numbers)

$$\left(\begin{array}{l} n=1: 1 = 1^2 \\ n=2: 1+3 = 4 = 2^2 \\ n=3: 1+3+5 = 9 = 3^2 \\ \vdots \end{array} \right)$$

P_n : "We have $\sum_{i=1}^n (2i-1) = n^2$ "

Prove by induction:

1) Base step: $n=1$ $\sum_{i=1}^1 (2i-1) = 2-1 = 1 = 1^2$
 $\Rightarrow P_1$ is true.

2) Induction step: Fix n and assume P_m is true for $m < n$.

In particular, for $m=n-1$ we assume that

$$\sum_{i=1}^{n-1} (2i-1) = (n-1)^2.$$

Then we get

$$\begin{aligned} \sum_{i=1}^n (2i-1) &= \sum_{i=1}^{n-1} (2i-1) + (2n-1) = (n-1)^2 + 2n-1 \\ &= n^2 - 2n + 1 + 2n - 1 = n^2. \end{aligned}$$

$\Rightarrow P_n$ is true.

17.2 Determinants

Definition 17.1 A pattern in an $n \times n$ -matrix is a choice of entries, such that precisely one entry from each row and each column is chosen.

Notation: $P = \{ (i_1, j_1), \dots, (i_n, j_n) \}$

row ↑
column ↑

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & P & 9 \end{pmatrix}$$

$$P = \{ (1,1), (2,3), (3,2) \}$$

Example 57 i) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2 \times 2}$

Then there are 2 patterns:

$$P_1 = \{ (1,1), (2,2) \}$$

$$P_2 = \{ (1,2), (2,1) \}$$

In general there are $n! = 1 \cdot 2 \cdot \dots \cdot n$ patterns.

$$\text{ii) } 3 \times 3 \text{ case: } A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \det(A) = a \cdot e \cdot i - a \cdot f \cdot h - b \cdot d \cdot i + b \cdot f \cdot g + c \cdot d \cdot h - c \cdot e \cdot g.$$

$$P_1 = \{(1,1), (2,2), (3,3)\} \quad P_4 = \{(1,2), (2,3), (3,1)\} \quad \times$$

$$P_2 = \{(1,1), (2,3), (3,2)\} \quad \times \quad P_5 = \{(1,3), (2,1), (3,2)\} \quad \times$$

$$P_3 = \{(1,2), (2,1), (3,3)\} \quad \times \quad P_6 = \{(1,3), (2,2), (3,1)\}$$

Definition 17.2 i) A bijective map $\sigma: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ is called a permutation of $\{1, \dots, n\}$.

ii) S_n denotes the set of all permutations of $\{1, \dots, n\}$.

Example 58

$$n=3, \quad \sigma: \{1, 2, 3\} \rightarrow \{1, 2, 3\}$$

$\sigma \in S_3$

$\sigma(1)=1$	1	•	—	•	1
$\sigma(2)=3$	2	•	—	•	2
$\sigma(3)=2$	3	•	—	•	3

We have a 1:1 correspondence

$$\xleftarrow{\hspace{1cm}} \text{Patterns of } n \times n \text{ matrices} \quad \xrightarrow{\hspace{1cm}} \text{Permutations of } \{1, \dots, n\}$$

$$P = \{(1, \sigma(1)), (2, \sigma(2)), \dots, (n, \sigma(n))\} \quad \xleftrightarrow{\hspace{1cm}} \quad \sigma$$

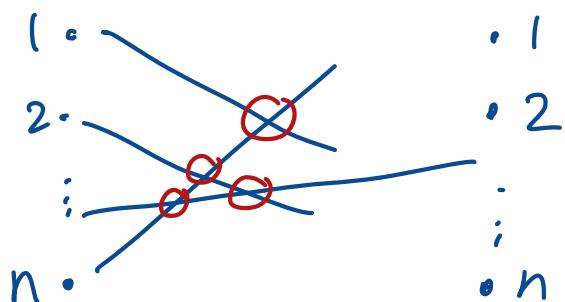
$$P = \{(1,1), (2,3), (3,2)\} \quad \xleftrightarrow{\hspace{1cm}} \quad \begin{array}{ccccccc} 1 & \bullet & \text{---} & \bullet & 1 \\ 2 & \bullet & \text{---} & \bullet & 2 \\ 3 & \bullet & \text{---} & \bullet & 3 \end{array}$$

Definition 17.3 i) The number of inversions of a permutation $\delta \in S_n$, denoted $\text{inv}(\delta)$, is the number of pairs $(i, \delta(i)), (j, \delta(j))$ with $i < j$ and $\delta(i) > \delta(j)$.

ii) The sign of a permutation $\delta \in S_n$ is

$$\text{sign}(\delta) = (-1)^{\text{inv}(\delta)} = \begin{cases} 1 & \text{inv}(\delta) \text{ even} \\ -1 & \text{inv}(\delta) \text{ odd} \end{cases}$$

To calculate $\text{inv}(\delta)$ count the number of intersections in the picture



Definition 17.4 The determinant of $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is defined by

$$\det(A) = \sum_{\delta \in S_n} \text{sign}(\delta) \prod_{i=1}^n a_{i, \delta(i)}.$$

Example 60:

i) In the case $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have

$$S_2 = \{\delta_1, \delta_2\}$$

$$\delta_1: \begin{array}{c|c} 1 & \dots \\ \hline 2 & \dots \end{array} \quad \text{inv}(\delta_1) = 0, \text{sign}(\delta_1) = 1$$

$$\delta_2: \begin{array}{c|c} 1 & \dots \\ \hline 2 & \cancel{\dots} \end{array} \quad \text{inv}(\delta_2) = 1, \text{sign}(\delta_2) = -1$$

$$\det(A) = \text{sign}(\delta_1) \cdot a_{1,\delta_1(1)} \cdot a_{2,\delta_1(2)}$$

$$+ \text{sign}(\delta_2) \cdot a_{1,\delta_2(1)} \cdot a_{2,\delta_2(2)}$$

$$= 1 \cdot a \cdot d + -1 \cdot b \cdot c = ad - bc$$

ii) Check that for $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ you get (by Example 57 ii)) "Sarrus rule"

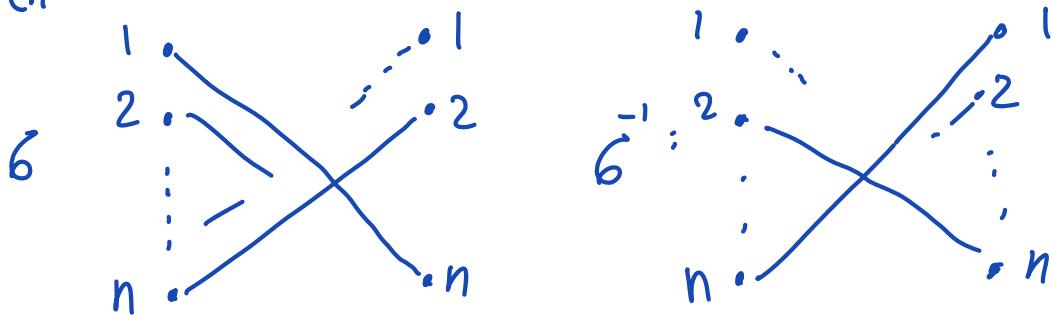
$$\det(A) = aei + bfg + cdh \quad \text{"\"/\"}"$$

$$- gec - hfa - idb \quad \text{"\"/\"}"$$

17.3 Properties of determinants

Lemma 17.5 For all $\sigma \in S_n$ we have $\text{inv}(\sigma) = \text{inv}(\sigma^{-1})$
 $\Rightarrow \text{sign}(\sigma) = \text{sign}(\sigma^{-1})$

Proof sketch:



$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \det(A) = ad - bc, \quad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \det(A^T) = ad - cb.$$

Proposition 17.6: For any $A \in \mathbb{R}^{n \times n}$ we have $\det(A) = \det(A^T)$

Proof: $A = (a_{i,j}), \quad A^T = (b_{i,j})$

we have $a_{i,j} = b_{j,i}$

$$\det(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^n a_{i, \sigma(i)}$$

$$\det(A^T) = \sum_{\tau \in S_n} \text{sign}(\tau) \prod_{j=1}^n b_{j, \tau(j)}$$

$\underbrace{b_{j, \tau(j)}}_{a_{\tau(j), j}}$
 $\underbrace{\tau}_{\tau'}$
 \underbrace{j}_{k}
 $\underbrace{\tau(k)}_{\tau'(k)}$

$$\kappa = \tau(j), \tau'(k) = j$$

$$= \sum_{\tau \in S_n} \text{sign}(\tau) \prod_{k=1}^n a_{\kappa, \tau'(k)} \quad \text{Lemma 4.5}$$

$$\tau' = \sigma \quad \sigma' = \tau \quad = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{k=1}^n a_{\kappa, \sigma(k)} \stackrel{\downarrow}{=} \det(A) \square$$